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F-BIHARMONIC CURVES IN THREE-DIMENSIONAL GENERALIZED SYMMETRIC SPACES

YASMINE BAHOUS, LAKEHAL BELARBI*, MANSOUR BELARBI AND HICHEM ELHENDI

Abstract. In this work we give the necessary and sufficient conditions for f-biharmonic curves in three-dimensional generalized symmetric spaces.

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1. Introduction

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two pseudo-Riemannian manifolds. We will call $e(\varphi)_x = \dfrac{1}{2}\sum_{i=1}^m h(d\varphi(E_i), d\varphi(E_i))$ the energy density of φ at x for any $\{E_i\}_{i=1}^m$ orthonormal basis of the tangent space T_xM can then be integrated over M, and with an eye toward the physical concept of kinetic energy $\dfrac{mv^2}{2}$ then the energy functional is defined by

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 dv_g. \tag{1}$$

A harmonic map φ is also a critical point of the energy functional E. The map φ being harmonic means that

$$\frac{d}{dt}E(\varphi_t)_{t=0} = 0,$$

holds for arbitrary smooth variation φ_t of φ, and denote the tension field $\tau(\varphi)$ of φ by

$$\tau(\varphi) = tr_g \nabla d\varphi = \sum_{i=1}^m \epsilon_i \Big(\nabla^{\varphi}_{E_i} E_i - d\varphi(\nabla_{E_i} E_i)\Big), \tag{2}$$

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*Corresponding author.

where $\epsilon_i = g(E_i, E_i) = \pm 1$.
The notion of harmonicity generalizes the usual one for mappings between Euclidean spaces, well known examples include geodesic curves. Next, the biharmonic maps $\varphi : (M^m, g) \longrightarrow (N^n, h)$ are defined as critical points of the bienergy functional:

$$E_2(\varphi) = \frac{1}{2}\int_M h(\tau(\varphi), \tau(\varphi))dv_g. \tag{3}$$

The Euler-Lagrange equation attached to the bienergy is

$$\tau_2(\varphi) = 0, \tag{4}$$

where $\tau_2(\varphi)$ is the bitension field given by

$$\tau_2(\varphi) = -(\Delta^\varphi \tau(\varphi) + tr_g R^N(\tau(\varphi), d\varphi)d\varphi), \tag{5}$$

where $\triangle = \text{trace}(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_\nabla)$ is the rough Laplacian on the sections of the pull-back bundle $\varphi^{-1}TN$, ∇^φ is the pull-back connection, and R^N is the curvature tensor on N.
Let $f : M \to \mathbb{R}$ be a smooth positive function on M. The f-energy functional of the map φ is given by

$$E_f(\varphi) = \frac{1}{2}\int_M fh(d\varphi(E_i), d\varphi(E_i))d\nu_g, \tag{6}$$

A map φ is called f-harmonic if it is a critical point of the energy functional E_f. The Euler-Lagrange equation attached to the f-energy is

$$\tau_f(\varphi) = 0,$$

where $\tau_f(\varphi)$ is the f-tension field given by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(gradf).$$

On the other hand the f-bienergy functional of the map φ is defined by

$$E_{2,f}(\varphi) = \frac{1}{2}\int_M fh(\tau(\varphi), \tau(\varphi))dv_g. \tag{7}$$

A map φ is called f-biharmonic if it is a critical point of the energy functional $E_{2,f}$. The Euler-Lagrange equation attached to the f-bienergy is

$$\tau_{2,f}(\varphi) = 0, \tag{8}$$

where $\tau_{2,f}(\varphi)$ is the f-bitension field given by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \triangle f\tau(\varphi) + 2\nabla^\varphi_{gradf}\tau(\varphi). \tag{9}$$

Clearly, any f-harmonic map was always a f-biharmonic map, and a proper f-biharmonic map can not be f-harmonic. (see [27]).
Biharmonic and nonharmonic submanifolds have been studied by many authors in [14, 12, 13, 16, 15, 30]. In [21], J. Inoguchi study biminimal submanifolds in contact 3-manifolds and biminimality of Legendre curves and Hopf cylinders. In [29], Ye-Lin Ou, the author introduced the concept of f-biharmonic submanifold. In [20], S. Güvenç and C. Özgür studied f-biharmonic Legendre curves in Sasakian manifold.

In this paper, we study the f-biharmonicity curves in three-dimensional generalized symmetric spaces and we give the necessary and sufficient conditions for $f-$biharmonic curves in $\mathbb{M}_3$.

2. Preliminaries

The $\mathbb{M}_3$ generalized symmetric space is the three-dimensional real space $\mathbb{R}^3$ endowed with the pseudo-Riemannian metric $g_{\epsilon,\lambda}$ given by

$$g_{\epsilon,\lambda} = \epsilon(e^{2t}dx^2 + e^{-2t}dy^2) + \lambda dt^2, \tag{10}$$

where $\epsilon = \pm 1$ and $\lambda \neq 0$ is a real constant. Depending on the values of ϵ and λ. We take the following orthonormal basis on $\mathbb{M}_3$

$$E_1 = e^{-t}\frac{\partial}{\partial x},\ E_2 = e^{t}\frac{\partial}{\partial y},\ E_3 = \frac{1}{\sqrt{|\lambda|}}\frac{\partial}{\partial t}. \tag{11}$$

The non-vanishing components of the Levi-Civita connection are given by:

$$\nabla_{E_1}E_1 = -\frac{\epsilon\epsilon_1}{\sqrt{|\lambda|}}E_3,\quad \nabla_{E_1}E_3 = \frac{1}{\sqrt{|\lambda|}}E_1,\quad \nabla_{E_2}E_2 = \frac{\epsilon\epsilon_1}{\sqrt{|\lambda|}}E_3,\quad \nabla_{E_2}E_3 = -\frac{1}{\sqrt{|\lambda|}}E_2, \tag{12}$$

where $\epsilon_1 = \dfrac{\lambda}{|\lambda|}$.

The non-vanishing Lie brackets are given by:

$$\Big[E_2, E_3\Big] = \frac{-1}{\sqrt{|\lambda|}}E_2,\quad \Big[E_1, E_3\Big] = \frac{1}{\sqrt{|\lambda|}}E_1. \tag{13}$$

The Riemannian curvature operator is given by

$$\mathbf{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{14}$$

The Riemannian curvature tensor is given by

$$\mathbf{R}(X,Y,Z,W) = -g(\mathbf{R}(X,Y)Z,W) \tag{15}$$

Moreover we put

$$\mathbf{R}_{abc} = \mathbf{R}(E_a,E_b)E_c,\qquad \mathbf{R}_{abcd} = \mathbf{R}(E_a,E_b,E_c,E_d) \tag{16}$$

By using equation 12, 13 and 14, we have

$$\begin{cases} \mathbf{R}(E_1,E_2)E_1 = -\frac{\epsilon}{\lambda}E_2, & \mathbf{R}(E_1,E_2)E_2 = \frac{\epsilon}{\lambda}E_1 \\ \mathbf{R}(E_1,E_3)E_1 = \frac{\epsilon}{\lambda}E_3, & \mathbf{R}(E_1,E_3)E_3 = -\frac{1}{|\lambda|}E_1 \\ \mathbf{R}(E_2,E_3)E_2 = \frac{\epsilon}{\lambda}E_3, & \mathbf{R}(E_2,E_3)E_3 = -\frac{1}{|\lambda|}E_2. \end{cases} \tag{17}$$

and

$$\begin{cases} \mathbf{R}_{1212} = \frac{1}{\lambda},\ \mathbf{R}_{1221} = -\frac{1}{\lambda}, \\ \mathbf{R}_{1313} = -\frac{\epsilon}{|\lambda|},\ \mathbf{R}_{1331} = \frac{\epsilon}{|\lambda|}, \\ \mathbf{R}_{2323} = -\frac{\epsilon}{|\lambda|},\ \mathbf{R}_{2332} = \frac{\epsilon}{|\lambda|}. \end{cases} \tag{18}$$

The non-vanishing Ricci curvature components $\{\mathbf{Ric}_{ij}\}$ are given by

$$\mathbf{Ric}_{33} = -\frac{2}{|\lambda|}. \tag{19}$$

3. Main results

3.1. In this section we give the necessary and sufficient conditions for $f-$biharmonic curves in $\mathbb{M}_3$. Suppose that $\gamma : I \to \mathbb{M}_3$ is a curve parameterized by arc-length. The Frenet orthonormal frame $\left\{\mathbf{T} = \gamma', \mathbf{N}, \mathbf{B}\right\}$ associated to the γ are following the Frenet formulas

$$\begin{cases} \nabla_{\gamma'}\mathbf{T} = \epsilon k\mathbf{N} \\ \nabla_{\gamma'}\mathbf{N} = -\epsilon k\mathbf{T} + \epsilon_1 \tau\mathbf{B} \\ \nabla_{\gamma'}\mathbf{B} = -\epsilon\tau\mathbf{N} \end{cases}$$

where $k = |\nabla_{\gamma'}\gamma'|$ is the geodesic curvature of γ and τ its the geodesic torsion and

$$g(\mathbf{T}, \mathbf{T}) = g(\mathbf{N}, \mathbf{N}) = \epsilon, g(\mathbf{B}, \mathbf{B}) = \epsilon_1.$$

We have

$$\begin{aligned} \tau(\gamma) &= \nabla^{\gamma}_{\frac{\partial}{\partial s}}(d\gamma(\frac{\partial}{\partial s})) - d\gamma(\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}) \\ &= \nabla^{\gamma}_{\frac{\partial}{\partial s}}(d\gamma(\frac{\partial}{\partial s})) = \nabla_{\gamma'}\gamma' = \nabla_{\mathbf{T}}\mathbf{T} = \epsilon k\mathbf{N}. \end{aligned} \tag{20}$$

and by definition we have

$$\mathbf{R}(\mathbf{T}, \mathbf{N}, \mathbf{T}, \mathbf{N}) = \frac{1}{\lambda}\left(2B_3^2 - 1\right), \tag{21}$$

$$\mathbf{R}(\mathbf{T}, \mathbf{N}, \mathbf{T}, \mathbf{B}) = -\frac{2}{\lambda}N_3B_3, \tag{22}$$

where N_3 and B_3 are the third components of the vectors $\mathbf{N}$ and $\mathbf{B}$ respectively.

By replacing equation 20 in the equation of bitension field 5, we get

$$\begin{aligned} \tau_2(\gamma) =& \nabla^3_{\mathbf{T}}\mathbf{T} - \mathbf{R}(\nabla_{\mathbf{T}}\mathbf{T}, \mathbf{T})\mathbf{T} \\ =& -3kk'\mathbf{T} + (\epsilon k'' - \epsilon k^3 - \epsilon_1 k\tau^2)\mathbf{N} \\ &+ (2\epsilon\epsilon_1 k'\tau + \epsilon\epsilon_1 k\tau')\mathbf{B} - \epsilon k\mathbf{R}(\mathbf{N}, \mathbf{T})\mathbf{T}. \end{aligned} \tag{23}$$

and

$$\nabla^{\gamma}_{gradf}\tau(\gamma) = \nabla^{\gamma}_{gradf}\epsilon k\mathbf{N} = f'\nabla_{\mathbf{T}}(\epsilon k\mathbf{N}) = f'(-k^2\mathbf{T} + \epsilon k'\mathbf{N} + \epsilon\epsilon_1 k\tau\mathbf{B}) \tag{24}$$

From equations 20, 23 and 24 we calculate the $f-$bitension field of the curve γ

$$\begin{aligned} \tau_{2,f}(\gamma) = & -3fkk'\mathbf{T} + f(\epsilon k'' - \epsilon k^3 - \epsilon_1 k\tau^2)\mathbf{N} + f(2\epsilon\epsilon_1 k'\tau + \epsilon\epsilon_1 k\tau')\mathbf{B} \\ & + f''\epsilon k\mathbf{N} + 2f'(-k^2\mathbf{T} + \epsilon k'\mathbf{N} + \epsilon\epsilon_1 k\tau\mathbf{B}) - f\epsilon kR(\mathbf{N}, \mathbf{T})\mathbf{T}. \end{aligned} \tag{25}$$

Now, by taking the inner product of equation 25 and using equations 21 and 22 respectively with $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$, we have:

Theorem 3.1. *The curve γ is f-biharmonic curve in $\mathbb{M}_3$ if and only if the following equations hold*

$$\begin{cases} 3fkk' = -2f'k^2, \\ fk'' - fk^3 - f\epsilon\epsilon_1 k\tau^2 + f\frac{1}{\lambda}k = 2f\frac{1}{\lambda}kB_3^2 - 2f'k' + f''k, \\ -2fk'\tau - fk\tau' = f\frac{2\epsilon_1}{\lambda}kN_3B_3 + 2f'k\tau. \end{cases} \tag{26}$$

3.2. In this section we discus the $f-$harmonicity of the curve γ, depending on the value of k and τ.

• If the curvature $k = c_1 \neq 0$ of γ is constant non null, the first equation in 26 gives that

$$f' = 0 \Rightarrow f = \texttt{constant} = c_2.$$

By replacing k and f in the second and the third equations in 26, we get the conditions of the $f-$biharmonicity of γ given by

$$\begin{cases} -c_1 - \epsilon\epsilon_1\tau^2 + \frac{1}{\lambda} = 2\frac{1}{\lambda}B_3^2, \\ \tau' = -\frac{2\epsilon_1}{\lambda}N_3B_3. \end{cases} \tag{27}$$

Corollary 3.2. *Let γ be a curve in $\mathbb{M}_3$ with non-null constant curvature in $\mathbb{M}_3$. Then γ is f-biharmonic curve if and only if the equations given in 27 are satisfied.*

• If the torsion $\tau = c_3 \neq 0$ of γ is constant non-null, the first equation in 26 gives that

$$f = c_4 k^{\frac{-3}{2}}. \tag{28}$$

And the third equation in 26 gives that

$$-2c_3(fk)' = \frac{2\epsilon_1}{\lambda}N_3B_3fk \Leftrightarrow f = \frac{c_5}{k}e^{\frac{-\epsilon_1}{c_3\lambda}\int N_3B_3ds}. \tag{29}$$

Now, combining both equations 28 and 29 gives

$$k = \sqrt{\frac{c_4}{c_5}e^{\frac{\epsilon_1}{2c_3\lambda}\int N_3B_3ds}}. \tag{30}$$

By replacing f given in 28 in the second equation in 26, we get the conditions of the $f-$biharmonicity of γ given by

$$c_4k''k^{\frac{-3}{2}} - c_4k^{\frac{3}{2}} + (-c_4c_3^2\epsilon\epsilon_1 + \frac{1}{\lambda}2c_4\frac{1}{\lambda}B_3^2)k^{\frac{-1}{2}} = 3c_4k'^2k^{\frac{-5}{2}} + \frac{15}{4}c_4k'^2k^{\frac{-5}{2}} - \frac{3}{2}c_4k''k^{\frac{-3}{2}}. \tag{31}$$

Corollary 3.3. *Let γ be a curve in $\mathbb{M}_3$ with a non-null constant torsion in $\mathbb{M}_3$. Then γ is f-biharmonic curve if and only if the curvature k of the curve γ given in the equation 30 satisfied the equations given in 31.*

• If the torsion of the curve γ is null($\tau = 0$), according to the value of f given in 28 and the second and third equations in 26, we get the conditions of the $f-$biharmonicity of γ given by

$$\begin{cases} c_4 k'' k^{\frac{-3}{2}} - c_4 k^{\frac{3}{2}} + (\frac{1}{\lambda} 2c_4 \frac{1}{\lambda} B_3^2) k^{\frac{-1}{2}} = 3c_4 k'^2 k^{\frac{-5}{2}} + \frac{15}{4} c_4 k'^2 k^{\frac{-5}{2}} - \frac{3}{2} c_4 k'' k^{\frac{-3}{2}} \\ N_3 B_3 = 0 \end{cases} \tag{32}$$

Corollary 3.4. *Let γ be a curve in $\mathbb{M}_3$ with null torsion. Then γ is f-biharmonic curve if and only if the curvature k of the curve γ given in the equation 30 satisfied the equations given in 32.*

• If γ be a curve with constant non-null curvature and constant non-null torsion. So immediately by the first equation in 26 gives that f is constant and the third equation in 26 gives $N_3 B_3 = 0$. We get the conditions of the $f-$biharmonicity of γ given by

$$\begin{cases} \texttt{If} B_3 \neq 0 \colon B_3^2 = \frac{\lambda}{2}(k + \tau^2 \epsilon \epsilon_1 - \frac{1}{\lambda}) \\ \texttt{If} B_3 = 0 \colon k + \tau^2 \epsilon \epsilon_1 - \frac{1}{\lambda} = 0 \end{cases} \tag{33}$$

Corollary 3.5. *Let γ be a curve in $\mathbb{M}_3$ with constant non-null curvature and constant non-null torsion in $\mathbb{M}_3$. Then γ is f-biharmonic curve if and only if the equations given in 33 are satisfied.*

Theorem 3.6. *Let γ be a curve in $\mathbb{M}_3$, then γ is proper f-biharmonic if and only if :*
(i) $\tau = 0$, $f = c_4 k^{-\frac{3}{2}}$ and the curvature k of the curve γ satisfied the equation

$$3(k')^2 - 2kk'' = 4k^2(k^2 + 2\frac{1}{\lambda} B_3^2 - \frac{1}{\lambda})$$

(ii) $\tau \neq 0$, $\frac{\tau}{k} = \frac{\exp^{\int \frac{-2N_3 B_3}{|\lambda| \tau}}}{{c_1}^2}$, and the curvature k satisfied the equation

$$3(k')^2 - 2kk'' = 4k^2(k^2(1 + \epsilon\epsilon_1 \frac{\exp^{\int \frac{-4N_3 B_3}{\tau|\lambda|}}}{{c_4}^4}) + 2\frac{1}{\lambda} B_3^2 - \frac{1}{\lambda})$$

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Yasmine Bahous received M.Sc. from Mostaganem University and Ph.D. student at University of Mostaganem. Her research interests include Differential Geometry and Topology.

Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (U.M.A.B.), B.P.227,27000, Mostaganem, Algeria.
e-mail: yasminebahous@gmail.com

Lakehal Belarbi received Magister Thesis from Mascara University and Ph.D at University of Sidi Bel Abbes. He is currently a professor at Abdelhamid Ibn Badis University of Mostaganem since 2012. His research interests include Differential Geometry and Topology.

Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (U.M.A.B.), B.P.227,27000, Mostaganem, Algeria.
e-mail: lakehalbelarbi@gmail.com

Mansour Belarbi received M.Sc. from Mostaganem University, and Ph.D. from Mascara University. He is currently a assistant professor at Mostaganem University since 2023. His research include Differential Geometry and Topology.

Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (U.M.A.B.), B.P.227,27000, Mostaganem, Algeria.
e-mail: belarbi_mansour@yahoo.com

Hichem Elhendi received Magister Thesis and Ph.D. from Oran 1 University. He is currently a professor at University of Bechar since 2011. His research include Differential Geometry and Topology.

Department of Mathematics, University of Bechar, PO Box 417, 08000, Bechar, Algeria.
e-mail: elhendihichem@yahoo.fr