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THE COINCIDENCE OF HYBRID HYPERIDEALS AND HYBRID INTERIOR HYPERIDEALS IN ORDERED HYPERSEMIGROUPS†

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Abstract. The concept of hybrid structures integrates two powerful mathematical tools: soft sets and fuzzy sets. This paper extends the application of hybrid structures to ordered hypersemigroups. We introduce the notions of hybrid interior hyperideals in ordered hypersemigroups and demonstrate their equivalence with hybrid hyperideals in certain classes, including regular, intra-regular, and semisimple ordered hypersemigroups. Furthermore, we provide a characterization of semisimple ordered hypersemigroups in terms of hybrid interior hyperideals.

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1. Introduction

One currently most explored and active hyperalgebraic structure is the concept of ordered hypersemigroups. These structures consist of a nonempty set with an associative binary hyperoperation and a partial order defined on the underlying set with certain properties. Introduced by Heidari and Davvaz [9] in 2011, ordered hypersemigroups are the subject of extensive investigation. The concept of hyperideals is essential to the investigation of ordered hypersemigroups (see [2, 5, 7]). Depending on the mathematical tools applied, various

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types of hyperideals can be extended in diverse ways. There are several mathematical structures that are widely studied nowadays besides ordered hypersemigroups, for example, LA-semihypergroups, ordered LA-semihypergroups, and ordered Γ-semihypergroups (see [12, 28, 32, 33]).

The pioneering concept capable of addressing uncertainties is fuzzy sets, first introduced by Zadeh [34] in 1965. Since its beginning, researchers across diverse scientific fields have conducted numerous studies utilizing fuzzy sets (see [6, 19, 20]). Another powerful tool for handling uncertainties, free from complications, is the concept of soft sets, introduced by Molodtsov [23] in 1999. Similar to fuzzy sets, soft sets find applications across various scientific disciplines. Consequently, researchers have been applying soft sets extensively across different scientific fields (see [17, 31, 35]). In numerous scenarios, relying only on either fuzzy sets or soft sets may prove inadequate due to the inherent limitations of each concept. In 2001, Maji et al. [21] introduced the idea of fuzzy soft sets, integrating the strengths of both approaches. Since then, researchers have extensively investigated this combined concept (see [3, 10, 24]). The combination of fuzzy sets and soft sets takes another form in hybrid structures, pioneered by Jun et al. [11] in 2018. Following its introduction, they applied this concept to analyze the structures of logical algebras, specifically BCK- and BCI-algebras. Beyond logical algebras, hybrid structures find application in some mathematical structures, for instance, semigroups, ordered semigroups, hypersemigroups, and ordered hypersemigroups (see [4, 18, 26, 29]).

Ordered hypersemigroups can be classified into different groups based on their properties. The concept of hyperideals is useful in categorizing these ordered hypersemigroups. Hyperideals and interior hyperideals are essential ideals in ordered hypersemigroups. It is known that every hyperideal is an interior hyperideal, but the converse is not always true. However, for some classes of ordered hypersemigroups, every interior hyperideal is also a hyperideal. Tiprachot and Pibaljommee [30] have shown that this is also the case for fuzzy hyperideals and interior hyperideals. Several authors investigate the equivalence between (hyper-)ideals and interior (hyper-)ideals in other algebraic systems (see [13, 14, 27]). The main focus of this paper is on hybrid structures in ordered hypersemigroups. We explore the relationship between hybrid hyperideals and hybrid interior hyperideals in these structures. Additionally, we characterize ordered hypersemigroups that are semisimple by hybrid hyperideals.

2. Preliminary

This section will review the basic terms and definitions from the theories of ordered hypersemigroups and hybrid structures. The paper will utilize the concepts of ordered hypersemigroups introduced by Kehayopulu [15] and hybrid structures introduced by Anis [4].

Definition 2.1 ([15])**.** A *hypergroupoid* is a structure $(H;\circ)$ consisting of a nonempty set H and a hyperoperation

$$\circ\colon H\times H\to\mathcal{P}^*(H)\mid(a,b)\mapsto a\circ b$$

defined on H.

Let $(H;\circ)$ is a hypergroupoid. We observe that the hyperoperation $\circ$ defined on H induces a binary operation

$$*\colon\mathcal{P}^*(H)\times\mathcal{P}^*(H)\to\mathcal{P}^*(H)\mid(A,B)\mapsto A*B$$

defined on $\mathcal{P}^*(H)$ assigned by

$$A*B:=\bigcup_{a\in A,b\in B}(a\circ b). \tag{1}$$

We note here that $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H.

Remark 2.1. From Equation (1), we see that $x\circ y=\{x\}*\{y\}$. Therefore, we simply write $\{x\}*\{y\}$ by $x*y$. That is, $x\circ y=x*y$ in this sense. We see that $A\subseteq B$ implies $A*C\subseteq B*C$ and $C*A\subseteq C*B$ for any nonempty subsets A,B and C of H.

Definition 2.2 ([15])**.** A hypergroupoid $(H;\circ)$ is called a *hypersemigroup* if

$$\{x\}*(y\circ z)=(x\circ y)*\{z\} \tag{2}$$

for every $x,y,z\in H$.

Applying Remark 2.1, Equation (2) could be identified as

$$x*(y*z)=(x*y)*z$$

and we denote the n-product $a*a*\cdots*a$ of an element a by a^n.

Let $(H;\leq)$ be a partial order set. We define a relation $\preceq$ on $\mathcal{P}^*(H)$ as follows: For two nonempty subsets A and B of H,

$$A\preceq B:=\{(x,y)\in A\times B\mid x\leq y,\forall x\in A,\exists y\in B\}.$$

Definition 2.3 ([9])**.** The structure $(H;\circ,\leq)$ is called an *ordered hypersemigroup* if the following conditions are satisfied:

(1) $(H;\circ)$ is a hypersemigroup;
(2) $(H;\leq)$ is a partial order set;
(3) for $a,b,c\in H$, if $a\leq b$ then $a*c\preceq b*c$ and $c*a\preceq c*b$.

For simplicity, we denoted an ordered hypersemigroup $(H;\circ,\leq)$ by its carrier set as a boldface letter $\mathbf{H}$.

Definition 2.4 ([9])**.** Let $\mathbf{H}$ be an ordered hypersemigroup. A nonempty subset A of H is called a *left* (*resp., right*) *hyperideal* of $\mathbf{H}$ if:

(1) $H*A\subseteq A$ (resp., $A*H\subseteq A$);
(2) for $a\in H,b\in A$, if $a\leq b$, then $a\in A$.

A nonempty subset A of H is called a *two-sided hyperideal*, or simply a *hyperideal* of $\mathbf{H}$ if it is both a left and a right hyperideal of $\mathbf{H}$.

Definition 2.5 ([30])**.** Let $\mathbf{H}$ be an ordered hypersemigroup. A nonempty subset A of H is called an *interior hyperideal* of $\mathbf{H}$ if:

(1) $H * A * H \subseteq A$;
(2) for $a \in H, b \in A$, if $a \leq b$, then $a \in A$.

Remark 2.2. Suppose that A is a hyperideal of an ordered hypersemigroup $\mathbf{H}$. Consider

$$H * A * H = H * (A * H) \subseteq H * A \subseteq A.$$

That is, A is also an interior hyperideal of $\mathbf{H}$. This means that any hyperideal is an interior hyperideal in ordered hypersemigroups.

Let A be a nonempty subset of H. Define

$$(A] := \{x \in H \mid x \leq a \text{ for some } a \in A\}.$$

Note that condition (2) in Definition 2.4 and Definition 2.5 are equivalent to $A = (A]$. If A and B are nonempty subsets of H, then we obtain the following conditions.

(1) $A \subseteq (A]$.
(2) $(A \cup B] = (A] \cup (B]$.
(3) $((A] * (B]] = (A * B]$.
(4) $(A] * (B] \subseteq (A * B]$.

For further information about ordered hypersemigroups, we refer the readers to [9, 15].

Let I be the unit interval, H a set of parameters and $\mathcal{P}(U)$ denote the power set of an initial universe set U. Hybrid structures are defined as follows.

Definition 2.6 ([4])**.** A *hybrid structure* in H over U is defined to be a mapping

$$f := (f^*, f^+)\colon H \to \mathcal{P}(U) \times I \mid x \mapsto (f^*(x), f^+(x)),$$

where

$$f^*\colon H \to \mathcal{P}(U) \qquad \text{and} \qquad f^+\colon H \to I$$

are mappings.

Let us denote by $Hyb_U(H)$ the set of all hybrid structures in H over U. We will define an operation on $Hyb_U(H)$ but first we have to define some important set as follows: Let a be an element in H. Then, we set

$$\mathbf{H}_a := \{(x, y) \in H \times H \mid a \in (x * y]\}.$$

Definition 2.7 ([4])**.** Let $f = (f^*, f^+)$ and $g = (g^*, g^+)$ be elements in $Hyb_U(H)$. Then, the hybrid products of f and g are denoted by $f \otimes g$ and is defined to be a hybrid structure

$$f \otimes g\colon H \to P(U) \times I \mid x \mapsto \left((f^* \odot g^*)(x), (f^+ \oplus g^+)(x)\right),$$

where

$$(f^* \odot g^*)(x) = \begin{cases} \bigcup\limits_{(a,b)\in \mathbf{H}_x} (f^*(a) \cap g^*(b)) & \text{if } \mathbf{H}_a \neq \emptyset \\ \emptyset & \text{if } \mathbf{H}_a = \emptyset \end{cases}$$

and

$$(f^+ \oplus g^+)(x) = \begin{cases} \bigwedge\limits_{(a,b)\in \mathbf{H}_x} \{\max\{f^+(a), g^+(b)\}\} & \text{if } \mathbf{H}_a \neq \emptyset \\ 1 & \text{if } \mathbf{H}_a = \emptyset. \end{cases}$$

By Definition 2.7, it is seen that the operation $\otimes$ satisfies the associative properties. Then, the structure $(Hyb_U(H); \otimes)$ becomes a semigroup. Now, we define a binary relation $\ll$ on $Hyb_U(H)$ as follows: For $f = (f^*, f^+)$ and $g = (g^*, g^+) \in Hyb_U(H)$,

$$f \ll g \qquad \text{if and only if} \qquad f^* \sqsubseteq g^* \text{ and } f^+ \succeq g^+,$$

where $f^* \sqsubseteq g^*$ means that $f^*(x) \subseteq g^*(x)$ and $f^+ \succeq g^+$ means that $f^+(x) \geq g^+(x)$ for all $x \in H$. Furthermore, $f = g$ if $f \ll g$ and $g \ll f$. It is routine to verify that the relation $\ll$ is compatible with the operation $\otimes$. This implies that the structure $(Hyb_U(H); \otimes, \ll)$ is an ordered semigroup.

Definition 2.8 ([4]). Let $f = (f^*, f^+)$ and $g = (g^*, g^+)$ be elements in $Hyb_U(H)$. Then, the hybrid intersection of f and g is denoted by $f \Cap g$ and is defined to be a hybrid structure

$$f \Cap g \colon H \to P(U) \times I \mid x \mapsto ((f^* \cap g^*)(x), (f^+ \vee g^+)(x)),$$

where

$$(f^* \cap g^*)(x) := f^*(x) \cap g^*(x) \quad \text{and} \quad (f^+ \vee g^+)(x) := \max\{f^+(x), g^+(x)\}.$$

We denote $\widetilde{H} := (H^*, H^+)$ the hybrid structure in H over U and is defined as follows:

$$\widetilde{H} \colon H \to P(H) \times I \mid x \mapsto (H^*(x), H^+(x)),$$

where

$$H^*(x) := U \qquad \text{and} \qquad H^+(x) := 0.$$

Let A be a nonempty subset of H. We denote by $\chi_A := (\chi_A^*, \chi_A^+)$ the characteristic hybrid structure of A in H over U which is defined to be a hybrid structure

$$\chi_A \colon H \to P(U) \times I \mid x \mapsto (\chi_A^*(x), \chi_A^+(x)),$$

where

$$\chi_A^*(x) = \begin{cases} U & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \qquad \text{and} \qquad \chi_A^+(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

for all $x \in H$. We see that $\chi_A = \widetilde{H}$ in the case that $A = H$.

Remark 2.3. Let $\mathbf{H}$ be an ordered hypersemigroup and A a nonempty subset of H. Suppose that one of the following conditions holds:

(1) $\chi_A^*(x) \supseteq \chi_A^*(y)$;
(2) $\chi_A^+(x) \leq \chi_A^+(y)$;

for all $x, y \in H$. We have that $x \in A$ if $y \in A$.

We obtain the following remark similarly.

Remark 2.4. Let $\mathbf{H}$ be an ordered hypersemigroup and A, B are nonempty subsets of H. Suppose that one of the following conditions holds:

(1) $\chi_B^*(x) \supseteq \chi_A^*(x)$;
(2) $\chi_B^+(x) \leq \chi_A^+(x)$;

for all $x, y \in H$. We have that $A \subseteq B$ if $x \in A$.

3. Main results

In this main section, we discuss the coincidence of hybrid hyperideals and hybrid interior hyperideals. Finally, we characterize semisimple ordered hypersemigroups in terms of hybrid interior hyperideals.

Definition 3.1 ([25, 26])**.** Let $\mathbf{H}$ be an ordered hypersemigroup. A hybrid structure $f := (f^*, f^+)$ in H over U is called a *hybrid right (resp., left) hyperideal* in $\mathbf{H}$ over U if for every $x, y \in H$:

(1) $\bigcap_{a \in x*y} f^*(a) \supseteq f^*(x)$ (resp., $\bigcap_{a \in x*y} f^*(a) \supseteq f^*(y)$);
(2) $\bigvee_{a \in x*y} f^+(a) \leq f^+(x)$ (resp., $\bigvee_{a \in x*y} f^+(a) \leq f^+(y)$);
(3) if $x \leq y$, then $f^*(x) \supseteq f^*(y)$ and $f^+(x) \leq f^+(y)$.

A hybrid structure f is called a *hybrid hyperideal* in $\mathbf{H}$ over U if it is both a hybrid left and a hybrid right hyperideal in $\mathbf{H}$ over U.

Example 3.2. Let $H = \{a, b, c\}$. We define a binary hyperoperation $\circ$ and a binary relation $\leq$ on H as follows:

$\circ$	a	b	c
a	{a}	{a}	{a}
b	{a}	{a}	{a}
c	H	H	H

and $\leq := \{(a, c), (b, c)\} \cup \Delta_H$, where Δ_H is an identity relation on H. Then, $\mathbf{H} := (H; \circ, \leq)$ is an ordered hypersemigroup. Let $U = \mathbb{N}$. Define a hybrid structure $f := (f^*, f^+)$ in H over U as follows:

H	$f^*(x)$	$f^+(x)$
a	$\mathbb{N}$	0.2
b	$2\mathbb{N}$	0.7
c	$4\mathbb{N}$	0.8

Then, f is a hybrid right hyperideal in $\mathbf{H}$ over U.

Example 3.3. Let $H = \{a, b, c\}$. We define a binary hyperoperation $\circ$ and a binary relation $\leq$ on H as follows:

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a, b\}$	$\{c\}$

and $\leq := \{(a, b)\} \cup \Delta_H$, where Δ_H is an identity relatity on H. Then, $H := (H; \circ, \leq)$ is an ordered hypersemigroup. Let $U = \mathbb{N}$. Define a hybrid structure $f := (f^*, f^+)$ in H over U as follows:

H	$f^*(x)$	$f^+(x)$
a	$2\mathbb{N}$	0.7
b	$\mathbb{N}$	0.2
c	$4\mathbb{N}$	0.8

Then, f is a hybrid left hyperideal in $\mathbf{H}$ over U.

Definition 3.4 ([29])**.** Let $\mathbf{H}$ be an ordered hypersemigroup. A hybrid structure $f := (f^*, f^+)$ in H over U is called a *hybrid interior hyperideal* in $\mathbf{H}$ over U if for every $x, y, z \in H$:

(1) $\bigcap\limits_{c \in x*y*z} f^*(c) \supseteq f^*(y)$;

(2) $\bigvee\limits_{c \in x*y*z} f^+(c) \leq f^+(y)$;

(3) if $x \leq y$, then $f^*(x) \supseteq f^*(y)$ and $f^+(x) \leq f^+(y)$.

Example 3.5. Let $H = \{a, b, c, d\}$. We define a binary hyperoperation $\circ$ and a binary relation $\leq$ on H as follows:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$

and $\leq := \{(a, b), (a, d)\} \cup \Delta_H$, where Δ_H is an identity relation on H. Then, $H := (H; \circ, \leq)$ is an ordered hypersemigroup. Let $U = \{1, 2, 3\}$. Define a hybrid structure $f := (f^*, f^+)$ in H over U as follows:

H	$f^*(x)$	$f^+(x)$
a	U	0.1
b	$\{1, 2\}$	0.5
c	$\{1\}$	0.6
d	$\emptyset$	1

Then, f is a hybrid interior hyperideal in $\mathbf{H}$ over U.

The following result illustrates that the concept of hybrid hyperideals is a special case of hybrid interior hyperideals.

Lemma 3.6 ([29]). *Let* $\mathbf{H}$ *be an ordered hypersemigroup. Every hybrid hyperideal in* $\mathbf{H}$ *over* U *is a hybrid interior hyperideal in* $\mathbf{H}$ *over* U.

The converse of Lemma 3.6, in general, is not true shown as follows.

Example 3.7. Let $H = \{a, b, c, d\}$. Define a binary hyperoperation $\circ$ on H as following table:

$\circ$	a	b	c	d
a	{a}	$\{a\}$	$\{a\}$	$\{a\}$
b	{a}	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$

We define an order relation $\leq$ on H as follows:

$$\leq := \{(a, b), (a, c), (a, d), (b, d), (c, d)\} \cup \Delta_H,$$

where Δ_H is an identity relation on H. Therefore $\mathbf{H} := (H; \circ, \leq)$ is an ordered hypersemigroup. Let $U = \{1, 2, 3\}$. Then, we define hybrid structure $f := (f^*, f^+)$ in H over U as follows:

H	$f^*(x)$	$f^+(x)$
a	$\emptyset$	1
b	U	0
c	$\{1\}$	0.8
d	$\{1, 2\}$	0.5

It is easy to see that f is a hybrid interior hyperideal in $\mathbf{H}$ over U. But, it is not a hybrid hyperideal in $\mathbf{H}$ over U since

$$\bigcap_{x \in c*d} f^*(x) = f^*(a) \cap f^*(b) = \emptyset \cap U = \emptyset \not\supseteq \{1, 2\} = f^*(d)$$

and

$$\bigvee_{x \in c*d} f^+(x) = \max\{f^+(a), f^+(b)\} = \max\{1, 0\} = 1 \not\leq 0.5 = f^+(d).$$

An ordered hypersemigroup $\mathbf{H}$ is called *regular* [15] if for each $a \in H$, there exists $x \in H$ such that $a \in (a * x * a]$.

Now, we show that in regular ordered hypersemigroups the concepts of hybrid interior hyperideals and hybrid hyperideals coincide.

Lemma 3.8. *Let* $\mathbf{H}$ *be a regular ordered hypersemigroup. Then, every hybrid interior hyperideal in* $\mathbf{H}$ *over* U *is a hybrid hyperideal in* $\mathbf{H}$ *over* U.

Proof. Let $f := (f^*, f^+)$ be a hybrid interior hyperideal in $\mathbf{H}$ over U. Suppose that $a, b \in H$. Since H is a regular ordered hypersemigroup, there exists $x \in H$ such that $a \in (a * x * a]$. That is, $a \leq c$ for some $c \in a * x * a$. By the condition

(3) of Definition 2.3, we have that $a*b \preceq c*b$. This means that for any $u \in a*b$, there exists $v \in c*b$ such that $u \leq v$. Thus,

$$f^*(u) \supseteq f^*(v) \supseteq \bigcap_{v\in c*d} f^*(v) \supseteq \bigcap_{v\in (a*x*a)*b} f^*(v) = \bigcap_{v\in (a*x)*a*b} f^*(v) \supseteq f^*(a),$$

and

$$f^+(u) \leq f^*(v) \leq \bigvee_{v\in c*d} f^+(v) \leq \bigvee_{v\in a*(x*a)*b} f^+(v) = \bigvee_{v\in (a*x)*a*b} f^+(v) \leq f^+(a).$$

Therefore,

$$\bigcap_{u\in a*b} f^*(u) \supseteq f^*(a) \qquad \text{and} \qquad \bigvee_{u\in a*b} f^+(u) \leq f^+(a).$$

Hence, f is a hybrid right hyperideal in $\mathbf{H}$ over U. Similarly, we can prove that f is a hybrid left hyperideal in $\mathbf{H}$ over U. Altogether, f is a hybrid hyperideal in $\mathbf{H}$ over U. $\square$

Combining Lemma 3.6 and 3.8, we obtain the following consequence.

Theorem 3.9. *Let* $\mathbf{H}$ *be a regular ordered hypersemigroup and* f *a hybrid structure in* H *over* U*. Then, the following statements are equivalent.*

(1) f *is hybrid hyperideal in* $\mathbf{H}$ *over* U*.*
(2) f *is hybrid interior hyperideal in* $\mathbf{H}$ *over* U*.*

An ordered hypersemigroup $\mathbf{H}$ is called *intra-regular* [15] if, for each $a \in H$, there exist $x, y \in H$ such that $a \in (x * a^2 * y]$.

As Lemma 3.8, the following theorem illustrates the equivalence of hybrid hyperideals and hybrid interior hyperideals in intra-regular ordered hypersemigroups.

Lemma 3.10. *Let* $\mathbf{H}$ *be an intra-regular ordered hypersemigroup. Then, every hybrid interior hyperideal in* $\mathbf{H}$ *over* U *is a hybrid hyperideal in* $\mathbf{H}$ *over* U*.*

Proof. Let $f := (f^*, f^+)$ be a hybrid interior hyperideal in $\mathbf{H}$ over U. Suppose that $a, b \in H$. Since $\mathbf{H}$ is an intra-regular ordered hypersemigroup, there exist $x, y \in H$ such that $a \in (x * a^2 * y]$. That is, $a \leq c$ for some $c \in x * a^2 * y$. By the condition (3) of Definition 2.3, we have that $a * b \preceq c * b$. This means that for any $u \in a * b$, there exists $v \in c * b$ such that $u \leq v$. Thus,

$$f^*(u) \supseteq f^*(v) \supseteq \bigcap_{v\in c*b} f^*(v) \supseteq \bigcap_{v\in (x*a^2*y)*b} f^*(v) = \bigcap_{v\in (x*a)*a*(y*b)} f^*(v) \subseteq f^*(a),$$

and

$$\begin{aligned} f^+(u) &\leq f^+(v) \\ &\leq \bigvee_{v\in c*b} f^+(v) \end{aligned}$$

$$\begin{aligned}&\leq \bigvee_{v\in(x*a^2*y)*b} f^+(v)\\&= \bigvee_{v\in(x*a)*a*(y*b)} f^+(v)\\&\leq f^+(a).\end{aligned}$$

Therefore,

$$\bigcap_{u\in a*b} f^*(u) \supseteq f^*(a) \qquad \text{and} \qquad \bigvee_{u\in a*b} f^+(u) \leq f^+(a).$$

Hence, f is a hybrid right hyperideal in $\mathbf{H}$ over U. Similarly, we can prove that f is a hybrid left hyperideal in $\mathbf{H}$ over U. Altogether, f is a hybrid hyperideal in $\mathbf{H}$ over U. $\square$

Combining Lemma 3.6 and 3.10, we obtain the following theorem.

Theorem 3.11. *Let $\mathbf{H}$ be an intra-regular ordered hypersemigroup and f a hybrid structure in H over U. Then, the following statements are equivalent.*

(1) *f is hybrid hyperideal in $\mathbf{H}$ over U.*
(2) *f is hybrid interior hyperideal in $\mathbf{H}$ over U.*

An ordered hypersemigroup $\mathbf{H}$ is called *semisimple* [16] if for each $a \in H$, there exist $x, y, z \in H$ such that $a \in (x * (a * y) * (a * z)]$.

In semisimple ordered hypersemigroup the concepts of hybrid interior hyperideals and hybrid hyperideals coincide as well as in regular and intra-regular ordered hypersemigroups as the following theorem.

Lemma 3.12. *Let $\mathbf{H}$ be a semisimple ordered hypersemigroup. Then every hybrid interior hyperideal in $\mathbf{H}$ over U is a hybrid hyperideal in $\mathbf{H}$ over U.*

Proof. Let $f := (f^*, f^+)$ be a hybrid interior hyperideal in $\mathbf{H}$ over U. Suppose that $a, b \in H$. Since $\mathbf{H}$ is a semisimple ordered hypersemigroup, there exist $x, y, z \in H$ such that $a \in (x * (a * y) * (a * y) * (a * z)]$. That is, $a \leq c$ for some $c \in x * (a * y) * (a * y) * (a * z)$. By the condition (3) of Definition 2.3, we have that $a * b \preceq c * b$. This means that for any $u \in a * b$, there exists $v \in c * b$ such that $u \leq v$. Thus,

$$\begin{aligned}f^*(u) &\supseteq f^*(v)\\&\supseteq \bigcap_{v\in c*b} f^*(v)\\&\supseteq \bigcap_{v\in x*(a*y)*(a*y)*(a*z)*b} f^*(v)\\&= \bigcap_{v\in(x*a)*(y*a)*y*a*(z*b)} f^*(v)\\&= f^*(a),\end{aligned}$$

and

$$\begin{aligned} f^+(u) &\leq f^*(v) \\ &\leq \bigvee_{v\in c*b} f^+(v) \\ &\leq \bigvee_{v\in x*(a*y)*(a*y)*(a*z)*b} f^+(v) \\ &= \bigvee_{v\in (x*a)*(y*a)*y*a*(z*b)} f^+(v) \\ &= f^+(a). \end{aligned}$$

Therefore,

$$\bigcap_{u\in a*b} f^*(u) \supseteq f^*(a) \qquad \text{and} \qquad \bigvee_{u\in a*b} f^+(u) \leq f^+(a).$$

Hence, f is a hybrid right hyperideal in $\mathbf{H}$ over U. Similarly, we can prove that f is a hybrid left hyperideal in $\mathbf{H}$ over U. Altogether, f is a hybrid hyperideal in $\mathbf{H}$ over U. $\square$

Combining Lemma 3.6 and 3.12, we have the following theorem.

Theorem 3.13. *Let $\mathbf{H}$ be a semisimple ordered hypersemigroup and f a hybrid structure in H over U. Then the following statements are equivalent.*

(1) *f is hybrid hyperideal in $\mathbf{H}$ over U.*
(2) *f is hybrid interior hyperideal in $\mathbf{H}$ over U.*

The next lemmas are important tools to give the connection between hyperideals of $\mathbf{H}$ and hybrid hyperideals in $\mathbf{H}$ over U in terms of the operation $\otimes$.

Lemma 3.14. *Let $\mathbf{H}$ be an ordered hypersemigroup and $f := (f^*, f^+)$ a hybrid structure in H over U. If f is a hybrid interior hyperideal in $\mathbf{H}$ over U, then we have $\widetilde{H} \otimes f \otimes \widetilde{H} \ll f$.*

Proof. Let f be a hybrid interior hyperideal in $\mathbf{H}$ over U. Suppose that $a \in H$. If $\mathbf{H}_a \neq \emptyset$, then we obtain

$$\begin{aligned} (H^* \odot f^* \odot H^*)(a) &= \bigcup_{(x,y)\in\mathbf{H}_a} [(H^* \odot f^*)(x) \cap H^*(y)] \\ &= \bigcup_{(x,y)\in\mathbf{H}_a} (H^* \odot f^*)(x) \\ &= \bigcup_{(x,y)\in\mathbf{H}_a} \left[\bigcup_{(u,v)\in\mathbf{H}_x} (H^*(u) \cap f^*(v)) \right] \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{(x,y)\in \mathbf{H}_a} \left[\bigcup_{(u,v)\in \mathbf{H}_x} f^*(v) \right] \\
&\subseteq \bigcup_{(x,y)\in \mathbf{H}_a} \left[\bigcup_{(u,v)\in \mathbf{H}_x} \left[\bigcap_{c\in u*(v*y)} f^*(c) \right] \right] \\
&= \bigcup_{(c,y)\in \mathbf{H}_a} \left[\bigcap_{c\in u*(v*y)} f^*(c) \right] \\
&\subseteq f^*(a),
\end{aligned}
$$

and

$$
\begin{aligned}
(H^+ \oplus f^+ \oplus H^+)(a) &= \bigwedge_{(x,y)\in \mathbf{H}_a} \max\{(H^+ \oplus f^+)(x), H^+(y)\} \\
&= \bigwedge_{(x,y)\in \mathbf{H}_a} (H^+ \oplus f^+)(x) \\
&= \bigwedge_{(x,y)\in \mathbf{H}_a} \left[\bigwedge_{(u,v)\in \mathbf{H}_x} \{\max\{H^+(u), f^+(v)\}\} \right] \\
&= \bigwedge_{(x,y)\in \mathbf{H}_a} \left[\bigwedge_{(u,v)\in \mathbf{H}_x} f^+(v) \right] \\
&\geq \bigwedge_{(x,y)\in \mathbf{H}_a} \left[\bigwedge_{(u,v)\in \mathbf{H}_x} \left[\bigvee_{c\in u*(v*y)} f^+(c) \right] \right] \\
&= \bigwedge_{(c,y)\in \mathbf{H}_a} \left[\bigvee_{c\in u*(v*y)} f^+(c) \right] \\
&\geq f^+(a).
\end{aligned}
$$

If $\mathbf{H}_a = \emptyset$, we obtain

$$(H^* \odot f^* \odot H^*)(a) = \emptyset \subseteq f^*(a),$$

and

$$(H^+ \oplus f^+ \oplus H^+)(a) = 1 \geq f^+(a).$$

Altogether, we have $\widetilde{H} \otimes f \otimes \widetilde{H} \ll f$. □

Lemma 3.15 ([25])**.** *Let $\mathbf{H}$ be an ordered hypersemigroup and A, B nonempty subsets of H. Then the following conditions hold.*

(1) *$A \subseteq B$ if and only if $\chi_A \ll \chi_B$.*
(2) *$\chi_A \Cap \chi_B = \chi_{A\cap B}$.*
(3) *$\chi_A \otimes \chi_B = \chi_{(A*B]}$.*

Now, we are ready to characterize interior hyperideals in ordered hypersemigroups in terms of hybrid interior hyperideals.

Lemma 3.16 ([29]). *Let $\mathbf{H}$ be an ordered hypersemigroup and A a nonempty subset of H. Then the following conditions are equivalent.*

(1) *A is an interior hyperideal of $\mathbf{H}$.*
(2) *χ_A is a hybrid interior hyperideal in $\mathbf{H}$ over U.*

Lemma 3.14, 3.15 and 3.16 are important for classifying some classes of ordered hypersemigroups. In order to demonstrate this, we will present additional results.

Lemma 3.17. *Let $\mathbf{H}$ be a semisimple ordered hypersemigroup. Suppose that $f := (f^*, f^+), g := (g^*, g^+)$ are hybrid interior hyperideals in $\mathbf{H}$ over U. Then $f \otimes g \ll f \Cap g$.*

Proof. Note that, by Lemma 3.12, f and g are hybrid hyperideals in $\mathbf{H}$ over U. Let $a \in H$. Since H is semisimple, there exist $x, y, z \in H$ such that $a \in (x * (a * y) * (a * z)]$. That is, $a \leq c$ for some $c \in x * (a * y) * (a * z)$. This implies that $\mathbf{H}_a \neq \emptyset$. Then,

$$\begin{aligned}
(f^* \odot g^*)(a) &= \bigcup_{(p,q)\in\mathbf{H}_a} [f^*(p) \cap g^*(q)] \\
&\subseteq \bigcup_{(p,q)\in\mathbf{H}_a} \left[\bigcap_{u\in p*q} f^*(u) \cap \bigcap_{u\in p*q} g^*(u)\right] \\
&\subseteq \bigcup_{(p,q)\in\mathbf{H}_a} \left[\bigcap_{u\in p*q} f^*(a) \cap \bigcap_{u\in p*q} g^*(a)\right] \\
&\subseteq \bigcup_{(p,q)\in\mathbf{H}_a} [f^*(a) \cap g^*(a)] \\
&= f^*(a) \cap g^*(a) \\
&= (f^* \cap g^*)(a)
\end{aligned}$$

and

$$\begin{aligned}
(f^+ \oplus g^+)(a) &= \bigwedge_{(p,q)\in\mathbf{H}_a} \{\max\{f^+(p), g^+(q)\}\} \\
&\geq \bigwedge_{(p,q)\in\mathbf{H}_a} \{\max\{\bigvee_{u\in p*q} f^+(u), \bigvee_{u\in p*q} g^+(u)\}\} \\
&\geq \bigwedge_{(p,q)\in\mathbf{H}_a} \{\max\{\bigvee_{u\in p*q} f^+(a), \bigvee_{u\in p*q} g^+(a)\}\} \\
&\geq \bigwedge_{(p,q)\in\mathbf{H}_a} \{\max\{f^+(a), g^+(a)\}\} \\
&= \max\{f^+(a), g^+(a)\}
\end{aligned}$$

$$= (f^+ \vee g^+)(a).$$

Therefore, $f \otimes g \ll f \Cap g$. □

The following result is obtained immediately.

Corollary 3.18. *Let* $\mathbf{H}$ *be a semisimple ordered hypersemigroup and* f *a hybrid interior hyperideal in* $\mathbf{H}$ *over* U*. Then* $f \otimes f \ll f$*.*

Lemma 3.19 ([8])**.** *Let* $\mathbf{H}$ *be an ordered hypersemigroup. Then the following conditions are equivalent.*

(1) $\mathbf{H}$ *is semisimple.*
(2) $A \cap B = (A * B]$ *for every hyperideals* A *and* B *of* $\mathbf{H}$*.*
(3) $A = (A * A]$ *for every hyperideal* A *of* $\mathbf{H}$*.*

Semisimple ordered hypersemigroups are characterized by hybrid hyperideals as follows.

Theorem 3.20. *Let* $\mathbf{H}$ *be an ordered hypersemigroup. Then the following conditions are equivalent.*

(1) $\mathbf{H}$ *is semisimple.*
(2) $f \otimes g = f \Cap g$ *for every hybrid interior hyperideals* $f := (f^*, f^+)$ *and* $g := (g^*, g^+)$ *in* $\mathbf{H}$ *over* U*.*

Proof. (1)⇒(2). Let f and g be hybrid interior hyperideals in $\mathbf{H}$ over U. Then, by Lemma 3.12, f and g are hybrid hyperideals in $\mathbf{H}$ over U. Let $a \in H$. Since $\mathbf{H}$ is semisimple, there exists $x, y, z \in H$ such that $a \in (x * (a * y) * (a * z)]$. That is, $a \leq c$ for some $c \in u * v$, where $u \in x * (a * y)$ and $v \in a * z$. This means that $\mathbf{H}_a \neq \emptyset$. Then,

$$\begin{aligned}
(f^* \odot g^*)(a) &= \bigcup_{(p,q) \in \mathbf{H}_a} [f^*(p) \cap g^*(q)] \\
&\supseteq f^*(u) \cap g^*(v) \\
&\supseteq \left(\bigcap_{u \in x*(a*y)} f^*(u) \right) \cap \left(\bigcap_{v \in a*z} g^*(v) \right) \\
&\supseteq f^*(a) \cap g^*(a) \\
&= (f^* \cap g^*)(a)
\end{aligned}$$

and

$$\begin{aligned}
(f^+ \oplus g^+)(a) &= \bigwedge_{(p,q) \in \mathbf{H}_a} \{\max\{f^+(p), g^+(q)\}\} \\
&\leq \max\{f^+(u), g^+(v)\} \\
&\leq \max \left\{ \bigvee_{u \in x*(a*y)} f^+(u), \bigcap_{v \in a*z} g^+(v) \right\}
\end{aligned}$$

$$\leq \max\{f^+(a), g^+(a)\}$$
$$= (f^+ \vee g^+)(a).$$

Therefore, $f \Cap g \ll f \otimes g$. By Lemma 3.17, we have $f \otimes g = f \Cap g$.

(2)⇒(1). Let A and B be hyperideals of $\mathbf{H}$. By Remark 2.2, A and B are interior hyperideals of $\mathbf{H}$. Moreover, by Lemma 3.16, we obtain χ_A and χ_B are hybrid interior hyperideals in $\mathbf{H}$ over U. By our hypothesis and Lemma 3.15, we have

$$\chi_{(A*B]} = \chi_A \otimes \chi_B = \chi_A \Cap \chi_B = \chi_{A\cap B}.$$

Again, by Lemma 3.15, $A \cap B = (A * B]$. Therefore, by Lemma 3.19, $\mathbf{H}$ is semisimple. □

By Theorem 3.20, we obtain the following corollary.

Corollary 3.21. *Let* $\mathbf{H}$ *be an ordered hypersemigroup. Then the following conditions are equivalent.*

(1) $\mathbf{H}$ *is semisimple.*
(2) $f = f \otimes f$ *for every hybrid hyperideal* $f := (f^*, f^+)$ *in* $\mathbf{H}$ *over* U.

4. Conclusions

Our focus in this paper was on hybrid interior hyperideals in ordered hypersemigroups. We discovered that the concepts of hybrid interior hyperideal and hybrid hyperideal are the same in regular, intra-regular, and semisimple ordered hypersemigroups. Therefore, these specific ordered hypersemigroups play a crucial role in investigating ordered hypersemigroups. We chose to consider the class of semisimple ordered hypersemigroups and gave a characterization of them using hybrid interior hyperideals. The ideas presented in this paper have potential applications in the theory of hypersemirings, hypersemirings, hypergroups, and BCI/BCK hyperalgebras.

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References

1. N. Abughazalah and N. Yaqoob, *A Novel Study on Ordered Anti-Involution LA-Semihypergroups*, J. Math. **2023** (2023), 6972483.
2. M.F. Ali and N.M. Khan, *Characterization of ordered semihypergroups by covered hyperideals*, Kragujevac J. Math. **47** (2023), 417-429.
3. T.M. Al-shami, J.C.R. Alcantud and A. Mhemdi, *New generalization of fuzzy soft sets: (a, b)-fuzzy soft sets*, AIMS Math. **8** (2023), 2995-3025.

4. S. Anis, M. Khan and Y.B. Jun, *Hybrid ideals in semigroups*, Cogent math. **4** (2017), 1352117.
5. T. Changphas and B. Davvaz, *Properties of hyperideals in ordered semihypergroups*, Ital. J. Pure Appl. Math. **33** (2014), 425-432.
6. J. de Andrés-Sánchez, *A systematic review of the interactions of fuzzy set theory and option pricing*, Expert Syst. Appl. **223** (2023), 119868.
7. Z. Gu and X. Tang, *Ordered regular equivalence relations on ordered semihypergroups*, J. Algebra **450** (2016), 384-397.
8. Z. Gu, *On hyperideals of ordered semihypergroups*, Ital. J. Pure Appl. Math. **40** (2018), 692-698.
9. D. Heidari and B. Davvaz, *On ordered hyperstructures*, UPB Sci. Bull. A: Appl. **73** (2011), 85-96.
10. N. Jan, J. Gwak and D. Pamucar, *Mathematical analysis of generative adversarial networks based on complex picture fuzzy soft information*, Appl. Soft Comput. **137** (2023), 110088.
11. Y.B. Jun, S.Z. Song and G. Muhiuddin, *Hybrid structures and applications*, Ann. Commun. Math. **1** (2018), 11-25.
12. A. Khan, M. Farooq and N. Yaqoob, *Uni-soft structures applied to ordered Γ-semihypergroups*, Proc. Natl. Acad. Sci. India-Phys. Sci. **90** (2020), 457-465.
13. N. Kehayopulu and M. Tsingelis, *Fuzzy interior ideals in ordered semigroups*, Lobachevskii J. Mathe. **21** (2006), 65-71.
14. N. Kehayopulu and M. Tsingelis, *Fuzzy ideals in ordered semigroups*, Quasigroups Related Systems **15** (2007), 279-289.
15. N. Kehayopulu, *From ordered semigroups to ordered hypersemigroups*, Turk. J. Math. **42** (2018), 2045-2060.
16. N. Kehayopulu, *From ordered semigroups to ordered hypersemigroups*, Turk. J. Math. **43** (2019), 21-35.
17. J.S. Lawrence and R. Manoharan, *New approach to DeMorgan's laws for soft sets via soft ideals*, Appl. Nanosci. **13** (2023), 2585-2592.
18. S. Lekkoksung, W. Samormob and K. Linesawat, *On hybrid hyperideals in hypersemigroups*, Proceeding of the 14th International Conference on Science, Technology and Innovation for Sustainable Well-Being (STISWB2022), Pakxe, Lao P.D.R., 2022, 144-148.
19. J. Lu, L. Zhu and W. Gao, *Remarks on bipolar cubic fuzzy graphs and its chemical applications*, Int. J. Math. Comp. Eng. **1** (2023), 1-10.
20. M. Luo and W. Li, *Some new similarity measures on picture fuzzy sets and their applications*, Soft Comput. **27** (2023), 6049-6067.
21. P.K. Maji, R. Biswas and A.R. Roy, *Fuzzy soft sets*, J. Fuzzy Math. **9** (2001), 589-602.
22. J. Mekwian and N. Lekkoksung, *Hybrid ideals in ordered semigroups and some characterizations of their regularities*, J. Math. Com. Sci. **11** (2021), 8075-8094.
23. D. Molodtsov, *Soft set theory first results*, Comput. Math. Appl. **37** (1999), 19-31.
24. V. Salsabeela, T.M. Athira, S.J. John and T. Baiju, *Multiple criteria group decision making based on q-rung orthopair fuzzy soft sets*, Granul. Comput. **8** (2023), 1067-1080.
25. H. Sanpan, P. Kitpratyakul and S. Lekkoksung, *On hybrid quasi-pure hyperideals in ordered hypersemigroups*, Ann. Fuzzy Math. Inform. **25** (2023), 161-173.
26. H. Sanpan, P. Kitpratyakul and S. Lekkoksung, *On hybrid pure hyperideals in ordered hypersemigroups*, Int. J. Anal. Appl. **21** (2023), 47.
27. M. Shabir and A. Khan, *Intuitionistic fuzzy interior ideals in ordered semigroups*, J. Appl. Math. and Informatics **59** (2010), 539-549.
28. J. Tang and N. Yaqoob, *A novel investigation on fuzzy hyperideals in ordered $*$-semihypergroups*, Comput. Appl. Math. **40** (2021), 43.
29. N. Tiprachot, N. Lekkkoksung and S. Lekkoksung, *On hybrid simple ordered hypersemigroups*, Ann. Fuzzy Math. Inform. **26** (2023), 125-140.

30. N. Tiprachot and B. Pibaljommee, *Fuzzy interior hyperideals in ordered semihypergroups*, Ital. J. Pure Appl. Math. **36** (2016), 859-870.
31. M.G. Voskoglou, *A Combined use of soft sets and grey numbers in decision making*, J. Comp. Cog. Eng. **2** (2023), 1-4.
32. N. Yaqoob, M. Gulistan, J. Tang and M. Azhar, *On generalized fuzzy hyperideals in ordered LA-semihypergroups*, Comp. Appl. Math. **38** (2019), 124.
33. N. Yaqoob, J. Tang and J. Tang, *Approximations of quasi and interior hyperfilters in partially ordered LA-semihypergroups*, AIMS Mathematics **6** (2021), 7944-7960.
34. L.A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965) 338-353.
35. F. Zhang, W. Ma and H. Ma, *Dynamic chaotic multi-attribute group decision making under weighted T-spherical fuzzy soft rough sets*, Symmetry **15** (2023), 307.

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