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HYERS-ULAM-RASSIAS STABILITY OF A GENERAL DECIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the stability of the general decic functional equation

$$\sum_{i=0}^{11} {}_{11}C_i(-1)^{11-i}f(x+iy)=0$$

in the sense of Rassias.

1. Introduction

In this paper, let V, X, Y be a real vector space, a real normed space, and a real Banach space, respectively. It is well known that the beginning of the study on the stability of functional equations began as an attempt to solve Ulam's question [25] about the stability of the group homomorphisms. As a partial answer to this question, Hyers [7] solved the stability of the Cauchy functional equation in the following year. Since then, many mathematicians [5, 6, 8, 23] have generalized Hyers' results by showing the stability of several functional equations. Today the term 'Hyers-Ulam-Rassias stability' refers to the generalization introduced by Rassias [23].

We call the following functional equation

$$\sum_{i=0}^{k} {}_{k}C_i(-1)^{k-i}f(x+iy)=0 \tag{1}$$

as a Jensen, general quadratic, general cubic, general quartic, general quintic, general sextic, general septic, general octic, general nonic, and general decic functional equation for $k=2,3,4,5,6,7,8,9,10,11$, respectively. And the solutions of the functional equation (1) are called Jensen, general quadratic, general cubic, general quartic, general quintic, general sextic, general septic, general octic, general nonic, and general decic mappings for $k=2,3,4,5,6,7,8,9,10,11$,

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respectively. The results of the stability of the functional equation (1) for $k \le 10$ can be found in [18, 15, 9, 14, 13, 16, 19, 17, 10, 2, 24, 3, 11, 12, 22, 4].

In this paper, we investigate the stability of the general decic functional equation

$$\sum_{i=0}^{11} {}_{11}C_i(-1)^{11-i} f(x+iy) = 0$$

in the sense of Rassias. Precisely, we decompose the given f by mappings, i.e., $f = f_1 + f_2 + \cdots + f_{10}$, and use the properties of these mappings in Lemma 2.3 and Lemma 2.4 to get the stability result in Theorem 3.1 and the partial hyperstability result in Theorem 4.1.

More detailed term for the concept of "a general decic mapping" can be found in Baker's paper [1] by the terms "generalized polynomial mappings of degree at most 10".

2. Preliminaries

Throughout this paper, we use the following definitions.

Definition 2.1. *For a given $f : V \to Y$, we define $Df : V^2 \to Y$ by*

$$Df(x,y) := \sum_{i=0}^{11} {}_{11}C_i(-1)^{11-i} f(x+iy)$$

for all $x, y \in V$, and the mappings $\tilde{f}, f_o, f_e, \Gamma f, \Delta f : V \to Y$ as

$$\tilde{f}(x) := f(x) - f(0),$$

$$f_o(x) := \frac{f(x) - f(-x)}{2}, \qquad f_e(x) := \frac{f(x) + f(-x)}{2},$$

$$\begin{aligned}\Gamma f(x) := {} & Df_o(-12x, 4x) + 11\, Df_o(-16x, 4x) + 66\, Df_o(-20x, 4x) \\ & + 220\, Df_o(-2x, 2x) + 2420\, Df_o(-4x, 2x) + 12408\, Df_o(-6x, 2x) \\ & + 39688\, Df_o(-8x, 2x) + 85040\, Df_o(-10x, 2x) + 17920\, Df_o(-x, x) \\ & + 197120\, Df_o(-2x, x) + 957440\, Df_o(-3x, x) + 2647040\, Df_o(-4x, x) \\ & + 4438016\, Df_o(-5x, x),\end{aligned}$$

$$\begin{aligned}\Delta f(x) := {} & Df_e(-12x, 4x) + 11\, Df_e(-16x, 4x) + 66\, Df_e(-20x, 4x) \\ & + 352\, Df_e(-2x, 2x) + 3872\, Df_e(-4x, 2x) + 19008\, Df_e(-6x, 2x) \\ & + 54208\, Df_e(-8x, 2x) + 84800\, Df_e(-10x, 2x) + 57344\, Df_e(-x, x) \\ & + 630784\, Df_e(-2x, x) + 2883584\, Df_e(-3x, x) + 6488064\, Df_e(-4x, x) \\ & + 5226496\, Df_e(-5x, x)\end{aligned}$$

for all $x \in V$, respectively.

Consider the family $\{f_1(x), f_3(x), f_5(x), f_7(x), f_9(x)\}$ which is the solution to the system of non-homogeneous linear equations

$$\begin{cases} f_1(x) + f_3(x) + f_5(x) + f_7(x) + f_9(x) = f_o(x) \\ 2\, f_1(x) + 8\, f_3(x) + 32\, f_5(x) + 128\, f_7(x) + 512\, f_9(x) = f_o(2x) \\ 2^2\, f_1(x) + 8^2\, f_3(x) + 32^2\, f_5(x) + 128^2\, f_7(x) + 512^2\, f_9(x) = f_o(4x) \\ 2^3\, f_1(x) + 8^3\, f_3(x) + 32^3\, f_5(x) + 128^3\, f_7(x) + 512^3\, f_9(x) = f_o(8x) \\ 2^4\, f_1(x) + 8^4\, f_3(x) + 32^4\, f_5(x) + 128^4\, f_7(x) + 512^4\, f_9(x) = f_o(16x) \end{cases},$$

and the family $\{f_2(x), f_4(x), f_6(x), f_8(x), f_{10}(x)\}$ of the solution to the following system

$$\begin{cases} f_2(x) + f_4(x) + f_6(x) + f_8(x) + f_{10}(x) = f_e(x) \\ 4\, f_2(x) + 16\, f_4(x) + 64\, f_6(x) + 256\, f_8(x) + 1024\, f_{10}(x) = f_e(2x) \\ 4^2\, f_2(x) + 16^2\, f_4(x) + 64^2\, f_6(x) + 256^2\, f_8(x) + 1024^2\, f_{10}(x) = f_e(4x) \\ 4^3\, f_2(x) + 16^3\, f_4(x) + 64^3\, f_6(x) + 256^3\, f_8(x) + 1024^3\, f_{10}(x) = f_e(8x) \\ 4^4\, f_2(x) + 16^4\, f_4(x) + 64^4\, f_6(x) + 256^4\, f_8(x) + 1024^4\, f_{10}(x) = f_e(16x) \end{cases}$$

for all $x \in V$, respectively. It can be noted that

$$M := \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 8 & 32 & 128 & 512 \\ 2^2 & 8^2 & 32^2 & 128^2 & 512^2 \\ 2^3 & 8^3 & 32^3 & 128^3 & 512^3 \\ 2^4 & 8^4 & 32^4 & 128^4 & 512^4 \end{vmatrix} \neq 0$$

and

$$M' := \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 16 & 64 & 256 & 1024 \\ 4^2 & 16^2 & 64^2 & 256^2 & 1024^2 \\ 4^3 & 16^3 & 64^3 & 256^3 & 1024^3 \\ 4^4 & 16^4 & 64^4 & 256^4 & 1024^4 \end{vmatrix} \neq 0.$$

So we can use the Cramer's rule to have the unique solutions $f_i : V \to Y$, $i = 1, 2, \cdots, 10$ as the following definition.

Definition 2.2. *For a given mapping $f : V \to Y$, we define the mappings $f_1, f_2, \ldots, f_{10} : V \to Y$ by*

$$f_1(x) := \frac{1}{M} \begin{vmatrix} f_o(x) & 1 & 1 & 1 & 1 \\ f_o(2x) & 8 & 32 & 128 & 512 \\ f_o(4x) & 8^2 & 32^2 & 128^2 & 512^2 \\ f_o(8x) & 8^3 & 32^3 & 128^3 & 512^3 \\ f_o(16x) & 8^4 & 32^4 & 128^4 & 512^4 \end{vmatrix} = \frac{1}{722925 \cdot 16} \big[f_o(16x)$$
$$-680\, f_o(8x) + 91392\, f_o(4x) - 2785280\, f_o(2x) + 16777216\, f_o(x) \big],$$

$$f_2(x) := \frac{1}{M'}\begin{vmatrix} f_e(x) & 1 & 1 & 1 & 1 \\ f_e(2x) & 16 & 64 & 256 & 1024 \\ f_e(4x) & 16^2 & 64^2 & 256^2 & 1024^2 \\ f_e(8x) & 16^3 & 64^3 & 256^3 & 1024^3 \\ f_e(16x) & 16^4 & 64^4 & 256^4 & 1024^4 \end{vmatrix} = \frac{1}{722925 \cdot 256}\big[f_e(16x)$$
$$-1360\, f_e(8x) + 365568\, f_e(4x) - 22282240\, f_e(2x) + 268435456\, f_e(x)\big],$$

$$f_3(x) := \frac{1}{M}\begin{vmatrix} 1 & f_o(x) & 1 & 1 & 1 \\ 2 & f_o(2x) & 32 & 128 & 512 \\ 2^2 & f_o(4x) & 32^2 & 128^2 & 512^2 \\ 2^3 & f_o(8x) & 32^3 & 128^3 & 512^3 \\ 2^4 & f_o(16x) & 32^4 & 128^4 & 512^4 \end{vmatrix} = \frac{-5440}{722925 \cdot 256^2}\big[f_o(16x)$$
$$-674\, f_o(8x) + 87360\, f_o(4x) - 2269184\, f_o(2x) + 4194304\, f_o(x)\big],$$

$$f_4(x) := \frac{1}{M'}\begin{vmatrix} 1 & f_e(x) & 1 & 1 & 1 \\ 4 & f_e(2x) & 64 & 256 & 1024 \\ 4^2 & f_e(4x) & 64^2 & 256^2 & 1024^2 \\ 4^3 & f_e(8x) & 64^3 & 256^3 & 1024^3 \\ 4^4 & f_e(16x) & 64^4 & 256^4 & 1024^4 \end{vmatrix} = \frac{-5440}{722925 \cdot 256^2 \cdot 16}\big[f_e(16x)$$
$$-1348\, f_e(8x) + 349440\, f_e(4x) - 18153472\, f_e(2x) + 67108864\, f_e(x)\big],$$

$$f_5(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & f_o(x) & 1 & 1 \\ 2 & 8 & f_o(2x) & 128 & 512 \\ 2^2 & 8^2 & f_o(4x) & 128^2 & 512^2 \\ 2^3 & 8^3 & f_o(8x) & 128^3 & 512^3 \\ 2^4 & 8^4 & f_o(16x) & 128^4 & 512^4 \end{vmatrix} = \frac{1428}{722925 \cdot 256^2}\big[f_o(16x)$$
$$-650\, f_o(8x) + 71952\, f_o(4x) - 665600\, f_o(2x) + 1048576\, f_o(x)\big],$$

$$f_6(x) := \frac{1}{M'}\begin{vmatrix} 1 & 1 & f_e(x) & 1 & 1 \\ 4 & 16 & f_e(2x) & 256 & 1024 \\ 4^2 & 16^2 & f_e(4x) & 256^2 & 1024^2 \\ 4^3 & 16^3 & f_o(ex) & 256^3 & 1024^3 \\ 4^4 & 16^4 & f_e(16x) & 256^4 & 1024^4 \end{vmatrix} = \frac{1428}{722925 \cdot 256^2 \cdot 16}\big[f_e(16x)$$
$$-1300\, f_e(8x) + 287808\, f_e(4x) - 5324800\, f_e(2x) + 16777216\, f_e(x)\big],$$

$$f_7(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & 1 & f_o(x) & 1 \\ 2 & 8 & 32 & f_o(2x) & 512 \\ 2^2 & 8^2 & 32^2 & f_o(4x) & 512^2 \\ 2^3 & 8^3 & 32^3 & f_o(8x) & 512^3 \\ 2^4 & 8^4 & 32^4 & f_o(16x) & 512^4 \end{vmatrix} = \frac{-85}{722925 \cdot 256^2}\big[f_o(16x)$$
$$-554\, f_o(8x) + 21840\, f_o(4x) - 172544\, f_o(2x) + 262144\, f_o(x)\big],$$

$$f_8(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & 1 & f_e(x) & 1 \\ 4 & 16 & 64 & f_e(2x) & 1024 \\ 4^2 & 16^2 & 64^2 & f_e(4x) & 1024^2 \\ 4^3 & 16^3 & 64^3 & f_e(8x) & 1024^3 \\ 4^4 & 16^4 & 64^4 & f_e(16x) & 1024^4 \end{vmatrix} = \frac{-85}{722925\cdot 256^2\cdot 16}\big[f_e(16x)$$
$$-1108\, f_e(8x) + 87360\, f_e(4x) - 1380352\, f_e(2x) + 4194304\, f_e(x)\big],$$

$$f_9(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & 1 & 1 & f_o(x) \\ 2 & 8 & 32 & 128 & f_o(2x) \\ 2^2 & 8^2 & 32^2 & 128^2 & f_o(4x) \\ 2^3 & 8^3 & 32^3 & 128^3 & f_o(8x) \\ 2^4 & 8^4 & 32^4 & 128^4 & f_o(16x) \end{vmatrix} = \frac{1}{722925\cdot 256^2}\big[f_o(16x)$$
$$-170\, f_o(8x) + 5712\, f_o(4x) - 43520\, f_o(2x) + 65536\, f_o(x)\big],$$

$$f_{10}(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & 1 & 1 & f_e(x) \\ 4 & 16 & 64 & 256 & f_e(2x) \\ 4^2 & 16^2 & 64^2 & 256^2 & f_e(4x) \\ 4^3 & 16^3 & 64^3 & 256^3 & f_e(8x) \\ 4^4 & 16^4 & 64^4 & 256^4 & f_e(16x) \end{vmatrix} = \frac{1}{722925\cdot 256^2\cdot 16}\big[f_e(16x)$$
$$-340\, f_e(8x) + 22848\, f_e(4x) - 348160\, f_e(2x) + 1048576\, f_e(x)\big]$$

for all $x \in V$.

Lemma 2.3. *Let $f : V \to Y$ be an arbitrarily given mapping. Then the equalities hold for all $x, y \in V$:*

$$f(x) = f_o(x) + f_e(x) = \sum_{i=1}^{10} f_i(x),$$
$$D\tilde{f}(x,y) = Df(x,y),$$
$$\Gamma\tilde{f}(x) = f_o(32x) - 682\, f_o(16x) + 92752\, f_o(8x) - 2968064\, f_o(4x),$$
$$+ 22347776\, f_o(2x) - 33554432\, f_o(x),$$
$$\Delta\tilde{f}(x) = \tilde{f}_e(32x) - 1364\,\tilde{f}_e(16x) + 371008\,\tilde{f}_e(8x) - 23744512\,\tilde{f}_e(4x)$$
$$+ 357564416\tilde{f}_e(2x) - 1073741824\tilde{f}_e(x).$$

Together with the lemma, we have

$$\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = \frac{-\Gamma\tilde{f}(x)}{722925\cdot 32}, \tag{2}$$

$$\tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = \frac{-\Delta\tilde{f}(x)}{722925\cdot 1024}, \tag{3}$$

$$\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = \frac{680\,\Gamma\tilde{f}(x)}{722925\cdot 65536}, \tag{4}$$

$$\tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} = \frac{340\,\Delta\tilde{f}(x)}{722925\cdot 1048576}, \tag{5}$$

$$\tilde{f}_5(x) - \frac{\tilde{f}_5(2x)}{32} = \frac{-357\,\Gamma\tilde{f}(x)}{722925\cdot 65536\cdot 8}, \tag{6}$$

$$\tilde{f}_6(x) - \frac{\tilde{f}_6(2x)}{64} = \frac{-357\,\Delta\tilde{f}(x)}{722925\cdot 1048576\cdot 16}, \tag{7}$$

$$\tilde{f}_7(x) - \frac{\tilde{f}_7(2x)}{128} = \frac{85\,\Gamma\tilde{f}(x)}{722925\cdot 65536\cdot 128}, \tag{8}$$

$$\tilde{f}_8(x) - \frac{\tilde{f}_8(2x)}{256} = \frac{85\,\Delta\tilde{f}(x)}{722925\cdot 1048576\cdot 256}, \tag{9}$$

$$\tilde{f}_9(x) - \frac{\tilde{f}_9(2x)}{512} = \frac{-\Gamma\tilde{f}(x)}{722925\cdot 65536\cdot 512}, \tag{10}$$

$$\tilde{f}_{10}(x) - \frac{\tilde{f}_{10}(2x)}{1024} = \frac{-\Delta\tilde{f}(x)}{722925\cdot 1048576\cdot 1024}. \tag{11}$$

Lemma 2.4. *If $f : V \to Y$ is a mapping such that $Df(x,y) = 0$ for all $x, y \in V\backslash\{0\}$, then*

$$Df(x,y) = 0 \quad \text{and} \quad \tilde{f}_i(2x) = 2^i\tilde{f}_i(x)$$

for $x, y \in V$ and each $i = 1, 2, \cdots, 10$.

Proof. Notice that $Df(x,0) = \sum_{i=0}^{11} {}_{11}C_i(-1)^{11-i}f(x) = 0$ for all $x \in V$, and $Df_o(x,y) = Df_e(x,y) = 0$ for all $x, y \in V\backslash\{0\}$. It follows from the latter that

$$\begin{aligned} Df(0,y) &= Df_o(0,y) + Df_e(0,y) \\ &= \sum_{i=0}^{11} {}_{11}C_i(-1)^{11-i}f_o(iy) + \sum_{i=0}^{11} {}_{11}C_i(-1)^{11-i}f_e(iy) \\ &= Df_o(-11y, y) + Df_e(11y, -y) \\ &= 0 \end{aligned}$$

for all $y \in V\backslash\{0\}$. We have shown that f is a general decic mapping, i.e.,

$$Df(x,y) = 0$$

for all $x, y \in V$. From the definitions of Γf and Δf, we easily get $\tilde{f}_i(2x) = 2^i\tilde{f}_i(x)$ for all $x \in V$ and each $i = 1, 2, \cdots, 10$.

3. Stability of a general decic functional equation

In this section, we show the stability result of the general decic functional equation.

Theorem 3.1. *Let $\theta \in \mathbb{R}$ and $p \neq 1,2,3,4,5,6,7,8,9,10$ be real numbers. Suppose that $f : X \to Y$ is a mapping such that*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{12}$$

for all $x, y \in X \backslash \{0\}$. Then there exists a unique mapping $F : X \to Y$ satisfying $DF(x,y) = 0$ for all $x, y \in X$ and

$$\|\tilde{f}(x) - F(x)\| \leq W\theta\|x\|^p \tag{13}$$

for all $x \in X$, where

$$\begin{aligned} W :=& \left(\frac{4096}{|2-2^p|} + \frac{5440}{|8-2^p|} + \frac{1428}{|32-2^p|} + \frac{85}{|128-2^p|} + \frac{1}{|512-2^p|}\right)K' \\ &+\left(\frac{4096}{|4-2^p|} + \frac{5440}{|16-2^p|} + \frac{1428}{|64-2^p|} + \frac{85}{|256-2^p|} + \frac{1}{|1024-2^p|}\right)K, \end{aligned}$$

and

$$\begin{aligned} K' :=& \frac{66 \cdot 20^p + 11 \cdot 16^p + 12^p + 85040 \cdot 10^p + 39688 \cdot 8^p + 12408 \cdot 6^p}{47377612800} \\ &+ \frac{4438016 \cdot 5^p + 2649538 \cdot 4^p + 957440 \cdot 3^p + 337116 \cdot 2^p + 8275456}{47377612800}, \\ K :=& \frac{66 \cdot 20^p + 11 \cdot 16^p + 12^p + 84800 \cdot 10^p + 54208 \cdot 8^p + 19008 \cdot 6^p}{758041804800} \\ &+ \frac{5226496 \cdot 5^p + 6492014 \cdot 4^p + 2883584 \cdot 3^p + 793376 \cdot 2^p + 15343616}{758041804800}. \end{aligned}$$

Moreover, if $p = 0$ then (13) is simplified to the following expression

$$\begin{aligned} \|\tilde{f}(x) - F(x)\| \leq& \left(4096 + \frac{5440}{7} + \frac{1428}{31} + \frac{85}{127} + \frac{1}{511}\right)K'\theta \\ &+ \left(\frac{4096}{3} + \frac{5440}{15} + \frac{1428}{63} + \frac{85}{255} + \frac{1}{1023}\right)K\theta \\ \leq& 2\theta \end{aligned}$$

for all $x \in X$.

Proof. By (12), it is clear that

$$\|Df_o(x,y)\| \leq \frac{1}{2}\left(\|Df(x,y)\| + \|Df(-x,-y)\|\right) \leq \theta(\|x\|^p + \|y\|^p),$$
$$\|Df_e(x,y)\| \leq \frac{1}{2}\left(\|Df(x,y)\| + \|Df(-x,-y)\|\right) \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$. Together with $\Gamma\tilde{f}(0) = \Delta\tilde{f}(0) = 0$, we have

$$
\begin{aligned}
&\|\Gamma\tilde{f}(x)\| \leq \big(12^p + 4^p + 11\cdot 16^p + 11\cdot 4^p + 66\cdot 20^p + 66\cdot 4^p + 440\cdot 2^p \\
&\quad + 2420\cdot 4^p + 2420\cdot 2^p + 12408\cdot 6^p + 12408\cdot 2^p + 39688\cdot 8^p + 39688\cdot 2^p \\
&\quad + 85040\cdot 10^p + 85040\cdot 2^p + 35840 + 197120\cdot 2^p + 197120 + 957440\cdot 3^p \\
&\quad + 957440 + 2647040\cdot 4^p + 2647040 + 4438016\cdot 5^p + 4438016\big)\theta\|x\|^p \\
&\leq 47377612800\, K'\theta\|x\|^p,
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&\|\Delta\tilde{f}(x)\| \leq \big(12^p + 4^p + 11\cdot 16^p + 11\cdot 4^p + 66\cdot 20^p + 66\cdot 4^p + 704\cdot 2^p \\
&\quad + 3872\cdot 4^p + 3872\cdot 2^p + 19008\cdot 6^p + 19008\cdot 2^p + 54208\cdot 8^p + 54208\cdot 2^p \\
&\quad + 84800\cdot 10^p + 84800\cdot 2^p + 114688 + 630784\cdot 2^p + 630784 + 2883584\cdot 3^p \\
&\quad + 2883584 + 6488064\cdot 4^p + 6488064 + 5226496\cdot 5^p + 5226496\big)\theta\|x\|^p \\
&\leq 758041804800\, K\theta\|x\|^p
\end{aligned} \tag{15}
$$

for all $x \in X$. We will show that each mapping $F^{(i)} : X \to Y$, $i = 1, 2, \cdots, 10$, where

$$
F^{(i)}(x) := \begin{cases} \lim_{n\to\infty} \frac{\tilde{f}_i(2^i x)}{2^{in}} & \text{if } p < i, \\ \lim_{n\to\infty} 2^{in}\tilde{f}_i(2^{-n}x) & \text{if } p > i, \end{cases}
$$

can be defined for all $x \in X$, and then $F(x) := \sum_{i=1}^{10} F^{(i)}(x)$ becomes the desired general decic mapping satisfying the inequality (13) by the following steps:

Step 1. Suppose that $p < 1$. It follows from (2) and (14) that

$$
\begin{aligned}
\left\| \frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m}x)}{2^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1}x)}{2^{i+1}} \right) \right\| \\
\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Gamma\tilde{f}(2^i x)}{722925\cdot 32\cdot 2^i} \right\| &\leq \sum_{i=n}^{n+m-1} \frac{4096\, K'\theta\|2^i x\|^p}{2^{i+1}}
\end{aligned}
$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. Together with $p < 1$, it leads us to show that the sequence $\left\{\frac{\tilde{f}_1(2^n x)}{2^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, we can define a mapping $F^{(1)} : X \to Y$ by

$$
F^{(1)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_1(2^n x)}{2^n}
$$

for all $x \in X$. Hence, letting $n = 0$ and passing the limit $m \to \infty$, we get the inequality

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \le \sum_{i=0}^{\infty} \frac{4096\, K' 2^{ip}\theta\|x\|^p}{2^{i+1}} = \frac{4096\, K'\theta\|x\|^p}{2 - 2^p} \tag{16}$$

for all $x \in X$.

Step 2. If $p > 1$, then it follows from (2) and (14) that

$$\begin{aligned}\left\|2^n \tilde{f}_1(2^{-n}x) - 2^{n+m}\tilde{f}_1(2^{-n-m}x)\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(2^i \tilde{f}_1(2^{-i}x) - 2^{i+1}\tilde{f}_1(2^{-i-1}x)\right)\right\| \\ &\le \sum_{i=n}^{n+m-1}\left\|\frac{2^{i+1}\,\Gamma\tilde{f}(2^{-i-1}x)}{722925\cdot 32}\right\| \le \sum_{i=n}^{n+m-1}\frac{4096\cdot 2^i\, K'\theta\|x\|^p}{2^{(i+1)p}}\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Together with $p > 1$, it leads us to prove that the sequence $\{2^n \tilde{f}_1(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, we can define a mapping $F^{(1)} : X \to Y$ by

$$F^{(1)}(x) := \lim_{n\to\infty} 2^n \tilde{f}_1(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$, we get the inequality

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \le \sum_{i=0}^{\infty} \frac{4096\cdot 2^i K'\,\theta\|x\|^p}{2^{(i+1)p}} = \frac{4096\, K'\theta\|x\|^p}{2^p - 2} \tag{17}$$

for all $x \in X$.

Step 3. Suppose that $p < 2$. It follows from (3) and (15) that

$$\begin{aligned}\left\|\frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_2(2^i x)}{4^i} - \frac{\tilde{f}_2(2^{i+1}x)}{4^{i+1}}\right)\right\| \\ &\le \sum_{i=n}^{n+m-1}\left\|\frac{\Delta\tilde{f}(2^i x)}{722925\cdot 1024\cdot 4^i}\right\| \le \sum_{i=n}^{n+m-1}\frac{4096\, K 2^{ip}\theta\|x\|^p}{4^{i+1}}\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 2$, we can define a mapping $F^{(2)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_2(2^n x)/4^n\}$, i.e.,

$$F^{(2)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_2(2^n x)}{4^n},$$

which satisfies the inequality

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \le \sum_{i=0}^{\infty} \frac{4096\, K 2^{ip}\theta\|x\|^p}{4^{i+1}} = \frac{4096\, K\theta\|x\|^p}{4 - 2^p} \tag{18}$$

for all $x \in X$.

Step 4. If $p > 2$, then it follows from (3) and (15) that

$$\left\|4^n \tilde{f}_2(2^{-n}x) - 4^{n+m}\tilde{f}_2(2^{-n-m}x)\right\| = \left\|\sum_{i=n}^{n+m-1}\left(4^i \tilde{f}_2(2^{-i}x) - 4^{i+1}\tilde{f}_2(2^{-i-1}x)\right)\right\|$$
$$\leq \sum_{i=n}^{n+m-1}\left\|\frac{4^{i+1}\Delta\tilde{f}_2(2^{-i-1}x)}{722925\cdot 1024}\right\| \leq \sum_{i=n}^{n+m-1}\frac{4096\cdot 4^i\, K\theta\|x\|^p}{2^{(i+1)p}}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. Together with $p > 2$, we can define a mapping $F^{(2)} : X \to Y$ as the limit of the Cauchy sequence $\{4^n \tilde{f}_2(2^{-n}x)\}$, i.e.,

$$F^{(2)}(x) := \lim_{n\to\infty} 4^n \tilde{f}_2(2^{-n}x),$$

which satisfies the inequality

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \sum_{i=0}^{\infty}\frac{4096\cdot 4^i\, K\theta\|x\|^p}{2^{(i+1)p}} = \frac{4096\, K\theta\|x\|^p}{2^p - 4} \tag{19}$$

for all $x \in X$.

Step 5. Suppose that $p < 3$. It follows from (4) and (14) that

$$\left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m}x)}{8^{n+m}}\right\| = \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_3(2^i x)}{8^i} - \frac{\tilde{f}_3(2^{i+1}x)}{8^{i+1}}\right)\right\|$$
$$\leq \sum_{i=n}^{n+m-1}\left\|\frac{680\,\Gamma\tilde{f}(2^i x)}{722925\cdot 65536\cdot 8^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{5440\, K' 2^{ip}\theta\|x\|^p}{8^{i+1}}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. Since $p < 3$, we can define a mapping $F^{(3)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_3(2^n x)/8^n\}$, i.e.,

$$F^{(3)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_3(2^n x)}{8^n},$$

which satisfies the inequality

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \leq \sum_{i=0}^{\infty}\frac{5440\, K' 2^{ip}\theta\|x\|^p}{8^{i+1}} = \frac{5440\, K'\theta\|x\|^p}{8 - 2^p} \tag{20}$$

for all $x \in X$.

Step 6. If $p > 3$, then it follows from (4) and (14) that

$$\left\|8^n \tilde{f}_3(2^{-n}x) - 8^{n+m}\tilde{f}_3(2^{-n-m}x)\right\| = \left\|\sum_{i=n}^{n+m-1}\left(8^i \tilde{f}_3(2^{-i}x) - 8^{i+1}\tilde{f}_3(2^{-i-1}x)\right)\right\|$$
$$\leq \sum_{i=n}^{n+m-1}\left\|\frac{680\cdot 8^{i+1}\,\Gamma\tilde{f}(2^{-i-1}x)}{722925\cdot 65536}\right\| \leq \sum_{i=n}^{n+m-1}\frac{5440\cdot 8^i\, K'\theta\|x\|^p}{2^{(i+1)p}}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Together with $p > 3$, we can define a mapping $F^{(3)} : X \to Y$ as the limit of the Cauchy sequence $\{8^n \tilde{f}_3(2^{-n}x)\}$, i.e.,

$$F^{(3)}(x) := \lim_{n\to\infty} 8^n \tilde{f}_3(2^{-n}x),$$

which satisfies the inequality

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \le \sum_{i=0}^{\infty} \frac{5440 \cdot 8^i\, K'\theta\|x\|^p}{2^{(i+1)p}} = \frac{5440\, K'\theta\|x\|^p}{2^p - 8} \tag{21}$$

for all $x \in X$.

Step 7. Suppose that $p < 4$. It follows from (5) and (15) that

$$\begin{aligned}\left\|\frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m}x)}{16^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_4(2^i x)}{16^i} - \frac{\tilde{f}_4(2^{i+1}x)}{16^{i+1}}\right)\right\| \\ &\le \sum_{i=n}^{n+m-1}\left\|\frac{340\,\Delta\tilde{f}(2^i x)}{722925 \cdot 1048576 \cdot 16^i}\right\| \le \sum_{i=n}^{n+m-1} \frac{5440\, K 2^{ip}\theta\|x\|^p}{16^{i+1}}\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 4$, we can define a mapping $F^{(4)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_4(2^n x)/16^n\}$, i.e.,

$$F^{(4)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_4(2^n x)}{16^n},$$

which satisfies the inequality

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \le \sum_{i=0}^{\infty} \frac{5440\, K 2^{ip}\theta\|x\|^p}{16^{i+1}} = \frac{5440\, K\theta\|x\|^p}{16 - 2^p} \tag{22}$$

for all $x \in X$.

Step 8. If $p > 4$, then it follows from (5) and (15) that

$$\begin{aligned}&\left\|16^n\, \tilde{f}_4(2^{-n}x) - 16^{n+m}\, \tilde{f}_4(2^{-n-m}x)\right\| \\ &= \left\|\sum_{i=n}^{n+m-1}\left(16^i\, \tilde{f}_4(2^{-i}x) - 16^{i+1}\, \tilde{f}_4(2^{-i-1}x)\right)\right\| \\ &\le \sum_{i=n}^{n+m-1}\left\|\frac{340 \cdot 16^{i+1}\Delta\tilde{f}(2^{-i-1}x)}{722925 \cdot 1048576}\right\| \le \sum_{i=n}^{n+m-1} \frac{5440 \cdot 16^i\, K\theta\|x\|^p}{2^{(i+1)p}}\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Together with $p > 4$, we can define a mapping $F^{(4)} : X \to Y$ as the limit of the Cauchy sequence $\{16^n \tilde{f}_4(2^{-n}x)\}$, i.e.,

$$F^{(4)}(x) := \lim_{n\to\infty} 16^n \tilde{f}_4(2^{-n}x),$$

which satisfies the inequality

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \leq \sum_{i=0}^{\infty} \frac{5440 \cdot 16^i\, K\theta\|x\|^p}{2^{(i+1)p}} = \frac{5440\, K\theta\|x\|^p}{2^p - 16} \tag{23}$$

for all $x \in X$.

Step 9. Suppose that $p < 5$. It follows from (6) and (14) that

$$\begin{aligned}\left\|\frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m}x)}{32^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_5(2^i x)}{32^i} - \frac{\tilde{f}_5(2^{i+1}x)}{32^{i+1}}\right)\right\| \\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{357\,\Gamma\tilde{f}(2^i x)}{722925 \cdot 65536 \cdot 8 \cdot 32^i}\right\| \leq \sum_{i=n}^{n+m-1} \frac{1428\, K' 2^{ip}\theta\|x\|^p}{32^{i+1}}\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 5$, we can define a mapping $F^{(5)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_5(2^n x)/32^n\}$, i.e.,

$$F^{(5)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_5(2^n x)}{32^n},$$

which satisfies the inequality

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \leq \sum_{i=0}^{\infty} \frac{1428\, K' 2^{ip}\theta\|x\|^p}{32^{i+1}} = \frac{1428\, K'\theta\|x\|^p}{32 - 2^p} \tag{24}$$

for all $x \in X$.

Step 10. If $p > 5$, then it follows from (6) and (14) that

$$\begin{aligned}&\left\|32^n\,\tilde{f}_5(2^{-n}x) - 32^{n+m}\,\tilde{f}_5(2^{-n-m}x)\right\| \\ &\quad = \left\|\sum_{i=n}^{n+m-1}\left(32^i\,\tilde{f}_5(2^{-i}x) - 32^{i+1}\,\tilde{f}_5(2^{-i-1}x)\right)\right\| \\ &\quad \leq \sum_{i=n}^{n+m-1}\left\|\frac{357 \cdot 32^{i+1}\,\Gamma\tilde{f}(2^{-i-1}x)}{722925 \cdot 65536 \cdot 8}\right\| \leq \sum_{i=n}^{n+m-1} \frac{1428 \cdot 32^i K'\theta\|x\|^p}{2^{(i+1)p}}\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Together with $p > 5$, we can define a mapping $F^{(5)} : X \to Y$ as the limit of the Cauchy sequence $\{32^n\,\tilde{f}_5(2^{-n}x)\}$, i.e.,

$$F^{(5)}(x) := \lim_{n\to\infty} 32^n\,\tilde{f}_5(2^{-n}x),$$

which satisfies the inequality

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \leq \sum_{i=0}^{\infty} \frac{1428 \cdot 32^i K'\theta\|x\|^p}{2^{(i+1)p}} = \frac{1428\, K'\theta\|x\|^p}{2^p - 32} \tag{25}$$

for all $x \in X$.

Step 11. Suppose that $p < 6$. It follows from (7) and (15) that

$$\left\|\frac{\tilde{f}_6(2^n x)}{64^n} - \frac{\tilde{f}_6(2^{n+m} x)}{64^{n+m}}\right\| = \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_6(2^i x)}{64^i} - \frac{\tilde{f}_6(2^{i+1} x)}{64^{i+1}}\right)\right\|$$
$$= \sum_{i=n}^{n+m-1}\left\|\frac{357\,\Delta\tilde{f}(2^i x)}{722925\cdot 1048576\cdot 16\cdot 64^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{1428\,K2^{ip}\theta\|x\|^p}{64^{i+1}}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. Since $p < 6$, we can define a mapping $F^{(6)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_6(2^n x)/64^n\}$, i.e.,

$$F^{(6)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_6(2^n x)}{64^n},$$

which satisfies the inequality

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \leq \sum_{i=0}^{\infty}\frac{1428\,K2^{ip}\theta\|x\|^p}{64^{i+1}} = \frac{1428\,K\theta\|x\|^p}{64-2^p} \tag{26}$$

for all $x \in X$.

Step 12. If $p > 6$, then it follows from (7) and (15) that

$$\left\|64^n\,\tilde{f}_6(2^{-n}x) - 64^{n+m}\,\tilde{f}_6(2^{-n-m}x)\right\|$$
$$= \left\|\sum_{i=n}^{n+m-1}\left(64^i\,\tilde{f}_6(2^{-i}x) - 64^{i+1}\,\tilde{f}_6(2^{-i-1}x)\right)\right\|$$
$$\leq \sum_{i=n}^{n+m-1}\left\|\frac{357\cdot 64^{i+1}\,\Delta\tilde{f}(2^{-i-1}x)}{722925\cdot 1048576\cdot 16}\right\| \leq \sum_{i=n}^{n+m-1}\frac{1428\cdot 64^i\,K\theta\|x\|^p}{2^{(i+1)p}}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. Together wtih $p > 6$, we can define a mapping $F^{(6)} : X \to Y$ as the limit of the Cauchy sequence $\{64^n\tilde{f}_6(2^{-n}x)\}$, i.e.,

$$F^{(6)}(x) := \lim_{n\to\infty} 64^n\,\tilde{f}_6(2^{-n}x),$$

which satisfies the inequality

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \leq \sum_{i=0}^{\infty}\frac{1428\cdot 64^i\,K\theta\|x\|^p}{2^{(i+1)p}} = \frac{1428\,K\theta\|x\|^p}{2^p-64} \tag{27}$$

for all $x \in X$.

Step 13. Suppose that $p < 7$. It follows from (8) and (14) that

$$\left\| \frac{\tilde{f}_7(2^n x)}{128^n} - \frac{\tilde{f}_7(2^{n+m} x)}{128^{n+m}} \right\| = \left\| \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_7(2^i x)}{128^i} - \frac{\tilde{f}_7(2^{i+1} x)}{128^{i+1}} \right) \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \left\| \frac{85\,\Gamma \tilde{f}(2^i x)}{722925 \cdot 65536 \cdot 128^{i+1}} \right\| \leq \sum_{i=n}^{n+m-1} \frac{85\, K' 2^{ip} \theta \|x\|^p}{128^{i+1}}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 7$, we can define a mapping $F^{(7)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_7(2^n x)/128^n\}$, i.e.,

$$F^{(7)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_7(2^n x)}{128^n},$$

which satisfies the inequality

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \sum_{i=0}^{\infty} \frac{85\, K' 2^{ip} \theta \|x\|^p}{128^{i+1}} = \frac{85\, K' \theta \|x\|^p}{128 - 2^p} \tag{28}$$

for all $x \in X$.

Step 14. If $p > 7$, then it follows from (8) and (14) that

$$\left\| 128^n\, \tilde{f}_7(2^{-n} x) - 128^{n+m}\, \tilde{f}_7(2^{-n-m} x) \right\|$$

$$= \left\| \sum_{i=n}^{n+m-1} \Big(128^i\, \tilde{f}_7(2^{-i} x) - 128^{i+1}\, \tilde{f}_7(2^{-i-1} x) \Big) \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \left\| \frac{85 \cdot 128^{i+1}\, \Gamma \tilde{f}(2^{-i-1} x)}{722925 \cdot 65536 \cdot 128} \right\| \leq \sum_{i=n}^{n+m-1} \frac{85 \cdot 128^i K' \theta \|x\|^p}{2^{(i+1)p}}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Together with $p > 7$, we can define a mapping $F^{(7)} : X \to Y$ as the limit of the Cauchy sequence $\{128^n \tilde{f}_7(2^{-n} x)\}$, i.e.,

$$F^{(7)}(x) := \lim_{n\to\infty} 128^n \tilde{f}_7(2^{-n} x),$$

which satisfies the inequality

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \sum_{i=0}^{\infty} \frac{85 \cdot 128^i K' \theta \|x\|^p}{2^{(i+1)p}} = \frac{85\, K' \theta \|x\|^p}{2^p - 128} \tag{29}$$

for all $x \in X$.

Step 15. Suppose that $p < 8$. It follows from (9) and (15) that

$$\begin{aligned}\left\|\frac{\tilde{f}_8(2^n x)}{256^n} - \frac{\tilde{f}_8(2^{n+m} x)}{256^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_8(2^i x)}{256^i} - \frac{\tilde{f}_8(2^{i+1} x)}{256^{i+1}}\right)\right\| \\ &\le \sum_{i=n}^{n+m-1}\left\|\frac{85\,\Delta\tilde{f}(2^i x)}{722925\cdot 1048576\cdot 256^{i+1}}\right\| \le \sum_{i=n}^{n+m-1}\frac{85\,K2^{ip}\theta\|x\|^p}{256^{i+1}}\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. Since $p < 8$, we can define a mapping $F^{(8)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_8(2^n x)/256^n\}$, i.e.,

$$F^{(8)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_8(2^n x)}{256^n},$$

which satisfies the inequality

$$\|\tilde{f}_8(x) - F^{(8)}(x)\| \le \sum_{i=0}^{\infty}\frac{85\,K2^{ip}\theta\|x\|^p}{256^{i+1}} = \frac{85\,K\theta\|x\|^p}{256 - 2^p} \tag{30}$$

for all $x \in X$.

Step 16. If $p > 8$, then it follows from (9) and (15) that

$$\begin{aligned}&\left\|256^n\,\tilde{f}_8(2^{-n}x) - 256^{n+m}\,\tilde{f}_8(2^{-n-m}x)\right\| \\ &= \left\|\sum_{i=n}^{n+m-1}\Big(256^i\,\tilde{f}_8(2^{-i}x) - 256^{i+1}\,\tilde{f}_8(2^{-i-1}x)\Big)\right\| \\ &\le \sum_{i=n}^{n+m-1}\left\|\frac{85\cdot 256^i\,\Delta\tilde{f}(2^{-i-1}x)}{722925\cdot 1048576}\right\| \le \sum_{i=n}^{n+m-1}\frac{85\cdot 256^i K\theta\|x\|^p}{2^{(i+1)p}}\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. Together with $p > 8$, we can define a mapping $F^{(8)} : X \to Y$ as the limit of the Cauchy sequence $\{256^n\tilde{f}_8(2^{-n}x)\}$, i.e.,

$$F^{(8)}(x) := \lim_{n\to\infty} 256^n\,\tilde{f}_8(2^{-n}x),$$

which satisfies the inequality

$$\|\tilde{f}_8(x) - F^{(8)}(x)\| \le \sum_{i=0}^{\infty}\frac{85\cdot 256^i K\theta\|x\|^p}{2^{(i+1)p}} = \frac{85\,K\theta\|x\|^p}{2^p - 256} \tag{31}$$

for all $x \in X$.

Step 17. Suppose that $p < 9$. It follows from (10) and (14) that

$$\begin{aligned} \left\| \frac{\tilde{f}_9(2^n x)}{512^n} - \frac{\tilde{f}_9(2^{n+m} x)}{512^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_9(2^i x)}{512^i} - \frac{\tilde{f}_9(2^{i+1} x)}{512^{i+1}} \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Gamma \tilde{f}(2^i x)}{722925 \cdot 65536 \cdot 512^{i+1}} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K' 2^{ip}\, \theta \|x\|^p}{512^{i+1}} \end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 9$, we can define a mapping $F^{(9)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_9(2^n x)/512^n\}$, i.e.,

$$F^{(9)}(x) := \lim_{n \to \infty} \frac{\tilde{f}_9(2^n x)}{512^n},$$

which satisfies the inequality

$$\|\tilde{f}_9(x) - F^{(9)}(x)\| \leq \sum_{i=0}^{\infty} \frac{K' 2^{ip}\, \theta \|x\|^p}{512^{i+1}} = \frac{K'\, \theta \|x\|^p}{512 - 2^p} \tag{32}$$

for all $x \in X$.

Step 18. If $p > 9$, then it follows from (10) and (14) that

$$\begin{aligned} &\left\| 512^n\, \tilde{f}_9(2^{-n} x) - 512^{n+m}\, \tilde{f}_9(2^{-n-m} x) \right\| \\ &= \left\| \sum_{i=n}^{n+m-1} \left(512^i\, \tilde{f}_9(2^{-i} x) - 512^{i+1}\, \tilde{f}_9(2^{-i-1} x) \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{512^i\, \Gamma \tilde{f}(2^{-i-1} x)}{722925 \cdot 65536} \right\| \leq \sum_{i=n}^{n+m-1} \frac{512^i\, K'\, \theta \|x\|^p}{2^{(i+1)p}} \end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. From $p > 9$, we can define a mapping $F^{(9)} : X \to Y$ as the limit of the Cauchy sequence $\{512^n \tilde{f}_9(2^{-n} x)\}$, i.e.,

$$F^{(9)}(x) := \lim_{n \to \infty} 512^n\, \tilde{f}_9(2^{-n} x),$$

which satisfies the inequality

$$\|\tilde{f}_9(x) - F^{(9)}(x)\| \leq \sum_{i=0}^{\infty} \frac{512^i\, K'\, \theta \|x\|^p}{2^{(i+1)p}} = \frac{K'\, \theta \|x\|^p}{2^p - 512} \tag{33}$$

for all $x \in X$.

Step 19. Suppose that $p < 10$. It follows from (11) and (15) that

$$\left\| \frac{\tilde{f}_8(2^n x)}{1024^n} - \frac{\tilde{f}_8(2^{n+m} x)}{1024^{n+m}} \right\| = \left\| \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_8(2^i x)}{1024^i} - \frac{\tilde{f}_8(2^{i+1} x)}{1024^{i+1}} \right) \right\|$$
$$\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Delta \tilde{f}(2^i x)}{722925 \cdot 1048576 \cdot 1024^{i+1}} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K\, 2^{ip} \theta \|x\|^p}{1024^{i+1}}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 10$, we can define a mapping $F^{(10)} : X \to Y$ as the limit of the Cauchy sequence $\{\tilde{f}_{10}(2^n x)/1024^n\}$, i.e.,

$$F^{(10)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_{10}(2^n x)}{1024^n},$$

which satisfies the inequality

$$\|\tilde{f}_{10}(x) - F^{(10)}(x)\| \leq \sum_{i=0}^{\infty} \frac{K\, 2^{ip} \theta \|x\|^p}{1024^{i+1}} = \frac{K\theta\|x\|^p}{1024 - 2^p} \tag{34}$$

for all $x \in X$.

Step 20. If $p > 10$, then it follows from (11) and (15) that

$$\left\| 1024^n\, \tilde{f}_{10}(2^{-n} x) - 1024^{n+m}\, \tilde{f}_{10}(2^{-n-m} x) \right\|$$
$$= \left\| \sum_{i=n}^{n+m-1} \left(1024^i\, \tilde{f}_{10}(2^{-i} x) - 1024^{i+1}\, \tilde{f}_{10}(2^{-i-1} x) \right) \right\|$$
$$\leq \sum_{i=n}^{n+m-1} \left\| \frac{1024^i\, \Delta \tilde{f}(2^{-i-1} x)}{722925 \cdot 1048576} \right\| \leq \sum_{i=n}^{n+m-1} \frac{1024^i K \theta \|x\|^p}{2^{(i+1)p}}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Together with $p > 10$, we can define a mapping $F^{(10)} : X \to Y$ as the limit of the Cauchy sequence $\{1024^n \tilde{f}_2(2^{-n} x)\}$, i.e.,

$$F^{(10)}(x) := \lim_{n\to\infty} 1024^n \tilde{f}_{10}(2^{-n} x),$$

which satisfies the inequality

$$\|\tilde{f}_{10}(x) - F^{(10)}(x)\| \leq \sum_{i=0}^{\infty} \frac{1024^i K \theta \|x\|^p}{2^{(i+1)p}} = \frac{K\theta\|x\|^p}{2^p - 1024} \tag{35}$$

for all $x \in X$.

Now, we have defined mappings $F^{(i)} : X \to Y$, $i = 1, 2, \cdots, 10$. In special case $p < 1$, it allows us to have that;

$$\begin{aligned}
\|DF^{(1)}(x,y)\| &= \lim_{n\to\infty} \left\| \frac{D\tilde{f}_1(2^n x, 2^n y)}{2^n} \right\| \\
&= \lim_{n\to\infty} \left\| \frac{D\tilde{f}_o(2^{n+4}x, 2^{n+4}y)}{11566800 \cdot 2^n} - \frac{680\, D\tilde{f}_o(2^{n+3}x, 2^{n+3}y)}{11566800 \cdot 2^n} \right. \\
&\quad + \frac{91392\, D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{11566800 \cdot 2^n} - \frac{2785280\, D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{11566800 \cdot 2^n} \\
&\quad \left. + \frac{16777216\, D\tilde{f}_o(2^n x, 2^n y)}{11566800 \cdot 2^n} \right\| \\
&\le \lim_{n\to\infty} \frac{(2^{(n+4)p} + 680 \cdot 2^{(n+3)p})\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} \\
&\quad + \lim_{n\to\infty} \frac{(91392 \cdot 2^{(n+2)p} + 2785280 \cdot 2^{(n+1)p})\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} \\
&\quad + \lim_{n\to\infty} \frac{16777216 \cdot 2^{np}\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} \\
&= 0
\end{aligned}$$

for all $x, y \in X\backslash\{0\}$. Similarly, in all cases $p \neq 1, 2, \cdots, 10$, $DF^{(i)}(x,y) = 0$ for all $x, y \in X\backslash\{0\}$ and $i = 1, 2, \cdots, 10$. By putting

$$F(x) := \sum_{i=1}^{10} F^{(i)}(x),$$

we have $DF(x,y) := \sum_{i=1}^{10} DF^{(i)}(x,y) = 0$ for all $x, y \in X\backslash\{0\}$. It follows that

$$DF(x,y) = 0$$

for all $x, y \in X$ by Lemma 2.4. Additionally, from (16)$\sim$ (35), F satisfies the inequality (13). Finally, to prove the uniqueness of F, let $G : X \to Y$ be another mapping satisfying $DG(x,y) = 0$ for all $x, y \in X$ and the inequality (13) for all $x \in X$. It is clear that

$$\|\tilde{f}_o(x) - G_o(x)\| \le W\theta\|x\|^p, \quad \|\tilde{f}_e(x) - G_e(x)\| \le W\theta\|x\|^p$$

for all $x \in X$. Since $G(0) = 0$, by Lemma 2.3 and Lemma 2.4, we have that

$$G(x) = \sum_{i=1}^{10} G_i(x) \quad \text{and} \quad G_i(2x) = 2^i G_i(x)$$

for all $x \in X$ and each $i \in \{1, 2, \ldots, 10\}$. In case $2 < p < 3$, we get the inequalities that;

$$\begin{aligned}
&\left\| 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - G_2(x) \right\| = \left\| 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - 4^n G_2\left(\frac{x}{2^n}\right) \right\| \\
&\quad = \frac{4^n}{722925 \cdot 256} \left\| (\tilde{f}_e - G_e)\left(\frac{16x}{2^n}\right) - 1360(\tilde{f}_e - G_e)\left(\frac{8x}{2^n}\right) \right. \\
&\qquad + 365568(\tilde{f}_e - G_e)\left(\frac{4x}{2^n}\right) - 22282240(\tilde{f}_e - G_e)\left(\frac{2x}{2^n}\right) \\
&\qquad \left. + 268435456(\tilde{f}_e - G_e)\left(\frac{x}{2^n}\right) \right\| \\
&\quad \leq \left(\frac{16^p + 1360 \cdot 8^p + 365568 \cdot 4^p + 22282240 \cdot 2^p + 268435456}{722925 \cdot 256} \right) \frac{4^n}{2^{np}} W\theta \|x\|^p
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \frac{\tilde{f}_3(2^n x)}{8^n} - G_3(x) \right\| = \left\| \frac{\tilde{f}_3(2^n x)}{8^n} - \frac{G_3(2^n x)}{8^n} \right\| \\
&= \frac{5440}{722925 \cdot 65536} \left\| \frac{(\tilde{f}_o - G_o)(2^{n+4}x) - 674((\tilde{f}_o - G_o)(2^{n+3}x))}{8^n} \right. \\
&\quad + \frac{87360((\tilde{f}_o - G_o)(2^{n+2}x)) - 2269184((\tilde{f}_o - G_o)(2^{n+1}x))}{8^n} \\
&\quad \left. + \frac{4194304((\tilde{f}_o - G_o)(2^n x))}{8^n} \right\| \\
&\leq \left(\frac{2^{4p} + 674 \cdot 2^{3p} + 87360 \cdot 2^{2p} + 2269184 \cdot 2^p + 4194304}{722925 \cdot 65536} \right) \frac{5440 \cdot 2^{np} W\theta \|x\|^p}{8^n}
\end{aligned}$$

for all $x \in X$ and all non-negative integers n. Taking the limit in the above inequalities as $n \to \infty$, since $2 < p < 3$, we obtain the equality

$$\begin{aligned}
G_2(x) &= \lim_{n \to \infty} 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) = F^{(2)}(x) \\
G_3(x) &= \lim_{n \to \infty} \frac{\tilde{f}_3(2^n x)}{8^n} = F^{(3)}(x)
\end{aligned}$$

for all $x \in X$. Also, in all cases $p \neq 1, 2, \cdots, 10$, we easily show that $G_i(x) = F^{(i)}(x)$ for all $x \in X$ and each $i = 1, 2, \cdots, 10$, by the similar method. It leads us that

$$F(x) = G(x)$$

for all $x \in X$ and we have shown the uniqueness of F.

4. Hyperstability of a general decic functional equation

In this section, we show the hyperstability result in special case $p < 0$ of the previous theorem.

Theorem 4.1. *Let $p < 0$ and θ be real constants. If $f : X \to Y$ satisfies*

$$\|Df(x,y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$, then $Df(x,y) = 0$ for all $x, y \in X$

Proof. Let $F : X \to Y$ be the general decic mapping satisfying (13) in Theorem 3.1. From the equations

$$D\tilde{f}(x,y) - DF(x,y) = \sum_{i=0}^{11} {}_{11}C_i(-1)^{11-i}\left(\tilde{f}(x+iy) - F(x+iy)\right)$$

and $DF((1-n)x, nx) = 0$ for all $x \in X\backslash\{0\}$, we have the equality

$$\begin{aligned} D\tilde{f}\,((1-n)x, nx) + 11\tilde{f}(x) - 11F(x) &= \tilde{f}((1-n)x) - F((1-n)x) \\ &+ \sum_{i=2}^{11} {}_{11}C_i(-1)^{11-i}\left(\tilde{f}((1-n)x + nix) - F((1-n)x+nix)\right) \end{aligned}$$

for all $x \in X\backslash\{0\}$. Therefore, we have

$$\begin{aligned} &11\|\tilde{f}(x) - F(x)\| \\ &\le \lim_{n\to\infty} \|D\tilde{f}((1-n)x, nx)\| + \lim_{n\to\infty} \|\tilde{f}((1-n)x) - F((1-n)x)\| \\ &\quad + \sum_{i=2}^{11} \lim_{n\to\infty} \|{}_{11}C_i(\tilde{f}(((i-1)n+1)x) - F(((i-1)n+1)x)\| \\ &\le \lim_{n\to\infty} ((n-1)^p + n^p) + M(n-1)^p + \sum_{i=2}^{11} {}_{11}C_i((i-1)n+1)^p\theta\|x\|^p \\ &= 0 \end{aligned}$$

for all $x \in X\backslash\{0\}$. Hence $Df(x,y) = D\tilde{f}x,y) = DF(x,y) = 0$ for all $x, y \in X\backslash\{0\}$. By Lemma 2.4, we conclude that $Df(x,y) = 0$ for all $x, y \in X$. □

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