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A RECURRENCE RELATION ASSOCIATED WITH UNIT-PRIMITIVE MATRICES

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Abstract. In this paper we obtained several properties that the characteristic polynomial of the unit-primitive matrix satisfies. In addition, using these properties we have shown that the recurrence relation given as in the formula (1) is true. In fact, Xin and Zhong ([4]) showed it earlier. However, we provide a simpler method here.

1. Introduction

The unit-primitive matrix comes naturally when computing discrete volumes of certain graph polytopes. Its related terms and results will be explained below. For a given positive integer m, let $B(m) = (b_{ij})$ $(1 \le i, j \le m)$ be the square matrix of size $m \times m$ satisfying

$$b_{ij} = \begin{cases} 1 & \text{if } i+j \le m+1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We call this type of an upper triangular matrix a **unit-primitive matrix** of size m. For example, a unit-primitive matrix of size 5 and its inverse matrix are as follows.

$$B(5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B(5)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

If M is a square matrix, we denote the sum of all entries of M by $s(M)$. So, $s(M) = u^t M u$ for the column vector u all of whose entries are 1. We define a

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$n \backslash m$	0	1	2	3	4	5	6	7	8	9	$\cdots$
0	1	1	1	1	1	1	1	1	1	1	$\cdots$
1	1	2	3	4	5	6	7	8	9	10	$\cdots$
2	1	3	6	10	15	21	28	36	45	55	$\cdots$
3	1	5	14	30	55	91	140	204	285	385	$\cdots$
4	1	8	31	85	190	371	658	1086	1695	2530	$\cdots$
5	1	13	70	246	671	1547	3164	5916	10317	17017	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

TABLE 1. Table of $b(n,m)$

bi-variate sequence $b(n,m)$ $(n,m \geq 0)$, as follows.

$$b(n,m) = \begin{cases} 1, & n\text{=0 or } m\text{=0}, \\ s\big(B(m+1)^{n-1}\big), & n \text{ and } m \text{ are positive.} \end{cases}$$

Our main concern is that the following recurrence relation holds for the sequence $(b(n,m))_{n,m\geq 0}$.

$$b(n,m) = b(n,m-1) + \sum_{k\geq 0} b(2k,m-1)b(n-1-2k,m). \tag{1}$$

This number appeared in chemistry as the number of Kekulé structures of the benzenoid hydrocarbons. (See [1], [4] for details.) It is also listed with an id A050446 in [3]. Let $F_m(x) = \sum_{n\geq 0} b(n,m)x^n$, $G_n(x) = \sum_{m\geq 0} b(n,m)x^m$, $Q_m(x)$ =det$\,(I - xB(m))$ and $R_m(x) = \det \begin{pmatrix} 0 & u^t \\ -u & I - xB(m) \end{pmatrix}$, where $u = (1,1,\cdots,1)^t \in \mathbb{R}^m$.

The sequence $(G_n(y))_{n\geq 0}$ of the Ehrhart series is well analyzed in detail by Xin and Zhong ([4]). We want to describe our main results in a slightly different way, however. Now, let M^* be an adjugate matrix of the square matrix M.

Lemma 1.1. $s(M^*) = \det \begin{pmatrix} 0 & u^t \\ -u & M \end{pmatrix}$ *for all matrix M of size $n \times n$.*

Proof. Let m_i be the ith column of the matrix M, and

$$M_i = \det(m_1, \cdots, m_{i-1}, u, m_{i+1}, \cdots, m_n).$$

Then

$$\begin{aligned}
s(M^*) =& u^t M^* u = u^t[M_1, M_2, \cdots, M_n]^t \\
=& \det(u, m_2, m_3, \cdots, m_n) + \det(m_1, u, m_3, \cdots, m_n) \\
& + \cdots + \det(m_1, m_2, m_3, \cdots, u) \\
=& \det(u, m_2, m_3, \cdots, m_n) - \det(u, m_1, m_3, \cdots, m_n) \\
& + \cdots + (-1)^{n+1} \det(u, m_1, m_2, \cdots, m_{n-1}) \\
=& \det \begin{pmatrix} 0 & u^t \\ -u & M \end{pmatrix}.
\end{aligned}$$

□

Theorem 1.2. *Let* $E_m(x) = \sum_{n\geq 0} b(n+1, m)x^n$. *Then*

$$E_m(x) = \frac{\det \begin{pmatrix} 0 & u^t \\ -u & I - xB(m+1) \end{pmatrix}}{\det(I_{m+1} - xB(m+1))} = \frac{R_{m+1}(x)}{Q_{m+1}(x)}.$$

Proof.

$$\begin{aligned}
E_m(x) &= \sum_{n\geq 0} b(n+1, m)x^n = \sum_{n\geq 0} s(B(m+1)^n)x^n \\
&= s\left(\sum_{n\geq 0} (xB(m+1))^n\right) = s\left[I - xB(m+1)^{-1}\right] \\
&= s\left(\frac{(I - xB(m+1))^*}{\det(I - xB(m+1))}\right) = \frac{1}{Q_{m+1}(x)} s((I - xB(m+1))^*) \\
&= \frac{1}{Q_{m+1}(x)} \left[u^t(I - xB(m+1))^* u\right] \\
&= \frac{1}{Q_{m+1}(x)} \det \begin{pmatrix} 0 & u^t \\ -u & I - xB(m+1) \end{pmatrix} = \frac{R_{m+1}(x)}{Q_{m+1}(x)}.
\end{aligned}$$

□

2. Properties of $Q_m(x)$

In this section we list the properties of $Q_m(x)$ and prove them.

Theorem 2.1. $Q_m(x) = \det(I - xB(m))$ *satisfies the following properties.*

(1) $Q_m(x) = -xQ_{m-1}(-x) + Q_{m-2}(x) \quad (m \geq 2),$ *and*
$Q_0(x) = 1, Q_1(x) = 1 - x.$

(2) $Q_m(x)Q_{m+1}(x) + Q_m(-x)Q_{m+1}(-x) = 2 \quad (m \geq 0).$

(3) $Q_{m+1}(x)Q_{m+1}(-x) - Q_{m+2}(x)Q_m(-x) = x \quad (m \geq 0).$

(4) $Q_m(x) + xR_m(x) = Q_{m-1}(-x) \quad (m \geq 1).$

Proof. (1) For $m \geq 2$,

$$Q_m(x) = \begin{vmatrix} 1-x & -x & -x & \cdots & -x & -x & -x \\ -x & 1-x & -x & \cdots & -x & -x & 0 \\ -x & -x & 1-x & \cdots & -x & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x & -x & 0 & \cdots & 0 & 1 & 0 \\ -x & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -x & -x & \cdots & -x & -x & -x \\ 0 & 1-x & -x & \cdots & -x & -x & 0 \\ 0 & -x & 1-x & \cdots & -x & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -x & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} -x & -x & -x & \cdots & -x & -x & -x \\ -x & 1-x & -x & \cdots & -x & -x & 0 \\ -x & -x & 1-x & \cdots & -x & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x & -x & 0 & \cdots & 0 & 1 & 0 \\ -x & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$

by the splitting of the first column

$$= \det(I - xB(m-2)) + \begin{vmatrix} -x & 0 & 0 & \cdots & 0 & 0 & 0 \\ -x & 1 & 0 & \cdots & 0 & 0 & x \\ -x & 0 & 1 & \cdots & 0 & x & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x & 0 & x & \cdots & x & 1+x & x \\ -x & x & x & \cdots & x & x & 1+x \end{vmatrix}$$

$$= Q_{m-2}(x) - x \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & x \\ 0 & 1 & \cdots & 0 & x & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x & \cdots & x & 1+x & x \\ x & x & \cdots & x & x & 1+x \end{vmatrix}$$

$$= Q_{m-2}(x) - x \begin{vmatrix} 1+x & x & \cdots & x & x & x \\ x & 1+x & \cdots & x & x & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x & x & \cdots & 0 & 1 & 0 \\ x & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$

in reverse order of rows and columns in the second determinant

$$= Q_{m-2}(x) - xQ_{m-1}(-x).$$

(2) Use the induction on m. Formula (2) holds for $m = 0$, since

$$Q_0(x)Q_1(x) + Q_0(-x)Q_1(-x) = (1)(1-x) + (1)(1+x) = 2.$$

We assume that formula (2) holds for $m \leq k$.

$$\begin{aligned} &Q_{k+1}(x)Q_{k+2}(x) + Q_{k+1}(-x)Q_{k+2}(-x) \\ &= Q_{k+1}(x)(-xQ_{k+1}(-x) + Q_k(x)) + Q_{k+1}(-x)(xQ_{k+1}(x) + Q_k(-x)) \\ &= Q_{k+1}(x)Q_k(x) + Q_{k+1}(-x)Q_k(-x) = 2. \end{aligned}$$

(3) Similarly to the previous one, we also use the induction on m. Formula (3) holds for $m = 0$, since

$$Q_1(x)Q_1(-x) - Q_2(x)Q_0(-x) = (1-x)(1+x) - (1-x-x^2)(1) = x.$$

We assume that formula (3) holds for $m \leq k$.

$$\begin{aligned} &Q_{k+2}(x)Q_{k+2}(-x) - Q_{k+3}Q_{k+1}(-x) \\ &= Q_{k+2}(x)(xQ_{k+1}(x) + Q_k(-x)) - (-xQ_{k+2}(-x) + Q_{k+1}(x))Q_{k+1}(-x) \\ &= x\left[Q_{k+1}(x)Q_{k+2}(x) + Q_{k+1}(-x)Q_{k+2}(-x)\right] + \left[Q_k(-x)Q_{k+2}(x) - Q_{k+1}(x)Q_{k+1}(-x)\right] \\ &= 2x - x = x, \end{aligned}$$

by the formula (2) and the induction assumption.

(4)

$$
\begin{aligned}
&Q_m(x) + xR_m(x)\\
&= \det(I - xB(m)) + x\det\begin{pmatrix} 0 & u^t \\ -u & I - xB(m)\end{pmatrix}\\
&= \det\begin{pmatrix} 1 & u^t \\ -xu & I - xB(m)\end{pmatrix}\\
&= \begin{vmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
-x & 1-x & -x & -x & \cdots & -x & -x & -x\\
-x & -x & 1-x & -x & \cdots & -x & -x & 0\\
-x & -x & -x & 1-x & \cdots & -x & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
-x & -x & -x & -x & \cdots & 1 & 0 & 0\\
-x & -x & -x & 0 & \cdots & 0 & 1 & 0\\
-x & -x & 0 & 0 & \cdots & 0 & 0 & 1
\end{vmatrix}\\
&= \begin{vmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & x\\
0 & 0 & 0 & 1 & \cdots & 0 & x & x\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots & 1+x & x & x\\
0 & 0 & 0 & x & \cdots & x & 1+x & x\\
0 & 0 & x & x & \cdots & x & x & 1+x
\end{vmatrix}\\
&= \begin{vmatrix}
1 & 0 & \cdots & 0 & 0 & x\\
0 & 1 & \cdots & 0 & x & x\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1+x & x & x\\
0 & x & \cdots & x & 1+x & x\\
x & x & \cdots & x & x & 1+x
\end{vmatrix}\\
&= \begin{vmatrix}
1+x & x & \cdots & x & x & x\\
x & 1+x & \cdots & x & x & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
x & x & \cdots & 1 & 0 & 0\\
x & x & \cdots & 0 & 1 & 0\\
x & 0 & \cdots & 0 & 0 & 1
\end{vmatrix}\\
&= Q_{m-1}(-x).
\end{aligned}
$$

□

Here is another interesting property on $(Q_m(x))_{m\geq 0}$. The generating function is given below. (See [2] for the proof.)

Theorem 2.2. *We have the following equation:*

$$\sum_{m\geq 0} Q_m(x)t^m = \frac{(1+t)(1-t^2-xt)}{(1-t^2)^2+(xt)^2}.$$

3. Recurrence Relation of the Sequence $b(n,m)$

Note that $F_m(x) = 1 + xE_m(x)$. By the property (4) of Theorem 2.1, we obtain the generating function $F_m(x)$ of the sequence $(b(n,m))_{n\geq 0}$.

Theorem 3.1. *The following formula holds:*

$$F_m(x) = \frac{Q_m(-x)}{Q_{m+1}(x)}.$$

Proof.

$$\begin{aligned} F_m(x) &= 1 + xE_m(x) = 1 + x\frac{R_{m+1}(x)}{Q_{m+1}(x)} \\ &= \frac{Q_{m+1} + xR_{m+1}(x)}{Q_{m+1}(x)} = \frac{Q_m(-x)}{Q_{m+1}(x)}. \end{aligned}$$

□

Similarly to the Chebyshev function of the second kind, $F_m(x)$ has an expression by a trigonometric function as in the next theorem. (For details, see references [2] and [4].)

Theorem 3.2. *For each positive integer m,*

$$F_m(x) = (-1)^{m+1}\frac{\cos(\frac{2m+1}{2})\theta}{\cos(\frac{2m+3}{2})\theta} = \frac{\sin(m+1)\theta - \sin m\theta}{\sin(m+2)\theta - \sin(m+1)\theta},$$

where $\theta = \cos^{-1}\left(\frac{(-1)^m x}{2}\right)$.

Theorem 3.3. *The generating function $F_m(x)$ satisfies the continued fraction property:*

$$F_m(x) = \frac{1}{-x + F_{m-1}(-x)},\ F_0(x) = \frac{1}{1-x}.$$

First three formulas are listed here:

$$F_1(x) = \frac{1}{-x + F_0(-x)} = \frac{1}{-x + \frac{1}{x+1}} = \frac{1+x}{1-x-x^2} = \frac{Q_1(-x)}{Q_2(x)},$$

$$F_2(x) = \frac{1}{-x + F_1(-x)} = \frac{1+x-x^2}{1-2x-x^2+x^3} = \frac{Q_2(-x)}{Q_3(x)}, \text{ and}$$

$$F_3(x) = \frac{1}{-x + F_2(-x)} = \frac{1+2x-x^2-x^3}{1-2x-3x^2+x^3+x^4} = \frac{Q_3(-x)}{Q_4(x)}.$$

Proof. For a positive integer $m \geq 2$, if we use the property (1) of Theorem 2.1. we obtain the following:

$$\begin{aligned} F_m(x) &= \frac{Q_m(-x)}{Q_{m+1}(x)} = \frac{Q_m(-x)}{-xQ_m(-x) + Q_{m-1}(x)} \\ &= \frac{1}{-x + \frac{Q_{m-1}(x)}{Q_m(-x)}} = \frac{1}{-x + F_{m-1}(-x)}. \end{aligned}$$

□

Theorem 3.4. *Let the sequence* $\{c(n,m)\}_{n,m\geq 0}$ *satisfy the recurrence relation stated as below:*

$$c(n,m) = c(n,m-1) + \sum_{k\geq 0} c(2k,m-1)c(n-1-2k,m), \tag{2}$$

where $c(n,0) = 1, \forall n \geq 0$. *Then* $c(n,m) = b(n,m)$ *for all nonnegative integers* n *and* m.

In fact, this has been proved by Xin and Zhong ([4]). However, we prove it another way by using properties (2) and (3) of Theorem 2.1 as below. Our proof seems to be shorter and simpler compared to their lengthy proof which amounts to several pages.

Proof. Let

$$C_m(x) = \sum_{n\geq 0} c(n,m)x^n.$$

The proof is done if we show that $C_m(x) = F_m(x)$ for all $m \geq 0$. Note that from the recurrence relation (2) we get the following formula:

$$C_m(x) = C_{m-1}(x) + xC^e_{m-1}(x)C_m(x) \text{ with } C_0(x) = \frac{1}{1-x}, \tag{3}$$

where

$$C^e_m(x) = \frac{1}{2}(C_m(x) + C_m(-x)).$$

From the equation (3), we obtain the following:

$$C_m(x) = \frac{C_{m-1}(x)}{1 - x(C_{m-1}(x) + C_{m-1}(-x))/2}. \tag{4}$$

We will use the mathematical induction on m to prove that $C_m(x) = F_m(x)$ for all $m \geq 0$. It is obvious that $C_0(x) = \frac{1}{1-x} = F_0(x)$. We assume that $C_i(x) = F_i(x)$ for all $i \leq m-1$. By the assumption, from the formula (4), we have

$$\begin{aligned}
C_m(x) &= \frac{F_{m-1}(x)}{1 - x(F_{m-1}(x) + F_{m-1}(-x))/2} \\
&= \frac{\frac{Q_{m-1}(-x)}{Q_m(x)}}{1 - x\left(\frac{Q_{m-1}(-x)}{Q_m(x)} + \frac{Q_{m-1}(x)}{Q_m(-x)}\right)/2} \\
&= \frac{\frac{Q_{m-1}(-x)}{Q_m(x)}}{1 - \frac{x}{2}\frac{Q_{m-1}(-x)Q_m(-x) + Q_{m-1}(x)Q_m(x)}{Q_m(x)Q_m(-x)}} \\
&= \frac{\frac{Q_{m-1}(-x)}{Q_m(x)}}{1 - \frac{x}{Q_m(x)Q_m(-x)}} = \frac{Q_{m-1}(-x)Q_m(-x)}{Q_m(x)Q_m(-x) - x} = \frac{Q_{m-1}(-x)Q_m(-x)}{Q_{m+1}(x)Q_{m-1}(-x)} \\
&= \frac{Q_m(-x)}{Q_{m+1}(x)} = F_m(x).
\end{aligned}$$

In the computation above, the properties (2) and (3) of Theorem 2.1 were used. □

4. Concluding Remarks

As we mentioned earlier, Xin and Zhong ([4]) analyzed the generating function $G_n(y) = \sum_{m \geq 0} b(n,m)y^m$, in detail. It is an Ehrhart series of the graph polytope for the linear graph which is the rational function of the form

$$G_n(y) = \frac{H_n(y)}{(1-y)^{n+1}},$$

where $H_n(y)$ is a polynomial of degree at most n. Consider

$$K(x,y) = \sum_{n \geq 0} G_n(y)x^n = \sum_{m \geq 0} F_m(x)y^m.$$

Our question is the following: **What is the form of the generating function $K(x,y)$ exactly?**

This bi-variate generating function $K(x, y)$ requires more analysis and understanding of us.

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