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MONOPHONIC PEBBLING NUMBER OF SOME NETWORK-RELATED GRAPHS

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Abstract. Chung defined a pebbling move on a graph G as the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. The monophonic pebbling number guarantees that a pebble can be shifted in the chordless and the longest path possible if there are any hurdles in the process of the supply chain. For a connected graph G a monophonic path between any two vertices x and y contains no chords. The monophonic pebbling number, $\mu(G)$, is the least positive integer n such that for any distribution of $\mu(G)$ pebbles it is possible to move on G allowing one pebble to be carried to any specified but arbitrary vertex using monophonic a path by a sequence of pebbling operations. The aim of this study is to find out the monophonic pebbling numbers of the sun graphs, $(C_n \times P_2) + K_1$ graph, the spherical graph, the anti-prism graphs, and an n-crossed prism graph.

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1. Introduction

Lagarias and Saks introduced the concept of pebbling in the graph theory and later Chung in [Chung [1]] gave the literature form for it. Since then the concept of pebbling in graph theory evolved and Hulbert in [[2]] gives a details report on the development of different areas in graph pebbling. The research on this area has been going on for the past 30 years. Let G be a connected graph the vertex set be $V(G)$ and the edge set be $E(G)$. We consider a configuration D on the vertices of G for which it is possible to shift a pebble to the desired vertex. Santhakumaran, A. P et al.in [[4]] introduced the monophonic distance. Lourdusamy et al.in [[3]] introduced the detour pebbling number and found the

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detour pebbling number for the standard graphs and various derived graphs using detour paths. Detour pebbling number guarantees that a pebble can be transferred even if there are any hurdles in the supply chain process. Similarly, the monophonic pebbling number guarantees that a pebble can be shifted in the chordless and the longest path possible if there are any hurdles in the process of the supply chain. The monophonic distance between x and y is the length of the longest x-y monophonic path, denoted as $d_m(x,y)$, in G. For a connected graph G a monophonic path between any two vertices x and y contains no chords.[[4]] A chord is the line segment that connects two points on a curve. Lourdusamy et al. in [[5]] introduced monophonic pebbling number and monophonic t-pebbling number "A monophonic pebbling number, $\mu(G,v)$, of a vertex v of a graph G is the smallest number $\mu(G,v)$ such that at least one pebble may be moved to v using a monophonic path by a sequence of pebbling moves for any placement of $\mu(G,v)$ pebbles on the vertices of G. A monophonic path between u and v is a u-v path that contains no chords. The maximum $\mu(G,v)$ over all the vertices of G is the monophonic pebbling number of a graph, denoted as $\mu(G)$. A monophonic t-pebbling number, $\mu_t(G,v)$, of a vertex v of a graph G is the smallest number $\mu_t(G,v)$ such that it is possible to transfer t pebbles to v using a monophonic path by a sequence of pebbling moves for any placement of $\mu_t(G,v)$ pebbles on the vertices of G. The maximum of $\mu_t(G,v)$ over all vertices of G is the monophonic t-pebbling number, denoted by $\mu_t(G)$." The monophonic pebbling number of some network-related graphs is determined in this study.

Theorem 1.1 (Lourdusamy et al.,[5]). *For the cycle C_n , $\mu(C_n)$ is $2^{n-2}+1$.*

Theorem 1.2 (Lourdusamy et al.,[5]). *For the path P_n , $\mu(P_n)$ is 2^{n-1}.*

Theorem 1.3 (LOurdusamy et al., [5]). *The monophonic pebbling number of the wheel graph W_n is $\mu(W_n) = 2^{n-2}+2$.*

Notation 1.1. The number of pebbles on the vertex x is denoted as $p(x)$ and $p^{\sim}(x)$ is considered as the number of pebbles on the vertex x that is not on the monophonic path. Let $A \subset V(G)$. By $p^{\sim}(A)$ we mean the total number of pebbles placed on $V(A)$. In this paper, we denote M_K as the monophonic path and $M_K^{\sim}$ be the vertices that are not on M_K, where K is a non-negative positive number. For $(x_i)\underrightarrow{t}(x_l)$ refers taking off at least $2t$ pebbles from (x_i) and placing at least t pebbles on (x_l). Throughout the paper, we use r to denote the destination vertex.

2. Main results

Theorem 2.1. *The monophonic pebbling number of $(C_n \times P_2) + K_1$ for $n \geq 7$ is $2^{n+2\lceil \frac{n-6}{4} \rceil} + n - 2\lceil \frac{n-6}{4} \rceil$.*

Proof: The graphs $(C_n \times P_2) + K_1$ are like double wheel graphs, but the vertices of the two wheels are joined pairwise. They could alternatively be thought of as a prism $C_n \times P_2$, with every vertex joined to a common point. Let

$n \geq 7$. The vertex set of $(C_n \times P_2) + K_1$ is $\{u_i, v_j, v_0\}$ where $1 \leq i, j, \leq n$. The edge set of $(C_n \times P_2) + K_1$ is $\{v_i u_j, v_i v_0, u_j v_0, v_i v_{i+1}, u_j u_{j+1}, v_1 v_n, u_1 u_n\}$ where $1 \leq i, j, \leq n-1$. The number of vertices is $2n+1$ and the edges are $5n$.

The monophonic distance from v_1 to any other vertex is at most $n + 2\lceil \frac{n-6}{4} \rceil$. Let this monophonic path be M_1. Placing $2^{n+2\lceil \frac{n-6}{4} \rceil} - 1$ pebbles on v_n and one pebble each on $p^{\sim}(M_1)$, We can not shift a pebble to v_1. Thus, $\mu((C_n \times P_2) + K_1) \geq 2^{n+2\lceil \frac{n-6}{4} \rceil} + n - 2\lceil \frac{n-6}{4} \rceil$

Distributing $2^{n+2\lceil \frac{n-6}{4} \rceil} + n - 2\lceil \frac{n-6}{4} \rceil$ pebbles on the vertices of $(C_n \times P_2) + K_1$ for the configuration of C, we prove $\mu((C_n \times P_2) + K_1) \leq 2^{n+2\lceil \frac{n-6}{4} \rceil} + n - 2\lceil \frac{n-6}{4} \rceil$.

Case 1: Let $r = v_i$ or u_j be the destination vertex where $1 \leq i, j \leq n$.

If $p(v_{i+1}, v_{i-1}) \geq 2$ or $p(u_{i+1}, u_{i-1}) \geq 2$ or $p(v_0) \geq 2$, the proof is trivial. Without loss of generality, let $w = u_n$ be the destination to reach a pebble. Let the monophonic path M_1 be $\{v_1, v_2, u_2, u_3, u_4, v_4, v_5, v_6, u_6 \cdots, u_n\}$. The monophonic path M_1 has $n + 2\lceil \frac{n-6}{4} \rceil + 1$ vertices and $M_1^{\sim}$ which are not on M_1 has $n - 2\lceil \frac{n-6}{4} \rceil$ vertices. If we place $2^{n+2\lceil \frac{n-6}{4} \rceil}$ pebbles on v_1 and one pebble each on the vertices of $M_1^{\sim}$ which are not on M_1 then without using the pebbles from $M_1^{\sim}$ we can transfer a pebble to r by using the monophonic path from v_1 to w. Suppose $p(V(M_1)) < 2^{n+2\lceil \frac{n-6}{4} \rceil}$ and $p^{\sim}(M_1) > n - 2\lceil \frac{n-6}{4} \rceil$, then moving as many pebbles as possible to M_1, we can transfer a pebble to r. If there exists $\frac{p^{\sim}(V(M_1))}{2} + p(V(M_1)) \geq 2^{n+2\lceil \frac{n-6}{4} \rceil}$ pebbles, we can transfer a pebble to r. Similarly, we can prove this for all the vertices, since the monophonic distance for all the vertices are same. Hence, $\mu((C_n \times P_2) + K_1) = 2^{n+2\lceil \frac{n-6}{4} \rceil} + n - 2\lceil \frac{n-6}{4} \rceil$.

Theorem 2.2. *The monophonic pebbling number of the sun graph, $\mu(S_n)$, is $2^3 + (2n-4)$.*

The sun graph, S_n, is the graph with 2_n vertices consisting of a central complete graph K_n with an outer ring of n vertices, each of which is joined to both endpoints of the closest outer ring of the central core. Let $V(S_n) = \{v_1, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and $E(S_n) = \{v_i v_{i+1}, v_1 v_n, v_i u_i, u_i v_{i+1}, v_n u_n, u_n v_1, v_i v_j\}$ where $1 \leq i, j \leq n-1$ and $i \neq j$. The degree of vi is $n+1$ and u_i is 2.

Let M_1 be the monophonic path from u_1 to u_n. Consider $M_1 = \{u_1, v_2, v_n, u_n, \}$. The monophonic distance from u_1 to any other vertices of S_n is at most 3. There are $2n-4$ vertices that do not pass by M_1. Placing $2^3 - 1$ pebbles on u_1 and distributing one pebbles each on the remaining vertices that are not on M_1, we can not transfer a pebble to $r = u_n$. Thus, $\mu(S_n) \geq 2^3 + (2n-4)$

Let C be the configuration of $2^3 + (2n-4)$ pebbles on the vertices of S_n. Now we prove the sufficient condition.

Case 1: Let $r = u_k,\ k \in \{1, 2, 3, \cdots, n\}$. Without loss of generality, consider $r = u_n$. Then we arrive at having the monophonic path of length at most 3 from u_1. If $p(V(M_1)) \geq 2^3$, we are done. Let $p(V(M_1)) < 2^3$, $p^{\sim}(V(M_1)) \geq 2n-3$ and $N(r) = 0$. If there exist $E,\ 2 \leq E \leq 3$, pebbles each on u_i where $i \neq 1, n$, we can shift $V(u_i) \underset{\rightarrow}{2} V(M_1))$. Thus, using at least 2 pebbles on u_1 we can transfer

$u_1 \xrightarrow{1+1} v_1 \xrightarrow{(1+1)} v_{(n)} \xrightarrow{1} r$. Total number of pebbles used for this configuration is at most $3(n-2)+2 = 3n-4$. If there exist $E,\ 2 \leq E \leq 3$, pebbles each on v_i where $i \neq 1, 2, n$, we can transfer $V(v_i) \xrightarrow{2} V(M_1)) \xrightarrow{2} r$. Using at least 4 pebbles we can transfer a pebbles to r.

If there exists S pebbles each, $4 \leq\ S \leq 5$, on any two vertices of u_j or one vertex of v_i where $i \neq 1, n$ and $j \neq 1, 2, n$, we can transfer a pebble to r. The total number of pebbles used to reach the target through M_1 is at least 8 pebbles if we place on u_i or 4 pebbles if the pebbles are on v_i. Similarly, we can prove for all u_k.

Case 2: Let $r = v_k,\ k \in \{1, 2, 3 \cdots, n\}$. Without loss of generality, let $r = v_n$. Let M_2 be the monophonic path from u_{n-1} to v_1. Consider $M_2 = \{u_{n-1}, v_{n-1}, v_1\}$. The monophonic distance from u_{n-1} to v_1 is at most 2. There are $2n-3$ vertices that do not pass by M_2. Placing 2^2 pebbles on u_{n-1} and distributing one pebbles each on the remaining vertices that are not on M_2, we can transfer a pebble to $r = v_1$.

Let $p(V(M_2)) < 2^2$, $p^{\sim}(V(M_2)) \geq 2n-2$ and $N(r) = 0$. If there exist any two vertices of u_j with $E, 2 \leq E \leq 3$, pebbles each then we can transfer $(u_j) \xrightarrow{(1+1)} V(v_i)$, where $i, j \neq 1, n$. Thus, we can transfer a pebble to r. Let $p(V(M_2)) < 2^2$, $p^{\sim}(V(M_2)) \geq 2n-2$ and $N(r) = 1$. If we place E pebble on any one of the vertices of u_j we are done. Similarly, we can prove for all v_k. Hence, $\mu(S_n)$, is $2^3 + (2n-4)$.

Theorem 2.3. *The monophonic pebbling number of the spherical graph $S_2^{(n)}$ is $2^{2(2^{n-1}+1)-4} + 3$.*

Proof. The spherical graph,$S_2^{(n)}$, is a connected graph with $2(2^{n-1}+1)$ vertices and 3×2^n edges $n \in N$ obtained from $C_{2^n} + \bar{K}_2$. Let $V(S_2^{(n)}) = \{v_1, v_2, \cdots, v_{2^n}, u_1, u_2\}$ and $E(S_2^{(n)}) = \{v_i v_{i+1}, v_1 v_{2^n}, u_1 v_i, u_2 v_i\}$ where $1 \leq i \leq 2^n - 1$.

Let M_1 be the monophonic path from $v_{2^n} n$ to v_2. Consider $M_1 = \{v_{2^n}, v_{2^{n-1}}, v_{2^{n-2}}, \cdots, v_3, v_2\}$. The monophonic distance from v_{2^n} to any other vertex is at most 2^{n-1}. There are 3 vertices that do not pass by M_1 are $\{u_1, u_2, v_1\}$. Placing $2^{2(2^{n-1}+1)-4}-1$ pebbles on v_{2^n} and distributing one pebble each on the remaining vertices that are not on M_1, we can not transfer a pebble to $r = v_2$. Thus, $\mu(S_2^n)$, is $2^{2(2^{n-1}+1)-4} + 3$

Let c be the configuration of $2^{2(2^{n-1}+1)-4} + 3$ pebbles on the vertices of $S_2^{(n)}$. Now we prove the sufficient condition.

Case 1: Let $r = v_k,\ k \in \{1, 2, 3, \cdots, 2^n\}$. Without loss of generality, consider $w = v_{2^n}$. Then we arrive at having the monophonic path of length at most 2^{n-1} from v_2 to any other vertex of the graph $S_2^{(n)}$. Let the monophonic path $M-2$ be $\{v_{2^n}, v_{2^{n-1}}, v_{2^{n-2}}, \cdots, v_3, v_2\}$. If $p(V(M_2)) \geq 2^{2(2^{n-1}+1)-4}$, we are done.

Let $p(V(M_2)) < 2^{2(2^{n-1}+1)-4}$, $p^{\sim}(V(M_1)) \geq 4$. If there exist $E, 2 \leq E \leq 3$ pebbles on any one of the vertices of $M_2^{\sim}$ then we can transfer a pebble to r. Similarly, we can prove for all v_k.

Case 2: Let $r = u_1$ or u_2.

Without loss of generality, let $r = u_1$. Let M_3 be the monophonic path from u_2 to u_1. Consider $M_3 = \{u_1, v_1, u_2\}$. The monophonic distance from u_1 to any other vertex is at most 2. There are 2^{n-1} vertices that do not pass by M_3 that are $\{v_{2^n}, v_{2^{n-1}}, v_{2^{n-2}}, \cdots, v_3, v_2\}$. Placing 4 pebbles on u_2 and distributing one pebble each on the remaining vertices that are not on M_3, we can transfer a pebble to $r = u_1$.

Let $p(V(M_3)) < 3$, $p^{\sim}(V(M_2)) \geq 2^{n-1} + 1$. If there exist $E, 2 \leq E \leq 3$, pebbles on any one of the vertices of $M_2^{\sim}$ then we can transfer a pebble to w. Similarly, we can prove for u_2 Hence, $\mu(S_2^{(n)})$, is $2^{2(2^{n-1}+1)-4} + 3$. □

Theorem 2.4. *The monophonic pebbling number of the closed sun graph,* $\mu((\bar{S_n}))$, *is* $2^{n-1} + n$.

Proof. The closed sun graph, $(\bar{S_n})$, is the graph obtained from $S_n \cup C_n$. Let $V(\bar{S_n}) = \{v_1, \cdots, v_n, u_1, u_2,$
$\cdots, u_n\}$ and $E(\bar{S_n}) = \{v_i v_{i+1}, v_1 v_n, u_i u_{i+1}, u_1 u_n, v_i u_i, u_i v_{i+1}, v_n u_n, u_n v_1, v_i v_j\}$ where $1 \leq i, j \leq n-1$ and $i \neq j$. The degree of vi is $n+1$ and u_i is 4.

Let M_1 be the monophonic path from v_1 to u_{n-1}. Consider $M_1 = \{v_1, v_2, u_2, u_3 \cdots u_{n-1},\}$. The monophonic distance from v_1 to any other vertices of $(\bar{S_n})$ is at most $n-1$. There are n vertices that do not pass by M_1. Placing $2^{n-1} - 1$ pebbles on v_1 and distributing one pebbles each on the remaining vertices that are not on M_1, we can not transfer a pebble to $r = u_{n-1}$. Thus, $\mu(\bar{S_n}) \geq 2^{n-1} + n$

Let C be the configuration of $2^{n-1} + n$ pebbles on the vertices of S_n. Now we prove the sufficient condition.

Case 1: Let $r = u_k$ or v_k , $k \in \{1, 2, 3, \cdots, n\}$. Without loss of generality, consider $r = u_n$. Then we arrive at having the monophonic path of length at most $n-1$ from v_2. If $p(V(M_1)) \geq 2^{n-1}$, we are done. Let $p(V(M_1)) < 2^{n-1}$, $p^{\sim}(V(M_1)) \geq n+1$. If there exist $E,\ 2 \leq E \leq 3$, pebbles each on any one of the vertices that do not pass through M_1 we can transfer a pebble to r. Hence, $\mu(\bar{S_n}) = 2^{n-1} + n$.

□

Theorem 2.5. *The monophonic pebbling number of the anti-prism graph,* $\mu(A_n)$, *is* $2^n + n - 1$.

Proof. **Proof**: We obtain the anti-prism graph A_n by joining two cycles of equal length. Let $V(A_n) = \{v_1, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and $E(A_n) = \{v_i v_{i+1}, v_1 v_n,$
$u_i u_{i+1}, u_1 u_n, v_i u_i, u_i v_{i+1}, v_n u_n, u_n v_1\}$ where $1 \leq i \leq n-1$. The degree of vi is 4 and u_i is 4.

Let M_1 be the monophonic path from v_1 to v_{n-1}. Consider $M_1 = \{v_1, v_2, u_2, u_3,$
$u_4, u_5, \cdots u_{n-1}, v_{n-1}\}$. The monophonic distance from v_1 to any other vertices

of A_n is at most n. There are $n-1$ vertices that do not pass by M_1. Let it be $M_1^{\sim}$. Placing $2^n - 1$ pebbles on v_1 and distributing one pebbles each on $M_1^{\sim}$, we can not transfer a pebble to $r = v_{n-1}$. Thus, $\mu(A_n) \geq 2^n + n - 1$

Let C be the configuration of $2^n + n - 1$ pebbles on the vertices of A_n. Now we prove the sufficient condition.

Case 1: Let $r = u_k$ or v_k , $k \in \{1, 2, 3, \cdots, n\}$. Without loss of generality, consider $r = u_n$. Then we arrive at having the monophonic path of length at most n from v_2. Let the monophonic path be M_2. If $p(V(M_2)) \geq 2^n$, we are done. Let $p(V(M_2)) < 2^n$, $p^{\sim}(V(M_2)) \geq n$ and $N(r) = 0$ If there exist $E,\ 2 \leq E \leq 3$, pebbles each on the vertices of $M_2^{\sim}$ other than u_1, v_n and v_1 we can transfer at most $n-4$ pebbles to $V(M_2)$. By using $(2^n+n-1)-3(n-4) = 2^n-2n-11$ pebbles on M_2 we can put a pebble on r. Hence, $\mu(A_n) = 2^n+n-1$. □

Theorem 2.6. *The monophonic pebbling number of an n-crossed prism graph, $\mu(R_n)$, is $2^n + n$.*

Proof. **Proof**: We obtain the n-crossed prism graph,R_n, when n is positive even vertices and considering two disjoint cycle graphs of the same length. Let $V(R_n) = \{v_1, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and $E(R_n) = \{v_i v_{i+1}, v_1 v_n, u_i u_{i+1}, u_1 u_n,$ $v_j u_{j+1}, v_k u_{k-1}, v_1 u_n, v_8 u_1\}$ where $j = 2, 4, , 6, \cdots, n-2$ and $k = 3, 5, 7 \cdots, n-1$. The degree of vi is 3 and u_i is 3.

Let M_1 be the monophonic path from v_2 to v_n. Consider $M_1 = \{v_2, v_3, v_4, u_5,$ $u_6, u_7, \cdots u_n, u_1, v_n\}$. The monophonic distance from v_2 to any other vertices of R_n is at most n. There are $n-1$ vertices that do not pass by M_1. Let it be $M_1^{\sim}$. Placing $2^n - 1$ pebbles on v_2 and distributing one pebbles each on $M_1^{\sim}$, we can not shift a pebble to $r = v_n$. Thus, $\mu(A_n) \geq 2^n + n - 1$

Let C be the configuration of $2^n + n - 1$ pebbles on the vertices of R_n. Now we prove the sufficient condition.

Case 1: Let $r = u_k$ or v_k , $k \in \{1, 2, 3, \cdots, n\}$. Without loss of generality, consider $r = u_n$. Then we arrive at having the monophonic path of length at most n from u_2. Let the monophonic path be M_2. If $p(V(M_2)) \geq 2^n$, we are done. Let $p(V(M_2)) < 2^n$, $p^{\sim}(V(M_2)) \geq n$ and $N(r) = 0$ If there exist $E,\ 2 \leq E \leq 3$, pebbles each on the vertices of $M_2^{\sim}$ other than u_1, u_{n-1} and v_1 we can shift at most $n-4$ pebbles to $V(M_2)$. By using $(2^n+n-1)-3(n-4) = 2^n-2n-11$ pebbles on M_2 we can put a pebble on r. Hence, $\mu(R_n) = 2^n + n - 1$. □

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References

1. F.R.K. Chung, *Pebbling in hypercubes*, SIAMJ. Disc. Math. **2(4)** (1989), 467-472.
2. G. Hurlbert, *A survey of graph pebbling*, Congressus Numerantium **139** (1999), 41-64.
3. A. Lourdusamy, S. Saratha Nellainayaki, *Detour pebbling on Path related Graphs*, Advances in Mathematics: Scientific **10** (2021), 2017-2024.
4. A.P. Santhakumaran and P. Titus, *Monophonic distance in graphs*, Discrete Mathematics, Algorithms, and Applications **3** (2011), 159-169. https://doi.org/10.1142/S1793830911001176
5. A. Lourdusamy, I. Dhivviyanandam and S. Kither Iammal, *Monophonic pebbling number and t-pebbling number of some graphs*, AKCE International Journal of Graphs and Combinatorics, **1-4** (2022). https://doi.org/10.1080/09728600.2022.2072789

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