J. Appl. Math. & Informatics Vol. 42(2024), No. 1, pp. 49 - 64
https://doi.org/10.14317/jami.2024.049

WEAK SOLUTION OF AN ARCH EQUATION ON A MOVING BOUNDARY†

DAEWOOK KIM, SUDEOK SHON, JUNHONG HA*

ABSTRACT. When setting up a structure with an embedded shallow arch, there is a phenomenon where the end of the arch moves. To study the so-called moving domain problem, one try to transform a considered non-cylindrical domain into the cylindrical domain using the transform operator, as well as utilizing the method of penalty and other approaches.

However, challenges arise when calculating time derivatives of solutions in a domain depending on time, or when extending the initial conditions from the non-cylindrical domain to the cylindrical domain.

In this paper, we employ the transform operator to prove the existence and uniqueness of weak solutions of the shallow arch equation on the moving domain as clarifying the time derivatives of solutions in the moving domain.

AMS Mathematics Subject Classification : 37L05, 35B40, 35Q74.
Key words and phrases : Shallow arch equation, moving domain, existence and uniqueness.

1. Introduction

Let a shallow arch or an extensible beam be positioned over the interval $[0, L]$. Suppose that $w_0(\xi),\ \xi \in [0, L]$ is a shape when no load ($q = 0$) is applied to it. The deflection $w(\xi, t)$ of the shallow arch or extensible beam at $\xi \in (0, L)$ and $\tau > 0$ can expressed by

$$\rho A \frac{\partial^2 w}{\partial \tau^2} - \left(a + b \int_0^L \left| \frac{\partial w}{\partial \xi} \right|^2 d\xi \right) \frac{\partial^2 w}{\partial \xi^2} + EI \frac{\partial^4 w}{\partial \xi^4} + c_d \frac{\partial w}{\partial \tau} = g(\xi, \tau) \tag{1.1}$$

Received August 21, 2023. Revised October 10, 2023. Accepted October 12, 2023.
*Corresponding author.
†This paper was supported by the Sabbatical Year Research Program of KOREATECH in 2023.

with

$$a = -\frac{EA}{2L}\int_0^L \left|\frac{\partial w_0}{\partial \xi}\right|^2 d\xi,\ b = \frac{EA}{2L} \text{ and } g(\xi,t) = EI\frac{\partial^4 w_0}{\partial \xi^4} + q(\xi,\tau),$$

where ρ is the mass density, A is the cross-section area of the arch, E is the Young's modulus, I is the moment of inertial of the cross-section, and c_d is the air damping coefficient, see [11, 12].

This equation (1.1) has attracted a lot of interest in both engineering and mathematics, and many papers examining its properties have been published. Extensive research in the field of engineering has been dedicated to investigating diverse phenomena, including chaotic motion, global dynamic behavior, resonance, and buckling under various loads. Among the vast array of engineering articles, we refer to see the review work [18] and references there.

Mathematical study has been started in [1, 2], and abstract study on infinite dimensional spaces was summarized in [19], and the author has established the fundamental study like the existence, uniqueness, regularity and stability of solutions for equation (1.1).

When setting up a structure with an embedded shallow arch, there is a phenomenon where the end of the arch moves. Studying issues related to the installation and removal of this structure with the moving end of the arch is crucial for enhancing efficiency and ensuring safety. It is clear that the properties of such the problem differ significantly from the problem with a fixed boundary.

In the studies conducted by [5] and [7], the primary focus was on investigating the existence, uniqueness and stability of solutions for (1.1) on a moving domain. They studied the problems utilizing a methodology that converts the non-cylindrical domain into a cylindrical domain. However, a notable issue arises when calculating the time derivatives of solutions within the moving domain depending on time. But, their studies didn't address of this ambiguity.

To avoid this ambiguity, two alternative approaches were proposed for differential equations other than (1.1). One approach involved the utilization of the penalty method initiated by J. L. Lions. The method were discussed in [4] and [20]. The other approach involved employing the Lie derivative through push-forwards and pull-backs of functions, which was suggested in [17]. Another general approach for utilizing the sub-differential operator on the moving domain was explored in [3] and [14]. This method, known as the penalty method for a single value operator as discussed in [6], [8], and [13]. Since the differential highest operator in theirs studies is Δ, there was no difficulty in extending the initial conditions from the non-cylindrical domain to the cylindrical domain. However, since (1.1) is containing Δ^2, the extension of the initial conditions cannot be carried out in a manner similar to the process involving Δ.

In this paper, we prove the existence and uniqueness of weak solutions by using the transform operator as the first research step. To utilize this method, certain conditions must be imposed to facilitate the calculation of time derivatives within the moving domain depending on time. It is enough that the velocity of the moving boundary is bounded, and we want to see (2.21) and description below it.

This paper establishes the uniqueness and existence of weak solutions for the initial boundary problem described on the moving domain. In Section 2, we introduce a model equation for the shallow arch defined on the time-dependent moving domain and derive its dimensionless equation on the cylindrical problem by using the transform operator. In Section 3, we gather relevant materials and their associated proofs and prove the existence and uniqueness of weak solutions in proper Hilbert spaces. In Section 4, we establish the existence and uniqueness of weak solutions for the proposed equations on both the cylindrical and non-cylindrical domains. Furthermore, we provide a numerical example that demonstrates the validity of the model equation.

2. Dimensionless model

Let us derive a dimensionless model equation by carefully considering the physical parameters. Let $\tilde{\Omega}_\tau = (\tilde{\alpha}(\tau), \tilde{\beta}(\tau)) \subset \tilde{\Omega} = (0, L)$ be a domain dependent of the time variable $\tau \in (0, \tilde{T})$. Put $\tilde{\theta}(\tau) = \tilde{\beta}(\tau) - \tilde{\alpha}(\tau)$.

$$\tilde{Q} = \bigcup_{0<\tau<\tilde{T}} \tilde{\Omega}_\tau \times \{\tau\} \text{ and } Q = \Omega \times (0, \tilde{T}).$$

By considering (1.1) we introduce the system in the moving domain $\tilde{Q}$ governed by

$$\rho A \frac{\partial^2 \tilde{w}}{\partial \tau^2} - \left(\tilde{a}(\tau) + \tilde{b}(\tau) \int_{\tilde{\alpha}(\tau)}^{\tilde{\beta}(\tau)} \left| \frac{\partial \tilde{w}}{\partial \tilde{\xi}} \right|^2 d\tilde{\xi} \right) \frac{\partial^2 \tilde{w}}{\partial \tilde{\xi}^2} + EI \frac{\partial^4 \tilde{w}}{\partial \tilde{\xi}^4} + c_d \frac{\partial \tilde{w}}{\partial \tau} = \tilde{g}(\tilde{\xi}, \tau) \quad (2.1)$$

with

$$\tilde{a}(\tau) = -\frac{EA}{2\tilde{\theta}(\tau)} \int_{\tilde{\alpha}(\tau)}^{\tilde{\beta}(\tau)} \left| \frac{\partial \tilde{w}_0}{\partial \tilde{\xi}} \right|^2 d\tilde{\xi}, \ \tilde{b}(\tau) = \frac{EA}{2\tilde{\theta}(\tau)}$$

and

$$\tilde{g}(\tilde{\xi}, \tau) = EI \frac{\partial^4 \tilde{w}_0}{\partial \tilde{\xi}^4} + \tilde{q}(\tilde{\xi}, \tau)$$

with the initial conditions

$$\tilde{w}(\tilde{\xi}, 0) = \tilde{w}_0, \ \frac{\partial \tilde{w}}{\partial \tau}(\tilde{\xi}, 0) = \tilde{w}_1 \text{ in } (\tilde{\alpha}(0), \tilde{\beta}(0)), \quad (2.2)$$

with the hinged boundary condition

$$\tilde{w}(\tilde{\alpha}(\tau), \tau) = \tilde{w}(\tilde{\beta}(t), \tau) = \tilde{w}_\xi(\tilde{\alpha}(\tau), \tau) = \tilde{w}_\xi(\tilde{\beta}(\tau), \tau) = 0 \text{ for all } \tau \geq 0, \quad (2.3)$$

or with the clamped boundary condition

$$\tilde{w}(\tilde{\alpha}(\tau), \tau) = \tilde{w}(\tilde{\beta}(\tau), \tau) = \tilde{w}_{\tilde{\xi}\tilde{\xi}}(\tilde{\alpha}(\tau), \tau) = \tilde{w}_{\tilde{\xi}\tilde{\xi}}(\tilde{\beta}(\tau), \tau) = 0 \text{ for all } \tau \geq 0. \quad (2.4)$$

To transform the equations (2.1)-(2.4) in $\tilde{Q}$ into an equations in Q, let's define the transformation operator $\mathcal{T} : \tilde{Q} \to Q$ by

$$\mathcal{T}(\tilde{\xi}, \tau) = (\xi, \tau) = \left(\frac{\tilde{\xi} - \tilde{\alpha}(t)}{\tilde{\theta}(t)} L, \tau \right).$$

Assume that the functions $\tilde{\alpha}(\tau)$ and $\tilde{\beta}(\tau)$ satisfy

$$\tilde{\alpha}, \tilde{\beta} \in C^2[0, \tilde{T}]. \tag{2.5}$$

Put $w(\xi, \tau) = (\tilde{w} \circ \mathcal{T}^{-1})(\xi, \tau) = \tilde{w}(\tilde{\xi}, \tau)$. All derivatives of$\tilde{w}$ with respective to $\tilde{\xi}$ and τ are formally calculated by

$$\tilde{w}_\tau = \xi' w_\xi + w_\tau, \tag{2.6}$$

$$\tilde{w}_{\tau\tau} = w_{\xi\xi}\xi'^2 + w_\xi \xi'' + 2w_{\tau\xi}\xi' + w_{\tau\tau}, \tag{2.7}$$

$$\partial^k_{\tilde{\xi}} \tilde{w} = \frac{L^k}{\tilde{\theta}^k} \partial^k_\xi w \text{ for } \; k = 1, 2, \cdots, \tag{2.8}$$

where

$$\xi' = -\frac{1}{\tilde{\theta}}(\xi\tilde{\theta}' + \tilde{\alpha}' L), \; \xi'' = \frac{1}{\tilde{\theta}^2} \left\{ 2\tilde{\theta}'(\xi\tilde{\theta}' + \tilde{\alpha}' L) - \tilde{\theta}(\xi\tilde{\theta}'' + \tilde{\alpha}'' L) \right\}.$$

Putting all derivatives (2.6)-(2.8) into (2.1) we have the equation for $w(\xi, \tau)$ on Q satisfying

$$\begin{aligned} &\rho A(w_{\xi\xi}\xi'^2 + w_\xi\xi'' + 2w_{\tau\xi}\xi' + w_{\tau\tau}) \\ &+ \frac{L^2}{\tilde{\theta}(\tau)^2} \left(a(\tau) + \frac{Lb(\tau)}{\tilde{\theta}(\tau)} \int_0^L \left| \frac{\partial w}{\partial \xi} \right|^2 d\xi \right) \frac{\partial^2 w}{\partial \xi^2} \\ &+ \frac{EIL^4}{\tilde{\theta}(\tau)^4} \frac{\partial^4 w}{\partial \xi^4} + c_d(w_\xi \xi' + w_\tau) = g(x, \tau) \end{aligned} \tag{2.9}$$

with

$$a(\tau) = -\frac{EAL}{2\tilde{\theta}(\tau)^2} \int_0^L \left| \frac{\partial w_0}{\partial \xi} \right|^2 d\xi, \; b(\tau) = \frac{EA}{2\tilde{\theta}(\tau)}$$

and

$$g(\xi, \tau) = \frac{EIL^4}{\tilde{\theta}(\tau)^4} \frac{\partial^4 w_0}{\partial \xi^4} + q(\xi, \tau), \; q(\xi, \tau) = \tilde{q}(\tilde{\theta}\xi + \tilde{\alpha}, \tau).$$

Change of variables as follows:

$$x = \frac{\xi}{L}, \; y = \frac{w}{r}, \; p = \frac{q}{EIr}, \; \omega_0 = \sqrt{\frac{EI}{\rho A}}, r = \sqrt{\frac{I}{A}}, \; t = \omega_0 \tau, \; \gamma = \frac{c_d}{\rho A \omega_0}.$$

Define $\alpha(t), \beta(t)$ and $\theta(t)$ on $[0, T]$ by

$$\alpha(t) = \tilde{\alpha}(\tau) = \tilde{\alpha}\left(\frac{t}{\omega_0} \right), \beta(t) = \tilde{\beta}(\tau), \theta(t) = \tilde{\theta}(\tau). \tag{2.10}$$

Using these the variables and the boundary functions we have

$$w_\tau = r\omega_0 y_t, w_{\tau\tau} = r\omega_0^2 y_{tt}, w_{\tau\xi} = \frac{r\omega_0}{L} y_{tx}, \partial_\xi^k w = \frac{r}{L^k}\partial_x^k y, \tag{2.11}$$

$$\tilde{\alpha}'(\tau) = \omega_0\alpha'(t), \tilde{\beta}'(\tau) = \omega_0\beta'(t), \tilde{\theta}'(\tau) = \omega_0\theta'(t) \tag{2.12}$$

and

$$\xi' = -\frac{L\omega_0}{\theta}(x\theta' + \alpha'),\ \xi'' = \frac{L\omega_0^2}{\theta^2}\left\{2\theta'(x\theta'+\alpha') - \theta(x\theta''+\alpha'')\right\}. \tag{2.13}$$

From (2.10)-(2.13), the first term and last term in (2) are written by

$$\begin{aligned}&\rho A(w_{\xi\xi}\xi'^2 + w_\xi\xi'' + 2w_{\tau\xi}\xi' + w_{\tau\tau}) \qquad (2.14)\\ &= r\omega_0^2\rho A\left[\frac{1}{\theta^2}(x\theta'+\alpha')^2 y_{xx} + \frac{1}{\theta^2}\left\{2\theta'(x\theta'+\alpha') - \theta(x\theta''+\alpha'')\right\} y_x\right.\\ &\left.-\frac{2}{\theta}(x\theta'+\alpha')y_{tx} + y_{tt}\right]\end{aligned}$$

and

$$c_d(w_\xi\xi' + w_\tau) = r\omega_0 c_d\left[-\frac{1}{\theta}(x\theta'+\alpha')y_x + y_t\right]. \tag{2.15}$$

Finally, putting (2.10)-(2.15) into we get the dimensionless equation

$$\begin{aligned}&y_{tt} + a_1(t)y_{xxxx} - \left(a_2(t) + a_3(t)\int_\Omega |y_x|^2 dx\right) y_{xx} + a_4(x,t)y_x \qquad (2.16)\\ &+a_5(x,t)y_{xx} + a_6(x,t)y_{xt} + \gamma y_t = f(x,t),\end{aligned}$$

where

- $a_1(t) = \dfrac{1}{\theta^4},\ a_2(t) = -\dfrac{1}{2\theta^4}\displaystyle\int_0^1 \left|\frac{\partial y_0}{\partial x}\right|^2 dx,\ a_3(t) = \dfrac{1}{2\theta^4},$
- $a_4(x,t) = \dfrac{1}{\theta^2}\left\{2\theta'(x\theta'+\alpha') - \theta(x\theta''+\alpha'')\right\} - \dfrac{\gamma}{\theta}(x\theta'+\alpha')$
- $a_5(x,t) = \dfrac{(\theta' x+\alpha')^2}{\theta^2},\ a_6(x,t) = -\dfrac{2(x\theta'+\alpha')}{\theta},$
- $f(x,t) = \frac{1}{\theta(t)^4}\frac{\partial^4 y_0}{\partial x^4} + p(x,t),\ p(x,t) = \frac{1}{r\omega_0^2\rho A}q(Lx, \omega_0^{-1}t).$

From now on, let $\Omega_t = (\alpha(t), \beta(t)) \subset \Omega = (0,1)$ be a domain dependent of the time variable $t \in (0,T), T = \omega_0\tilde{T}$, and

$$\hat{Q} = \bigcup_{0<t<T} \Omega_t \times \{t\} \text{ and } Q = \Omega \times (0,T).$$

We consider the initial and boundary valued problem as follows:

$$\begin{aligned}&y_{tt} + a_1(t)y_{xxxx} - \left(a_2(t) + a_3(t)\int_\Omega |y_x|^2 dx\right) y_{xx} + a_4(x,t)y_x \qquad (2.17)\\ &+a_5(x,t)y_{xx} + a_6(x,t)y_{xt} + \gamma y_t = f(x,t) \text{ in } Q\end{aligned}$$

with the initial conditions

$$y(x,0) = y_0(x),\ y_t(x,0) = y_1(x) \text{ in } \Omega_0, \tag{2.18}$$

with the hinged boundary condition

$$y(0,t) = y(1,t) = y_x(0,t) = y_x(1,t) = 0 \text{ for all } t \geq 0, \tag{2.19}$$

or with the clamped boundary condition

$$y(0,t) = y(1,t) = y_{xx}(0,t) = y_{xx}(1,t) = 0 \text{ for all } t \geq 0, \tag{2.20}$$

In (2.18), $y_0(x) = w_0(\xi)$ and $y_1(x) = w_1(\xi)$ for $\xi \in \Omega_0 = (\alpha(0), \beta(0))$.

We need specific conditions necessary to ensure the validity of all derivatives (2.6)-(2.8). Condition $\tilde{\Omega}_\tau \subset \tilde{\Omega}_{\tau+h}, h > 0$ would be sufficient, but it may not always hold. We assume that for any $\tilde{\xi} \in \tilde{\Omega}_\tau$ there is $\hat{h} > 0$ such that

$$\tilde{\xi} \in \tilde{\Omega}_{\tau+h} \text{ for } 0 < h < \hat{h}. \tag{2.21}$$

Assumption (2.21) is obtainable because $|\tilde{\alpha}'(\tau)|$ and $|\tilde{\beta}'(\tau)|$ are bounded from (2.5). Let $\tilde{\xi} \in \tilde{\Omega}_\tau$ and consider $\tilde{\alpha}(\tau)$ only. Since $\tilde{\Omega}_\tau$ is open, $d = \tilde{\xi} - \tilde{\alpha}(\tau) > 0$. Integrating $\tilde{\alpha}'(t)$ over the interval $[\tau, \tau+h]$ yields

$$|\tilde{\alpha}(\tau+h) - \tilde{\alpha}(\tau)| \leq h \max_{\tau \in [0,\tilde{T}]} |\tilde{\alpha}'(\tau)|.$$

We can take $\hat{h}$ to give the inequality

$$|\tilde{\alpha}(\tau+h) - \tilde{\alpha}(\tau)| \leq \frac{d}{2} \text{ for } 0 < h < \hat{h}.$$

This inequality means $\tilde{\xi} \in \tilde{\Omega}_{\tau+h}$ for sufficiently small $h > 0$.

Assumption (2.5) implies that there is a constant c independent of x and t satisfying

$$|a_i(t)|, |a_i'(t)|, |a_i(x,t)|, |a_i'(x,t)| \leq c \text{ for all } (x,t) \in [0,1] \times [0,T]. \tag{2.22}$$

3. Problem setup

Let us introduce Hilbert spaces for solving (2.17)-(2.18) with (2.19) and (2.17)-(2.18) with (2.20). Let the Hilbert space $H = L^2(\Omega)$ have the norm $|u|$, and the inner product $(u,v) = \int_\Omega u(x)v(x)dx$. For the hinged boundary conditions we choose $V = H_0^1(\Omega) \cap H^2(\Omega)$, and for the clamped boundary conditions let $V = H_0^2(\Omega)$. In both cases V is a Hilbert space with the inner product $((u,v)) = (u_{xx}, v_{xx})$, and the norm $\|u\| = |u_{xx}|,\ u, v \in V$. This norm is equivalent to the standard norm in $H^2(\Omega)$, see [9]. It implies that $((u,v)) = 0$ deduces $(u_x v_x) = 0$.

Since $C_0^\infty(\Omega)$ is dense in H, it follows that V is densely embedded in H. In fact, the embedding is compact and continuous. Identifying H with its dual gives a Gelfand five fold $V \subset H \subset V'$, where the duality pairing $\langle \cdot, \cdot \rangle$ between V and its dual V' is consistent with the inner product in H.

First of all, we introduce the operator $\Delta_c^2 u = \Delta^2 u,\ u \in \mathcal{D}(\Delta_c) = H^4(\Omega) \cap H_0^2(\Omega)$ for the clamped boundary condition, and the operator $\Delta_h^2 u = \Delta^2 u,\ u \in \mathcal{D}(\Delta_h) = \{u \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta u = 0 \text{ on } \Gamma\}$ for the hinged boundary condition. Then Δ_c^2 and Δ_h^2 generate coercive bilinear forms on $V = H_0^2(\Omega)$ and $V = H_0^1(\Omega) \cap H^2(\Omega)$, respectively. Hence, $A = \Delta_c^2 = \Delta_h^2$ defined by

$$\langle Au, v\rangle = \int_\Omega \Delta u \; \Delta v \, dx, \quad u, v \in V \tag{3.1}$$

is a self-adjoint, strictly positive operator on $V = H_0^2(\Omega)$ or $V = H_0^1(\Omega) \cap H^2(\Omega)$. It is clear that A is a linear continuous operator from V to V', which denotes $A \in \mathcal{L}(V, V')$.

Since operator A is a self-adjoint, strictly positive operator in V and (unbounded) in H, see [19, Section 2.2.1]. Its inverse A^{-1} is also self-adjoint in H. Since the injection of V into H is compact, A^{-1} is a compact operator in H. Accordingly, there exists a complete orthonormal sequence of eigenfunctions $\{\varphi_k\}_{k=1}^\infty \subset \mathcal{D}(A)$. The corresponding eigenvalues $\mu_k,\ k \in \mathbb{N}$ satisfy $A\varphi_k = \mu_k \varphi_k,\ k \in \mathbb{N}$.

Using the interpolation theory ([19, section 2.2.1]), one can define various fractional powers of A. In particular, one can show that $A^{1/2} = -B \in \mathcal{L}(V, H)$. This positive operator has the same eigenfunctions φ_k as A, and its eigenvalues λ_k satisfy $\lambda_k^2 = \mu_k,\ k \in \mathbb{N}$.

To set up the weak formulation of the problem we introduce operators $A_i, i = 4, 5, 6$ and G, corresponding to the terms of equation (2.17).

Through this paper, constant c denotes various constants in all estimates, and it is independent of t.

Lemma 3.1 ([10])**.** *(i) Define operator $A_i(t), i = 4, 6$ by*

$$(A_i(t)u, v) \quad = \quad \int_\Omega a_i(x,t) u_x \; v \, dx, \quad u, v \in V.$$

Then $A_i(t) \in \mathcal{L}(V, H)$ and $A_6(t) \in \mathcal{L}(H, V')$.

(ii) Define operator $A_5(t)$ by

$$(A_5(t)u, v) = \int_\Omega a_5(x,t) u_{xx} \; v \; dx, \quad u, v \in V.$$

Then $A_5(t) \in \mathcal{L}(V, H)$.

(iii) Define operator G by

$$Gu = |\nabla u|^2 Bu, \quad u \in V.$$

Then G is a nonlinear continuous operator from V into H, which is Lipschitz continuous on bounded subsets of V.

$$|Gu - Gv| \le c\big(\|u\|^2 + \|v\|^2\big)\|u - v\|. \tag{3.2}$$

Considered as an operator from V into V', operator G is Lipschitz continuous on bounded subsets of V.

$$\|Gu - Gv\|_{V'} \le c\big(\|u\|^2 + \|v\|^2\big)|\nabla(u - v)|^2. \tag{3.3}$$

Furthermore, operator G map weakly convergent sequences in V into strongly convergent sequences in V'.

Proof. (i) and (ii) are clear. See [10] for (iii). □

When y is considered as a function with values in a Banach space X, let $\dot{y}$ denote its derivative with respect to t in an appropriate sense. Let

$$W[0,T] = \{y \ : \ y \in L^2(0,T;V), \quad \dot{y} \in L^2(0,T;H), \quad \ddot{y} \in L^2(0,T;V')\},$$

where the derivatives are understood in the sense of distributions with the values in V, H and V', see [16]. Space $W[0,T]$ becomes a Hilbert space, when its inner product is set to be the sum of the inner products in the constituent spaces.

Definition 3.2. *Let $y_0 \in V$, $y_1 \in H$, $T > 0$, and $f \in L^2(0,T;H)$. Function $y \in W[0,T]$ is called a* weak solution *of the problem* (2.17)-(2.18) *with* (2.19) *or* (2.20)*, if $y \in L^\infty(0,T;V)$, $\dot{y} \in L^\infty(0,T;H)$, equation*

$$\ddot{y} + a_1(t)Ay + a_2(t)By + a_3(t)Gy + A_4(t)y + A_5(t)y + A_6(t)\dot{y} + \gamma\dot{y} = f \quad (3.4)$$

is satisfied in V' a.e. on $[0,T]$, and the initial conditions

$$y(0) = y_0, \quad \dot{y}(0) = y_1 \quad (3.5)$$

are satisfied in V and H correspondingly.

To shorten (3.4), we write (3.4) as follows.

$$\ddot{y} + a_1(t)Ay + a_3(t)Gy + \mathcal{A}_1(t)y + \mathcal{A}_2(t)\dot{y} = f.$$

Then $\mathcal{A}_1(t) = a_2(t)B + A_4(t) + A_5(t) \in \mathcal{L}(V,V')$ and $\mathcal{A}_2(t) = A_6(t) + \gamma \in \mathcal{L}(H,V')$. Since $y \in W[0,T]$, $\mathcal{A}_1(t)y \in L^2(0,T;V')$ and $\mathcal{A}_2(t)\dot{y} \in L^2(0,T;V')$. Also, G is Lipschitz continuous on bounded subsets of V, $a_3(t)Gy \in L^2(0,T;V')$. So equation (3.4) makes sense. Also, by Lemma 3.4, functions y and $\dot{y}$ are weakly continuous in V and H correspondingly. Therefore conditions (3.5) make sense as well.

Definition 3.3. *Let X be a Banach space. Function $y : [0,T] \to X$ is called weakly continuous with values in X, if scalar functions $t \to \langle y(t), w\rangle$ are continuous for any $w \in X'$.*

Lemma 3.4 ([10])**.** *Suppose that $y \in L^\infty(0,T;V)$, and $\dot{y} \in L^\infty(0,T;H)$. Then, after a modification on a set of measure zero in $[0,T]$,*

(i) Function $y \in C([0,T];H)$. It is weakly continuous with values in V.
(ii) Function $\dot{y} \in C([0,T];V')$. It is weakly continuous with values in H.
(iii) Function $t \to By(t)$ is weakly continuous with values in H.
(iv) Function $t \to |\nabla y(t)|^2$ is absolutely continuous on $[0,T]$, and

$$\frac{1}{2}\frac{d}{dt}|\nabla y(t)|^2 = (By, \dot{y}), \quad \text{a.e. on } [0,T]. \quad (3.6)$$

(v) Function $t \to |\nabla y(t)|^2$ is absolutely continuous on $[0,T]$, and

$$\frac{1}{4}\frac{d}{dt}|\nabla y(t)|^4 = |\nabla y(t)|^2(By, \dot{y}), \quad \text{a.e. on } [0,T]. \quad (3.7)$$

Since $\dot{y} \in L^2(0,T;H)$, $\quad \ddot{y} \in L^2(0,T;V')$ the duality pairing $\langle \ddot{y}, \dot{y} \rangle$ loses its meaning. Lemma 3.5 makes up for deficiency of regularity of solutions.

Lemma 3.5. *Let $A : V \to V'$ be defined by (3.1). Assume $a_1(t)$ is positive and continuous on $[0,T]$ and assume that $y \in L^2(0,T;V)$, $\dot{y} \in L^2(0,T;H)$, and $\ddot{y} + a_1(t)Ay \in L^2(0,T;H)$. Then, after a modification on a set of measure zero, $y \in C([0,T];V)$, $\dot{y} \in C([0,T];H)$ and, in the sense of distributions on $(0,T)$ one has*

$$(\ddot{y} + a_1(t)Ay, \dot{y}) = \frac{1}{2}\frac{d}{dt}|\dot{y}|^2 + \frac{1}{2}a_1(t)\frac{d}{dt}\|y\|^2. \tag{3.8}$$

Proof. Equality (3.8) can be proved similar to Lemma 2.4.1 in [19] using the truncation functions and the mollifiers.

Assumptions imply $y \in L^2(0,T;V)$, $\dot{y} \in L^2(0,T;H)$, $\ddot{y} \in L^2(0,T;V')$. In Lemma 2.3.1 in [19], y is weakly continuous from $[0,T]$ into V and $\dot{y}$ is weakly continuous from $[0,T]$ into H.

Equality (3.8) is modified as

$$2(\ddot{y} + a_1(t)Ay, \dot{y}) + 2\dot{a}_1(t)\|y\|^2 = \frac{d}{dt}\left[|\dot{y}|^2 + a_1(t)\|y\|^2\right]. \tag{3.9}$$

By assumptions, $2(\ddot{y} + a_1(t)Ay, \dot{y}) + 2\dot{a}_1(t)\|y\|^2 \in L^1(0,T)$, function $\phi(t) = |\dot{y}(t)|^2 + a_1(t)\|y(t)\|^2$ is continuous on $[0,T]$, and

$$\begin{aligned}
&\phi(t) - \phi(s) = |\dot{y}(t) - \dot{y}(s)|^2 + 2(\dot{y}(s), \dot{y}(t) - \dot{y}(s)) \\
&+a_1(t)\|y(t) - y(s)\|^2 + 2a_1(t)\langle Ay(s), y(t) - y(s)\rangle \\
&+[a_1(t) - a_1(s)]\|y(s)\|^2.
\end{aligned}$$

This equality with weak convergence and continuity of $\phi(t)$ and $a_1(t) > 0$ implies $y \in C([0,T];V)$, $\dot{y} \in C([0,T];H)$. □

Our method requires the following analysis of weakly convergent sequences.

Lemma 3.6 ([10])**.** *Let X, Y be Banach spaces, and $L : X \to Y$ be a continuous linear operator. Suppose that $x_n \rightharpoonup x$ weakly in X, and the image of this sequence $\{Lx_n\}_{n=1}^{\infty}$ is precompact in Y. Then $Lx_n \to Lx$ strongly in Y, as $n \to \infty$.*

Lemma 3.7 on the embedding of $L^p(0,T;V)$ spaces can be deduced from general results established in [16, 10].

Lemma 3.7 ([16, 10])**.** *Let $h \in L^1(0,T)$. Suppose that $y_k \in L^\infty(0,T;V), \dot{y}_k \in L^\infty(0,T;H)$, with $\|y_k(t)\| \le 1$, and $|\dot{y}_k(t)| \le h(t)$ a.e. on $[0,T]$, $k \in \mathbb{N}$. Suppose that $y_k \rightharpoonup y$, weakly in $L^2(0,T;V)$, as $k \to \infty$. Then, after a modification on a set of measure zero in $[0,T]$,*

(i) $y_k \to y$ in $C([0,T];H)$, as $k \to \infty$,
(ii) $|\nabla(y_k - y)| \to 0$ as $k \to \infty$.

4. Existence and Uniqueness of solutions

We prove the existence and uniqueness of weak solutions of (3.4)-(3.5) using Lion's method of Galerkin approximations in [15]. First, we prove the energy estimate, and establish the uniqueness of solutions. Then approximate solutions are constructed, and their weak limit is shown to be the solution of the problem.

Lemma 4.1. *Let $\gamma \in \mathbb{R}$, $y_0 \in V$, $y_1 \in H$, $T > 0$, and $f \in L^2(0,T;H)$. Assume that y is a weak solution of the problem* (3.4)-(3.5). *Then $y \in W[0,T] \cap C([0,T];V)$, $\dot{y} \in C([0,T];H)$, and*

$$|\dot{y}(t)|^2 + \|y(t)\|^2 \le c\left(|y_1|^2 + \|y_0\|^2 + \|y_0\|^4 + \int_0^t |f(s)|^2\,ds\right) \tag{4.1}$$

for any $t \in [0,T]$. The constant c is dependent only on t.

Proof. Rewrite equation (3.4) as

$$\ddot{y} + a_1(t)Ay = f - a_3(t)Gy - \mathcal{A}_1(t)y - \mathcal{A}_2(t)\dot{y}. \tag{4.2}$$

Lemma 3.1 and $y \in L^\infty(0,T;V)$ imply that

$$a_3(t)Gy, \mathcal{A}_1(t)y \in L^2(0,T;H),$$

and $\dot{y} \in L^2(0,T;V)$ imply that

$$\mathcal{A}_2(t)\dot{y} \in L^2(0,T;H).$$

In addition, $f \in L^2(0,T;H)$. Therefore $\ddot{y} + a_1(t)Ay \in L^2(0,T;H)$, and Lemma 3.5 is applicable. Thus $y \in W[0,T] \cap C([0,T];V)$, $\dot{y} \in C([0,T];H)$.

Take the inner product of (4.2) with $\dot{y}$, and use Lemma 3.5 to get

$$\frac{1}{2}\frac{d}{dt}|\dot{y}|^2 + \frac{1}{2}a_1(t)\frac{d}{dt}\|y\|^2 = (f - a_3(t)Gy - \mathcal{A}_1(t)y - \mathcal{A}_2(t)\dot{y}, \dot{y}).$$

By (3.7) we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{|\dot{y}|^2 + a_1(t)\|y\|^2 + \frac{1}{2}a_3(t)|\nabla y|^4\right\} + \gamma|\dot{y}|^2 \\ &= \frac{1}{2}\dot{a}_1(t)\|y\|^2 + \frac{1}{4}\dot{a}_3(t)|\nabla y|^4 - (\mathcal{A}_1(t)y, \dot{y}) - (A_6(t)\dot{y}, \dot{y}) + (f, \dot{y}). \end{aligned} \tag{4.3}$$

Integrating (4.3) over $[0,t]$ we get

$$\begin{aligned} &|\dot{y}|^2 + \|y\|^2 + |\nabla y|^4 \\ &= \frac{c_2}{c_1}\left(\|y_0\|^2 + |y_1|^2 + |\nabla y_0|^4 + \int_0^t \|y\|^2 + |\nabla y|^4 + |\dot{y}|^2 + |f|^2\right) ds, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} c_1 &= \frac{1}{4}\min\{a_1(t), a_3(t), 1\}, \\ c_2 &= \frac{1}{2}\max\{|a_1(t)|, |\dot{a}_1(t)|, |a_2(t)|, |a_3(t)|, |\dot{a}_3(t)|, |a_4(x,t)|, |a_5(x,t)|, |\gamma|, 1\}. \end{aligned}$$

Here $(A_6(t)\dot{y},\dot{y})=0$ is used. Now $|\nabla y_0|^2 \le c_3\|y_0\|^2$, and applying Gronwall's inequality to (4.4) gives (4.1). $\square$

Lemma 4.2. *Let $\gamma\in\mathbb{R}$, $y_{0,i}\in V$, $y_{1,i}\in H$, $T>0$, and $f_i\in L^2(0,T;H)$.*

(i) Let y_i, $i=1,2$ be two solutions of the abstract problem

$$\ddot{y}_i + a_1(t)Ay + a_3(t)Gy_i + \mathcal{A}_1(t)y_i + \mathcal{A}_2(t)\dot{y}_i = f_i,$$
$$y_i(0)=y_{0,i}\in V, \quad \dot{y}_i(0)=y_{1,i}\in H.$$

Then $z=y_2-y_1$ satisfies

$$|\dot{z}(t)|^2 + \|z(t)\|^2 \le C\left(|z_1|^2+\|z_0\|^2+\|z_0\|^4+\|f_2-f_1\|^2_{L^2(0,T;H)}\right). \tag{4.5}$$

(ii) The solution of the problem (3.4)-(3.5) is unique.

Proof. The difference $z=y_2-y_1$ satisfies

$$\begin{aligned}&\ddot{z}+a_1(t)Az \\ &= -a_2(t)z - a_3(t)(Gy_2-Gy_1) - \mathcal{A}_1(t)z - \mathcal{A}_2(t)\dot{z} + f_2 - f_1.\end{aligned} \tag{4.6}$$

Since $y_i\in L^\infty(0,T;V), f_i\in L^2(0,T;H)$, $\dot{y}_i\in L^2(0,T;V_1)$ and

$$|Gy_2(t)-Gy_1(t)| \le C\|y_2(t)-y_1(t)\|, \quad t\in[0,T],$$

the right term of (4.6) belongs to $L^2(0,T;H)$.

According to the same argument as the proof of Lemma 4.1 it is easily to obtain inequality (4.5). The uniqueness follows from (4.5). $\square$

Lemma 4.3 ([10])**.** *System $\{\varphi_k\}_{k=1}^\infty$ is an orthonormal basis in H. System $\{\frac{1}{\lambda_k}\varphi_k\}_{k=1}^\infty$ is an orthonormal basis in V.*

Lemma 4.4 ([10])**.** *Let $m\in\mathbb{N}$, and the operator $P_m : H\to H$ be defined by*

$$P_m h = \sum_{k=1}^m (h,\varphi_k)\varphi_k, \quad h\in H. \tag{4.7}$$

(i) Operator P_m is an orthogonal projection in H, with $|P_m h|\le |h|$ for any $h\in H$. Also, $|P_m h - h|\to 0$, as $m\to\infty$.

(ii) Operator P_m is an orthogonal projection in V, with $\|P_m v\|\le\|v\|$ for any $v\in V$. Also, $\|P_m v - v\|\to 0$, as $m\to\infty$.

Definition 4.5. *Let $m\in\mathbb{N}$. Function y_m is called an approximate solution of the problem (3.4)-(3.5), if $y_m\in W[0,T]\cap L^\infty(0,T;V)$, $\dot{y}_m\in L^\infty(0,T;H)$, equation*

$$\ddot{y}_m + a_1(t)Ay_m + a_3(t)Gy_m + \mathcal{A}_1(t)y_m + \mathcal{A}_2(t)\dot{y}_m = P_m f \tag{4.8}$$

is satisfied in V' a.e. on $[0,T]$, and the initial conditions

$$y_m(0) = P_m y_0, \quad \dot{y}_m(0) = P_m y_1 \tag{4.9}$$

are satisfied in V and H correspondingly.

Lemma 4.6. *Let $\gamma \in \mathbb{R}$, $y_0 \in V$, $y_1 \in H$, $T > 0$, and $f \in L^2(0,T;H)$. Then there exists a unique solution y_m of the approximate problem (4.8)-(4.9). This solution satisfies $y_m, \dot{y}_m \in C([0,T];V_m)$, $\ddot{y}_m \in L^2(0,T;V_m)$, where $V_m = span\{\varphi_k,\ k = 1,2,\dots m\}$. Furthermore, for any $t \in [0,T]$*

$$|\dot{y}_m(t)|^2 + \|y_m(t)\|^2 \le c\left(|y_1|^2 + \|y_0\|^2 + \|y_0\|^4 + \int_0^t |f(s)|^2\,ds\right), \qquad (4.10)$$

where the constant c is independent of m. Also, there exist $C = C(u_0, v_0, f)$ independent of m, such that

$$\|\ddot{y}_m\|_{L^2(0,T;V')} \le C \qquad (4.11)$$

for any $m \in \mathbb{N}$.

Proof. Let $m \in \mathbb{N}$. Arguing as in Lemma 4.2, we conclude that the solution of the problem (4.8)-(4.9) is unique. Let

$$y_m(t) = \sum_{j=1}^{m} g_{jm}(t)\varphi_j,$$

where functions $g_{j,m}(t)$, $j = 1,2,\dots,m$ are the solutions of the following system of m equations

$$\begin{aligned}&(\ddot{y}_m + a_1(t)Ay_m + a_3(t)Gy_m + \mathcal{A}_1(t)y_m + \mathcal{A}_2(t)\dot{y}_m, \varphi_k) = (P_m f, \varphi_k), \qquad (4.12)\\ &(y_m(0), \varphi_k) = (P_m y_0, \varphi_k), \quad (\dot{y}_m(0), \varphi_k) = (P_m y_1, \varphi_k),\end{aligned}$$

where $k = 1,2,\dots,m$. Since $(\nabla\varphi_k, \nabla\varphi_j) = 0$ for $k \ne j$, and $|\nabla\varphi_k|^2 = \lambda_k$, we get an explicit expression for (4.12)

$$\begin{aligned}&\ddot{g}_{km} + \sum_{j=1}^{m}(a_6(x,t)\varphi_j, \varphi_k)\dot{g}_{jm}(t) + \gamma\dot{g}_{km} + a_1(t)\lambda_k^2 g_{km} + a_2(t)\lambda_k g_{km}\\ &\quad + a_3(t)\lambda_k^2\left(\sum_{j=1}^{m}|g_{jm}|^2\right)g_{km} + \sum_{j=1}^{m}([a_4(x,t) + a_5(x,t)]\varphi_j, \varphi_k)g_{jm} = (f, \varphi_k)\\ &g_{km}(0) = (y_0, \varphi_k), \quad \dot{g}_{km}(0) = (y_1, \varphi_k),\end{aligned}$$

where $k = 1,2,\dots,m$. This initial value problem for the system of m ODE has unique solutions satisfying $g_{km}, \dot{g}_{km} \in C[0,T]$, $\ddot{g}_{k,m} \in L^2[0,T]$. Thus $y_m, \dot{y}_m \in C([0,T];V_m)$, $\ddot{y}_m \in L^2(0,T;V_m)$.

Now we show that function y_m is a solution of equation (4.8). This equation has to be satisfied in V'. Since $span\{\varphi_k,\ k \in \mathbb{N}\}$ is dense in V', it is enough to check that equations (4.8) are satisfied for any φ_k, $k \in \mathbb{N}$. However, for $1 \le k \le m$ equations (4.8) are satisfied by the construction of y_m, and for $k > m$ equations (4.8) become $0 = 0$.

Inequality (4.10) is derived as in Lemma 4.1, using $\|P_m y_0\| \le \|y_0\|$, $|P_m y_1| \le |y_1|$, and $|P_m f| \le |f|$. Since $\dot{y}_m \in C([0,T];V_m)$, $a_3(t)Gy_m + \mathcal{A}_1(t)y_m + \mathcal{A}_2(t)\dot{y}_m \in L^2(0,T;H)$. So, we can use Lemma 3.5, and inequality (4.10) is derived as in Lemma 4.1.

It remains to derive inequality (4.11). Let $\varphi \in V$. Lemma 3.1 and (2.22) imply

$$|\langle a_1(t)Ay_m, \varphi\rangle| \le c\|y_m\|\ \|\varphi\|,\ |(\mathcal{A}_1(t)y_m, \varphi)| \le c\|y_m\|\ \|\varphi\|,$$
$$|(a_3(t)Gy_m, \varphi)| \le c\|y_m\|^3\ \|\varphi\|,\ |(A_6(t)\dot{y}_m, \varphi)| \le c|\dot{y}_m|\ \|\varphi\|,$$
$$|(\gamma\dot{y}_m, \varphi)| \le c|\dot{y}_m|\ \|\varphi\|.$$

By (2.22) we have

$$|\langle \ddot{y}_m, \varphi\rangle| \le C(|\dot{y}_m| + \|y_m\| + \|y_m\|^3 + |f|)\|\varphi\|,$$

that is,

$$\|\ddot{y}_m\|_{V'} \le C(|\dot{y}_m| + \|y_m\| + \|y_m\|^3 + |f|).$$

Square both sides of this inequality, integrate it from 0 to T, and use estimate (4.10) to get (4.11). □

Theorem 4.7. *Let $\gamma \in \mathbb{R},\ y_0 \in V,\ y_1 \in H,\ T > 0,$ and $f \in L^2(0,T;H)$.*

(i) There exists a unique weak solution y of the problem (3.4)-(3.5). *Solution y satisfies $y \in W[0,T] \cap C([0,T];V),\ \dot{y} \in C([0,T];H)$, and*

$$|\dot{y}(t)|^2 + \|y(t)\|^2 \le c\left(|y_1|^2 + \|y_0\|^2 + \|y_0\|^4 + \int_0^t |f(s)|^2\, ds\right) \tag{4.13}$$

for any $t \in [0,T]$.

(ii) Solution y and its approximation y_m satisfy

$$|\dot{y}(t) - \dot{y}_m(t)|^2 + \|y(t) - y_m(t)\|^2$$
$$\le c\left(|y_1 - P_m y_1|^2 + \|y_0 - P_m y_0\|^2 + \|y_0 - P_m y_0\|^4 + \int_0^t |f(s) - P_m f(s)|^2\, ds\right) \tag{4.14}$$

for any $t \in [0,T]$.

Proof. By Lemma 4.6, the sequence of the approximate solutions $y_m,\ m \in \mathbb{N}$ is bounded in $W[0,T]$. Since $W[0,T]$ is a reflexive space, we can find a subsequence of y_m (still denoted by y_m) such that it and the derivatives $\dot{y}_m, \ddot{y}_m$ are weakly convergent in the spaces $L^2(0,T;V),\ L^2(0,T;V_1)$, and $L^2(0,T;V')$ correspondingly. Since the derivatives are taken in the distributional sense, it follows that there exists $y \in W[0,T]$ such that

$$y_m \rightharpoonup y, \quad \dot{y}_m \rightharpoonup \dot{y}, \quad \ddot{y}_m \rightharpoonup \ddot{y}$$

weakly in the corresponding spaces. Estimate (4.11) also shows that the sequence y_m is bounded in $L^\infty(0,T;V)$, and the sequence $\dot{y}_m$ is bounded in $L^\infty(0,T;H)$. Therefore y satisfies estimate (4.13).

We are going to show that y satisfies the problem (3.4)-(3.5).

By Lemma 4.6 we have

$$\ddot{y}_m + a_1(t)y_m + a_3(t)Gy_m + \mathcal{A}_1(t)y_m + \mathcal{A}_2(t)\dot{y}_m = P_m f \tag{4.15}$$

in V', a.e. on $[0,T]$, and

$$y_m(0) = P_m y_0, \quad \dot{y}(0) = P_m y_1. \tag{4.16}$$

Clearly, we can pass to the limit in V' for $\ddot{y}_m, \mathcal{A}_1(t)y_m, Gy_m, \mathcal{A}_2(t)\dot{y}_m$, and $P_m f$, as $m \to \infty$. According to estimate (3.3) we have

$$\|Gy_m - Gy\|_{V'} \leq c(\|y_m\|^2 + \|y\|^2)|\nabla(y_m - y)|^2.$$

The norms $\|y_m\|$ and $\|y\|$ are bounded by estimates (4.10) and (4.13). By Lemma 3.7, the weak convergence of y_m to y in $L^2(0,T;V)$ implies that $y_m \to y$ in $C([0,T];H)$, and $|\nabla(y_m - y)|^2 \to 0$ as $m \to \infty$. Thus $Gy_m \to Gy$ in $L^2(0,T;V')$, and the passage to the limit as $m \to \infty$ in (4.15) is justified.

Concerning the initial conditions (4.16), it was also argued in Lemma 3.7 that the weak convergence of y_m to y in $L^2(0,T;V)$ implies that $y_m(t) \rightharpoonup y(t)$ weakly in V for any $t \in [0,T]$. Since $y_m(0) = P_m y_0 \to y_0$ in V, we conclude that $y(0) = y_0$. A straightforward modification of Lemma 3.7 shows that $\dot{y}_m \rightharpoonup \dot{y}$ weakly in H for any $t \in [0,T]$. Therefore $\dot{y}(0) = y_1$.

The uniqueness of weak solutions is easily obtained from (4.5).

Part (ii) is obtained by using Lemma 4.2 with $f_1 = f$, $f_2 = P_m f$, and the corresponding initial conditions. □

For $\tilde{\Omega}_\tau$ we introduce notations and inner products and norms similar to V, H and V', this is, $H_\tau = L^2(\tilde{\Omega}_\tau)$, $V_\tau = H_0^1(\tilde{\Omega}_\tau) \cap H^2(\tilde{\Omega}_\tau)$ for the hinged boundary condition and $V_\tau = H_0^2(\tilde{\Omega}_\tau)$ for the clamped boundary condition.

Theorem 4.8. *Let $c_d \in \mathbb{R}$, $\tilde{w}_0 \in V_0$, $\tilde{w}_1 \in H_0$, $\tilde{T} > 0$, and $\tilde{g} \in L^2(0,\tilde{T};H_\tau)$. Then function $\tilde{w}$ satisfies* (2.1)-(2.4)

$$\tilde{w} \in L^2(0,\tilde{T};V_\tau), \quad \dot{\tilde{w}} \in L^2(0,\tilde{T};H_\tau), \quad \ddot{\tilde{w}} \in L^2(0,\tilde{T};V_\tau'). \tag{4.17}$$

Proof. Equations (2.1)-(2.4) can be obtained by (2.6)-(2.8). Also, it is clear that $\tilde{w} \in V_\tau$ for τ. Now, $\tilde{w} \in L^2(0,\tilde{T};V_\tau)$ is from

$$\begin{aligned}\|\tilde{w}\|^2_{L^2(0,\tilde{T};V_\tau)} &= \int_0^{\tilde{T}} \int_{\tilde{\alpha}(\tau)}^{\tilde{\beta}(\tau)} \tilde{w}^2_{\tilde{\xi}\tilde{\xi}} d\tilde{\xi} d\tau \\ &= \frac{r^2}{\omega_0} \int_0^T \frac{1}{\theta(t)^3} \int_0^1 \{y_{xx}(x,t)\}^2 dx dt \leq c\|y\|^2_{L^2(0,T;V)} < \infty.\end{aligned}$$

The remaining ones can be derived from (2.6)-(2.7) and (2.11) using the similar way above. □

For the operator A with the clamped boundary condition, $V_m = span\{\varphi_k = \sqrt{2}\sin(\pi x),\ k = 1, 2, \ldots m\}$.

Example 4.9. *All data are given by*

$$y_0 = h\varphi_1, y_1 = 0, \alpha(t) = 0, \beta(t) = 0.9 + 0.1\sin(t), p = h\varphi_1, h = 2, \gamma = 0.$$

Then since $y_0, y_0, p \in V_1$, we have $y = Y_1(t)\varphi_1(x)$ by Theorem 4.7. Figure 1 illustrates the graph showing the maximum value of $w(\xi,\tau)$ for each τ along *with the graph of* $\beta(\tau)$. *As the domain diminishes due to the movement of the boundary, we can observe that at each time* τ*, the maximum values within region* $(0, \beta(\tau))$ *are moving in the opposite direction. This aligns precisely with our intended outcome.*

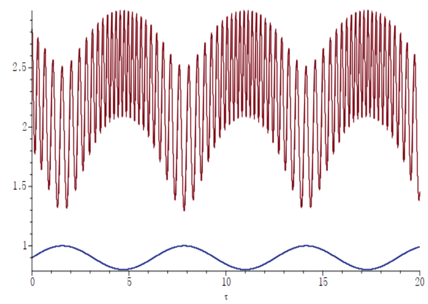

FIGURE 1. The graphs of $\max_\xi w(\xi, \tau)$ for each τ and $\beta(\tau)$

Conflicts of interest : The authors declare that there is no conflict of interests.

Data availability : Not applicable

Acknowledgments : D. Kim was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2021R1F1A1047079).

REFERENCES

1. J.M. Ball, *Initial-boundary value problems for an extensible beam*, J. Math. Anal. and Applications **42** (1973), 61-90.
2. J.M. Ball, *Stability theory for an extensible beam*, J. Diff. Equations **14** (1973), 399-418.
3. V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer-Verlag, New York, 2010.
4. C. Gonzales, J. Limaco and R. Barreto, *The penalty method and beam evolution equations*, Pro Mathematica **20** (2006), 39-40.
5. H.R. Clark and M.A. Rincon and R.D. Rodrigues, *Beam equation with weak-internal damping in domain with moving boundary*, Appl. Num. Mathematics **47** (2003), 139-157.
6. C.S.Q. Caldas, J. Limaco, R.K. Barreto, P. Gamboa, *About the Benjamin-Bona-Mahony Equation in Domains with Moving Boundary*, Trends in Com. and Appl. Mathematics **8** (2007), 1329-339.
7. J.B.L. Ferrel and L.A. Medeiros, *Vibrations of elastic membranes with moving boundaries*, Nonlinear Analysis **45** (2001), 363-382.
8. H. Fujota and N. Sauer, *On existence of weak solution of the Navier-Stokes equation in regions with moving boundaries*, J. the faculty of science, Univ. of Tokyo, XVII, Part 1 and 2 (1970), 403-420.
9. P. Grisvard, *Singularities in Boundary Value Problems*, Springer, New York, 1992.

10. S. Gutman and J. Ha and S. Lee, *Parameter identification for weakly damped shallow arches*, J. Math. Anal. and Applications **403** (2013), 297-313.
11. C.S. Hsu, *The effects of various parameters on the dynamic stability of a shallow arch*, ASME. J. Appl. Mechanics **34** (1967), 340-358.
12. J.S. Humphreys, *Dynamic snap buckling of shallow arches*, Amer. Inst. of Aero. and Astr. Journal **4** (1966), 878-886.
13. I.F. Lopez, G.O. Antunes, M.D.G. Da Silva and L.A.J. Medeiros, *Remarks on a nonlinear wave equation in a noncylindrical domain*, TEMA **16** (2015), 195-207.
14. M.H. Kubo, *Second order evolution equations with time-dependent subdifferentials*, J. Evol. Equations **7** (2007), 701-717.
15. J.L. Lions, *Optimal control of systems governed by partial differential equations*, Springer-Verlag, New York, 1971.
16. J.L. Lions and E. Magenes, *Nonhomogeneous boundary value problems and applications*, Springer-Verlag, New York, 1972.
17. P. Nägele and M. Růžička and D. Lengeler, *Functional setting for unsteady problems in moving domains and applications*, Comp. Vari. and Elli. Equations **62** (2017), 66-97.
18. M. Spagnuolo and U. Andreaus, *A targeted review on large deformations of planar elastic beams: extensibility, distributed loads, buckling and post-buckling*, Math. and Mech. of Solids **24** (2019), 258-280.
19. R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1997.
20. G. Zhou, *The fictitious domain method with penalty for the parabolic problem in moving-boundary domain: The error estimate of penalty and the finite element approximation*, Appl. Nume. Mathematics **115** (2017), 42-67.

Deawook Kim received M.Sc. from Pusan National University and Ph.D. at Pusan National University. He is currently a professor at Seowon University since 2015. His research interests include mathematical modelling, optimization of partial differential equations.

Department of Mathematics Education, Seowon University, Cheongju 28674, South Korea.
e-mail: `kdw@seowon.ac.kr`

Sudeok Shon received M.Sc. from Kyungpook National University, and Ph.D. from Kyungpook National University. He is currently a research professor at KOREATECH since 2011. His research interests include nonlinear analysis, the design and dynamic instability of long-span structures, shape control in tension structures, and optimization algorithms.

Department of Architectural Engineering, KOREATECH, Cheonan 31253, South Korea.
e-mail: `sdshon@koreatech.ac.kr`

Junhong Ha received M.Sc. from Pusan National University, and Ph.D. from Kobe University. He is currently a professor at KOREATECH since 1999. His research interests are computational mathematics and optimization of partial differential equations.

School of Liberal Arts, KOREATECH, Cheonan 31253, South Korea.
e-mail: `hjh@koreatech.ac.kr`