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SUFFICIENT CONDITIONS FOR ANALYTIC FUNCTIONS TO BE STARLIKE OF RECIPROCAL ORDER

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Abstract. A normalized analytic function f, defined on the unit disk $\mathbb{D}$, is starlike of reciprocal order $\alpha > 1$ if the real part of $f(z)/(zf'(z))$ is less than α for all $z \in \mathbb{D}$. By utilizing the theory of differential subordination, we establish several sufficient conditions for analytic functions defined on $\mathbb{D}$ to be starlike of reciprocal order. Additionally, we investigate the conditions under which the function $f(z)/(zf'(z))$ is subordinate to the function $1 + (\alpha - 1)z$. This subordination, in turn, is sufficient for the function f to be starlike of reciprocal order $\alpha > 1$.

1. Introduction

Throughout this paper, we shall be interested in functions belonging to the class $\mathcal{H}[a, n]$, which consists of all analytic functions f defined on the unit disk $\mathbb{D}$ having Taylor series expansion $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. In particular, we define $\mathcal{A}$ as the set $\{zg : g \in \mathcal{H}[1, 1]\}$ representing all analytic functions $f : \mathbb{D} \to \mathbb{C}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The condition $f'(0) = 1$ shows that the function f is locally univalent at the origin. In this paper, we shall consider functions f that are locally univalent in $\mathbb{D}$; these are precisely the functions whose derivative is non-vanishing in $\mathbb{D}$. The subclass of $\mathcal{A}$ consisting of all the functions univalent in $\mathbb{D}$ is denoted by $\mathcal{S}$. A function $f \in \mathcal{A}$ is *starlike* if $f(\mathbb{D})$ is starlike with respect to the origin and is called *convex* if $f(\mathbb{D})$ is convex. Analytically, a function $f \in \mathcal{A}$ is starlike if it satisfies the inequality $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all $z \in \mathbb{D}$ and it is *convex* if $\operatorname{Re}(1+zf''(z)/f'(z)) > 0$. The subclasses $\mathcal{A}$ of all starlike functions and convex functions are respectively denoted by $\mathcal{ST}$ and $\mathcal{CV}$. One way to generalize these two classes is to require the corresponding quantities

$$Q_{ST}(z) := \frac{zf'(z)}{f(z)} \quad \text{and} \quad Q_{CV}(z) := 1 + \frac{zf''(z)}{f'(z)}$$

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to lie in some half-plane; it can be either $H_\alpha = \{w \in \mathbb{C} : \operatorname{Re} w > \alpha\}$ when $\alpha < 1$ or $H^\alpha = \{w \in \mathbb{C} : \operatorname{Re} w < \alpha\}$ when $\alpha > 1$. These choices gives respectively the class $\mathcal{ST}(\alpha)$ of all functions starlike of reciprocal order α, $0 \leqslant \alpha < 1$, that satisfy the inequality $\operatorname{Re}(f(z)/(zf'(z))) > \alpha$ for all $z \in \mathbb{D}$ and the class $\mathcal{M}(\alpha)$ of all analytic functions starlike of order α, $\alpha > 1$, satisfying the inequality $\operatorname{Re}(zf'(z)/f(z)) < \alpha$. Though we have called the functions in the classes $\mathcal{ST}(\alpha)$ and $M(\alpha)$ as starlike functions of order α, the choice of symbol is historical; with the notation $M(\alpha)$, it was introduced and studied by Uralegaddi, Ganigi and Sarangi [13]. Apart from the notations, the main difference is the univalence of the functions. The functions in the class $\mathcal{ST}(\alpha)$ are univalent while the functions in the class $M(\alpha)$ need not be univalent. Yet another way is to consider the reciprocal of Q_{ST} to lie in the half-plane H^α or H_α. This leads to the classes $\mathcal{ST}_R(\alpha)$, studied in [7], and $M_R(\alpha)$ of all *starlike functions of reciprocal order* α, $\alpha > 1$. The latter class $\mathcal{M}_{\mathcal{R}}(\alpha)$ consists of all analytic functions satisfying $\operatorname{Re}(f(z)/zf'(z)) < \alpha$, $\alpha > 1$. For a function p with positive real part, it is easy to see that $\operatorname{Re}(1/p(z)) \leqslant 1/(\operatorname{Re} p(z))$ and applying this with $p(z) = zf'(z)/f(z)$, we see that every starlike function of order $1/\alpha$, $\alpha > 1$, is starlike of reciprocal order $\alpha > 1$. However, it is worth mentioning that a function in the class $\mathcal{M}_{\mathcal{R}}(\alpha)$ need not be starlike. Several authors [2, 3, 1, 4, 6, 7, 8, 5, 12] have examined the required criteria for starlikeness, coefficient estimates, and subordination results for various classes of analytic functions of reciprocal order.

Our primary interest in this paper is to give several sufficient condition for functions to belong to the class $M(\alpha)$. We make use of the theory of differential subordination developed by Miller and Mocanu [11]. We first investigate admissibility conditions for functions to belong to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$, and apply it obtained various sufficient conditions for functions to be in class $\mathcal{M}_{\mathcal{R}}(\alpha)$. We also obtain some criteria for the subordination $f(z)/(zf'(z)) \prec 1 + (\alpha - 1)z$.

2. Admissibility Conditions

Though the field of complex numbers is unordered, the concept of subordination in the complex plane play an analogues role of inequalities on the real line. For two analytic functions f and g on open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the function f is said to be *subordinate* to the function g, written as $f \prec g$, if there is a Schwarz function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. If g is a univalent function, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Let p be an analytic function defined in the unit disk and let the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$. For given domains Ω and Δ in $\mathbb{C}$, the theory of differential subordination deals with the following implication:

$$(1) \qquad \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{D}\} \subset \Omega \implies \{p(z) : z \in \mathbb{D}\} \subset \Delta.$$

If Ω and Δ are simply connected domains that are not the whole complex plane, then by Riemann mapping theorem, there exists univalent functions h and q defined on $\mathbb{D}$, that maps $\mathbb{D}$ respectively onto Ω and Δ such that $h(0) = \psi(p(0), 0, 0; 0)$ and $q(0) = p(0)$. In addition if $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic then the implication (1) can be written as follows:

$$\psi(p(z), zp'(z), z^2p''(z) \prec h(z) \implies p(z) \prec q(z).$$

The subordination $\psi(p(z), zp'(z), z^2p''(z)) \prec h(z)$ is known as second order differential subordination. If p is an analytic function and satisfies the second-order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z) \prec h(z), \tag{2}$$

then the function p is called a solution of the differential subordination (2). If q is a univalent function and the subordination $p \prec q$ holds for all p satisfying (2), then the function q is said to be a dominant of all solutions of the differential subordination (2). Milller and Mocanu have developed a comprehensive theory of differential subordination and its applications, see[9, 10, 11].

Definition 2.1. [11] *The set Q denotes the set of all functions q that are analytic and injective on $\overline{\mathbb{D}} \setminus E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(q)$.

Definition 2.2. [11] *Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$, consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$, that satisfy the admissibility condition:*

$$\psi(r, s, t; z) \notin \Omega,$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geqslant m \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \mathbb{D} \setminus E(q)$ and $m \geqslant n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

For the function $q(z) = (1 + z)/(1 - z)$, Definition 2.2 gives the following admissibility criteria:

Definition 2.3. [11] *Let Ω be a subset of $\mathbb{C}$. The class $\mathcal{P}_n(\Omega)$ of admissible functions consists of all the functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(i\rho, \sigma, \mu + i\nu; z) \notin \Omega,$$

when $\rho \in \mathbb{R}$, $\sigma \leqslant -n(1 + \rho^2)/2$, $\sigma + \mu \leqslant 0$ and $z \in \mathbb{D}$.

For the function $w(z) = z$, we obtain the following admissibility conditions from Definition 2.2.

Definition 2.4. [11] *Let Ω be a subset of $\mathbb{C}$. The class $\mathcal{B}_n(\Omega)$ of admissible functions consists of all functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(e^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega,$$

when $K \geqslant n$, $\mathrm{Re}(Le^{i\theta}) \geqslant (n-1)K$, $\theta \in \mathbb{R}$ and $z \in \mathbb{D}$.

The following theorem is a fundamental finding in the study of first and second order differential subordination.

Theorem 2.5. [11]*[Miller-Mocanu Theorem] Let the function ψ be in the class $\Psi_n[h, q]$ with $q(0) = a$. If the function p is in the class $\mathcal{H}[a, n]$, the function $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in $\mathbb{D}$, and satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{3}$$

then the function p is subordinate to the function q.

The class $\mathcal{P}$ of *Carathéodory functions* consists of all functions $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ satisfying $\mathrm{Re}(p(z)) > 0$. For the function $q(z) = (1 + z)/(1 - z)$, Theorem 2.5 becomes

Theorem 2.6. [11] *Let the function ψ be in the class $\mathcal{P}_n(\Omega)$. If the function p is in the class $\mathcal{H}[1, n]$ satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \quad (z \in \mathbb{D}),$$

then p belongs to the class $\mathcal{P}$.

For the function $q(z) = z$, Theorem 2.5 becomes

Theorem 2.7. [11] *Let the function ψ be in the class $\in \mathcal{B}_n(\Omega)$. If the function w is in the class $\mathcal{H}[0, n]$, satisfies*

$$\psi(w(z), zw'(z), z^2w''(z); z) \in \Omega \quad (z \in \mathbb{D}),$$

then w belongs to the class $\mathcal{B}_0$.

We need the following definition of admissibility function to prove the subordination result for the class $\mathcal{M}_{\mathcal{R}}(\alpha)$. .

Definition 2.8. *Let Ω be a subset of $\mathbb{C}$. The class $\mathcal{M}_{\mathcal{R}}(\Omega)$ of admissible functions consists of all the functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissible condition*

$$\psi(r, s; z) \notin \Omega, \tag{4}$$

where

$$\frac{1}{r} = \alpha + ir_0 \quad \text{and} \quad \frac{s}{r} \leqslant \frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}, \tag{5}$$

when $\alpha > 1$, $r_0 \in \mathbb{R}$ and $z \in \mathbb{D}$.

Our principal result is the following theorem which gives an easy mechanism to check whether a function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Theorem 2.9. *Let the function ψ be in the class $\mathcal{M}_{\mathcal{R}}(\Omega)$. If the locally univalent function $f \in \mathcal{A}$ such that, satisfies*

$$\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega \quad (z \in \mathbb{D}), \tag{6}$$

then the function f is in the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by

$$p(z) := \frac{1}{1-\alpha}\left(\frac{f(z)}{zf'(z)} - \alpha\right).$$

Then p is analytic in $\mathbb{D}$ and $f \in \mathcal{M}_{\mathcal{R}}(\alpha)$ is equivalent to $p \in \mathcal{P}$. From equations (7) and (8), we have

$$Q_{ST}(z) := \frac{zf'(z)}{f(z)} = \frac{1}{(1-\alpha)p(z)+\alpha}. \tag{7}$$

A computation shows that

$$Q_{CV}(z) := 1 + \frac{zf''(z)}{f'(z)} = \frac{1-(1-\alpha)zp'(z)}{(1-\alpha)p(z)+\alpha}. \tag{8}$$

Define the transformation from $\mathbb{C}^2$ to $\mathbb{C}^2$ by

$$r = \frac{1}{(1-\alpha)u+\alpha} \quad \text{and} \quad s = \frac{1-(1-\alpha)v}{(1-\alpha)u+\alpha}.$$

Let

$$\varphi(u, v; z) = \psi(r, s; z) = \psi\left(\frac{1}{(1-\alpha)u+\alpha}, \frac{1-(1-\alpha)v}{(1-\alpha)u+\alpha}; z\right).$$

Thus, using (7) and (8) we get

$$\varphi(p(z), zp'(z); z) = \psi(Q_{ST}(z), Q_{CV}(z); z).$$

By (6), we see that $\varphi(p(z), zp'(z); z) \in \Omega$. Note that

$$\varphi(i\rho, \sigma; z) = \psi\left(\frac{1}{(1-\alpha)i\rho+\alpha}, \frac{1-(1-\alpha)\sigma}{(1-\alpha)i\rho+\alpha}; z\right) = \psi(r, s; z).$$

Hence, we have

$$\frac{1}{r} = \alpha + (1-\alpha)i\rho = \alpha + ir_0,$$

which gives $\rho = r_0/(1-\alpha) \in \mathbb{R}$. Further, we have $s/r = 1-(1-\alpha)\sigma$, so using equation (5), we see that $\rho \leq -(1+\rho^2)/2$. Hence, we get the corresponding admissibility conditions given in (5). Thus, by Theorem 2.6, it follows that $\varphi \in \mathcal{P}_1(\Omega)$. Therefore, we obtain $\mathrm{Re}(p(z)) > 0$ or equivalently, f is in the class $\mathcal{M}_{\mathcal{R}}(\alpha)$. □

3. Sufficient Conditions

In this section, we apply Theorem 2.9 to obtain several sufficient condition for function to be starlike of reciprocal order. These results in terms of combinations of Q_{ST} and Q_{CV}.

Theorem 3.1. *For $\alpha > 1$, let $\Omega = \mathbb{C} - (-\infty, (3-\alpha)/2]$. If the locally univalent function $f \in \mathcal{A}$ satisfies $Q_{CV}(z)/Q_{ST}(z) \in \Omega$ for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.*

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r, s) = s/r$ and $\Omega = \mathbb{C} - (-\infty, (3-\alpha)/2]$. Then, the function ψ satisfies

$$\psi(r,s) = \frac{s}{r} \leqslant \frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)},$$

whenever $r_0 \in \mathbb{R}$. The function $g : [0,\infty) \to \mathbb{R}$ defined by

$$g(t) = \frac{3-\alpha}{2} + \frac{t}{2(1-\alpha)}$$

is a decreasing function of t and hence its maxima is attainted at $t = 0$. Hence, we have $g(t) \leqslant g(0) = (3-\alpha)/2$ and so

$$\psi(r,s) = \frac{s}{r} \leqslant \frac{3-\alpha}{2}.$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and therefore $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows from the Theorem 2.9.

□

Corollary 3.2. *Let $\alpha > 1$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality $\operatorname{Re}(Q_{CV}/Q_{ST}) > (3-\alpha)/2$ for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.*

Theorem 3.3. *For $\alpha > 1$, let $\Omega = \mathbb{C} - [2/(3-\alpha), \infty)$. If the locally univalent function $f \in \mathcal{A}$ satisfies $Q_{ST}(z)/Q_{CV}(z) \in \Omega$, for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.*

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r, s) = r/s$ and $\Omega = \mathbb{C} - [2/(3-\alpha), \infty)$. Then, the function ψ satisfies

$$\begin{aligned}\psi(r,s) = \frac{r}{s} &\geqslant \frac{1}{(3-\alpha)/2 + r_0^2/2(1-\alpha)} \\ &= \frac{2(1-\alpha)}{3(1-\alpha) - \alpha(1-\alpha) + r_0^2} \\ &= \frac{2(1-\alpha)}{(\alpha-1)(\alpha-3) + r_0^2},\end{aligned}$$

whenever $r_0 \in \mathbb{R}$. The function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(r_0) = \frac{2(1-\alpha)}{(\alpha-1)(\alpha-3)} + r_0^2$$

has minima at $r_0 = 0$. Hence, we have

$$\psi(r,s) = \frac{r}{s} \geqslant \frac{2(1-\alpha)}{(\alpha-1)(\alpha-3)} = \frac{2}{3-\alpha}.$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and so $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. □

Corollary 3.4. *Let $\alpha > 1$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality $\operatorname{Re}(Q_{ST}/Q_{CV}) < 2/(3-\alpha)$ for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.*

Theorem 3.5. *Let $1 < \alpha < 2$ and $\gamma > 1/(2-\alpha)$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}\left(\frac{Q_{CV}(z)}{Q_{ST}(z)}\left(1 + \frac{\gamma}{Q_{ST}(z)}\right)\right) > \frac{(3-\alpha)(1+\gamma\alpha)}{2}$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r,s) = s/r(1 + (\gamma/r))$. Then, for r, s satisfying (5), we have $\operatorname{Re}(\psi(r,s)) = s/r + \alpha\gamma s/r$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > ((3-\alpha)(1+\alpha\gamma)/2)\}$. For $1 < \alpha < 2$ and $\gamma > 1/(2-\alpha)$ the function ψ satisfies

$$\begin{aligned}
\operatorname{Re}\psi(r,s) &= \frac{s}{r} + \frac{\alpha\gamma s}{r} \\
&\leqslant \left(\frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}\right) + \alpha\gamma\left(\frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}\right) \\
&= (1+\alpha\gamma)\left(\frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}\right),
\end{aligned}$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(r_0) = (1 + \alpha\gamma)((3-\alpha)/2) + (r_0^2/2(1-\alpha))$. By second derivative test we get that maxima will occur at $r_0 = 0$. Hence

$$\operatorname{Re}\psi(r,s) \leqslant (1+\alpha\gamma)\left(\frac{3-\alpha}{2}\right).$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5), and so $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. □

Theorem 3.6. *Let $\alpha > 1$ and $2(1-\alpha) < \gamma < 0$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}\left(\frac{\gamma Q_{CV}(z)}{Q_{ST}(z)} + \frac{1}{Q_{ST}^2(z)}\right) < \gamma\left(\frac{3-\alpha}{2}\right) + \alpha^2$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r,s) = \gamma s/r + 1/r^2$. Then for r, s satisfying (5), we have $\operatorname{Re}\psi(r,s) = \gamma s/r + (\alpha^2 - r_0^2)$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w < \gamma(3-\alpha)/2 + \alpha^2\}$. Then, the function ψ satisfies

$$\begin{aligned}\operatorname{Re}\psi(r,s) &= \frac{\gamma s}{r} + \operatorname{Re}\left(\frac{1}{r^2}\right)\\ &\geqslant \gamma\left(\frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}\right) + (\alpha^2 - r_0^2)\\ &= \gamma\left(\frac{3-\alpha}{2}\right) + \alpha^2 + \left(\frac{\gamma}{2(1-\alpha)} - 1\right) r_0^2,\end{aligned}$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : [0,\infty) \to \mathbb{R}$ be defined by $g(t) = \gamma((3-\alpha)/2) + \alpha^2 + (\gamma/2(1-\alpha) - 1)t$. The function g is an increasing function of t if $\gamma > 2(1-\alpha)$. So minimum value of g will occur at $t = 0$, Hence, we have $g(t) \geqslant g(0) = \gamma(3-\alpha)/2 + \alpha^2$ and so

$$\operatorname{Re}\psi(r,s) \geqslant \gamma\left(\frac{3-\alpha}{2}\right) + \alpha^2.$$

Thus, it follows that for all r, s satisfying (5), $\psi(r,s) \notin \Omega$, and therefore $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. □

Theorem 3.7. *Let $\alpha > 2$. If the locally univalent function $f \in \mathcal{A}$ satisfies either*

$$\operatorname{Re}\left(\frac{\gamma Q_{CV}(z)}{Q_{ST}^2(z)} + \frac{1}{Q_{ST}(z)}\right) > \alpha + \gamma\alpha\left(\frac{3-\alpha}{2}\right) \quad \gamma > \frac{2}{\alpha-2}$$

or

$$\operatorname{Re}\left(\frac{\gamma Q_{CV}(z)}{Q_{ST}^2(z)} + \frac{1}{Q_{ST}(z)}\right) < \alpha + \gamma\alpha\left(\frac{3-\alpha}{2}\right) \quad \gamma < 0,$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r,s) = \gamma s/r^2 + 1/r$. Then for r, s satisfying (5), we have $\operatorname{Re}\psi(r,s) = \gamma\alpha s/r + \alpha$.

Case-1: For $\gamma > 2/(\alpha - 2)$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > \alpha + (\gamma\alpha(3-\alpha))/2\}$. The function ψ satisfies

$$\begin{aligned}\operatorname{Re}\psi(r,s) &= \alpha\gamma\left(\frac{s}{r}\right) + \alpha \\ &\leqslant \alpha\gamma\left(\frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}\right) + \alpha \\ &= \alpha + \frac{\alpha\gamma(3-\alpha)}{2} + \frac{\alpha\gamma r_0^2}{2(1-\alpha)},\end{aligned}$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : [0,\infty) \to \mathbb{R}$ be defined by

$$g(t) = \alpha + \frac{\alpha\gamma(3-\alpha)}{2} + \frac{\alpha\gamma t}{2(1-\alpha)}.$$

The function g is a decreasing function of t, as $g'(t) < 0$, so we see that maxima is at $t = 0$. Hence, we have $g(t) \leqslant g(0) = \alpha + \alpha\gamma((3-\alpha)/2)$ and so

$$\operatorname{Re}\psi(r,s) \leqslant \alpha + \alpha\gamma\left(\frac{3-\alpha}{2}\right).$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r,s satisfying (5) and therefore $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows from the Theorem 2.9.

Case-2: For $\gamma < 0$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w < \alpha + (\gamma\alpha(3-\alpha))/2\}$. The function ψ satisfies

$$\begin{aligned}\operatorname{Re}\psi(r,s) &= \alpha\gamma\left(\frac{s}{r}\right) + \alpha \\ &\geqslant \alpha\gamma\left(\frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}\right) + \alpha \\ &= \alpha + \frac{\alpha\gamma(3-\alpha)}{2} + \frac{\alpha\gamma r_0^2}{2(1-\alpha)},\end{aligned}$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : [0,\infty) \to \mathbb{R}$ be defined by

$$g(t) = \alpha + \frac{\alpha\gamma(3-\alpha)}{2} + \frac{\alpha\gamma t}{2(1-\alpha)}.$$

The function g is an increasing function of t, so we get that minima will occur at $t = 0$. Hence, we have $g(t) \geqslant g(0) = \alpha + \alpha\gamma((3-\alpha)/2)$ and so

$$\operatorname{Re}\psi(r,s) \geqslant \alpha + \alpha\gamma\left(\frac{3-\alpha}{2}\right).$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r,s satisfying (5) and so $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. $\square$

Theorem 3.8. *Let $1 < \alpha < 3$. If the locally univalent function $f \in \mathcal{A}$ satisfies, either*

$$\operatorname{Re}\left(\frac{\gamma Q_{ST}(z)}{Q_{CV}(z)} + \frac{1}{Q_{ST}(z)}\right) < \alpha + \frac{2\gamma}{3-\alpha}, \quad \gamma > 0$$

or

$$\operatorname{Re}\left(\frac{\gamma Q_{ST}(z)}{Q_{CV}(z)} + \frac{1}{Q_{ST}(z)}\right) > \alpha + \frac{2\gamma}{3-\alpha}, \quad \gamma < \alpha - 3,$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r,s) = \gamma(r/s) + 1/r$. Then for r, s satisfying (5), we have $\operatorname{Re}\psi(r,s) = \gamma(r/s) + \alpha$.

Case-1: For $\gamma > 0$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w < \alpha + (2\gamma/(3-\alpha))\}$. The function ψ satisfies

$$\begin{aligned}\operatorname{Re}\psi(r,s) &= \gamma\left(\frac{r}{s}\right) + \alpha \\ &\geqslant \frac{\gamma}{(3-\alpha)/2 + r_0^2/2(1-\alpha)} + \alpha \\ &= \frac{2\gamma(1-\alpha)}{(3-\alpha)(1-\alpha) + r_0^2} + \alpha,\end{aligned}$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(r_0) = (2\gamma(1-\alpha))/((3-\alpha)(1-\alpha) + r_0^2) + \alpha$. By second derivative test we get that minima will occur at $r_0 = 0$. Hence,

$$\operatorname{Re}\psi(r,s) \geqslant \frac{2\gamma(1-\alpha)}{(3-\alpha)(1-\alpha)} + \alpha = \frac{2\gamma}{3-\alpha} + \alpha.$$

Thus, it follows that $\psi(r,s) \notin \Omega$ for all r, s satisfying (5) and so $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows from the Theorem 2.9.

Case-2: For $\gamma < \alpha - 3$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > \alpha + (2\gamma/(3-\alpha))\}$. The function ψ satisfies

$$\begin{aligned}\operatorname{Re}\psi(r,s) &= \gamma\left(\frac{r}{s}\right) + \alpha \\ &\leqslant \frac{\gamma}{(3-\alpha)/2 + r_0^2/2(1-\alpha)} + \alpha \\ &= \frac{2\gamma(1-\alpha)}{(3-\alpha)(1-\alpha) + r_0^2} + \alpha,\end{aligned}$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(r_0) = (2\gamma(1-\alpha))/((3-\alpha)(1-\alpha) + r_0^2) + \alpha$. By second derivative test we get that maxima will occur at $r_0 = 0$. Hence,

$$\operatorname{Re}\psi(r,s) \leqslant \frac{2\gamma(1-\alpha)}{(3-\alpha)(1-\alpha)} + \alpha = \frac{2\gamma}{3-\alpha} + \alpha.$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and therefore $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. □

Theorem 3.9. *Let $\alpha > 1$. If the locally univalent function $f \in \mathcal{A}$ satisfies either*

$$\operatorname{Re}\left(\frac{1}{Q_{ST}(z)}+\frac{\gamma Q_{CV}(z)}{Q_{ST}(z)}\right) > \alpha+\gamma\left(\frac{3-\alpha}{2}\right), \quad \gamma > 2$$

or

$$\operatorname{Re}\left(\frac{1}{Q_{ST}(z)}+\frac{\gamma Q_{CV}(z)}{Q_{ST}(z)}\right) < \alpha+\gamma\left(\frac{3-\alpha}{2}\right), \quad \gamma < 0,$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r,s) = 1/r + \gamma s/r$. Then for r, s satisfying (5), we have $\operatorname{Re}\psi(r,s) = \alpha + \gamma s/r$.

Case-1: For $\gamma > 2$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > \alpha + \gamma((3-\alpha)/2)\}$. The function ψ satisfies

$$\operatorname{Re}\psi(r,s) = \alpha+\gamma\left(\frac{s}{r}\right) \leqslant \alpha+\gamma\left(\frac{3-\alpha}{2}+\frac{r_0^2}{2(1-\alpha)}\right),$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : [0,\infty) \to \mathbb{R}$ be defined by $g(t) = \alpha + \gamma(((3-\alpha)/2) + t/(2(1-\alpha)))$. The function g is a decreasing function of t as $g'(t) < 0$, so we obtain that maxima will occur at $t = 0$. Hence, we have $g(t) \leq g(0) = \alpha + \gamma((3-\alpha)/2)$ and so

$$\operatorname{Re}\psi(r,s) \leqslant \alpha+\gamma\left(\frac{3-\alpha}{2}\right).$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and so $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows from the Theorem 2.9.

Case-2: For $\gamma < 0$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w < \alpha + \gamma((3-\alpha)/2)\}$. The function ψ satisfies

$$\operatorname{Re}\psi(r,s) = \alpha+\gamma\left(\frac{s}{r}\right) \geqslant \alpha+\gamma\left(\frac{3-\alpha}{2}+\frac{r_0^2}{2(1-\alpha)}\right),$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : [0,\infty) \to \mathbb{R}$ be defined by $g(t) = \alpha + \gamma((3-\alpha)/2) + t/2(1-\alpha))$. The function g is an increasing function of t as $g'(t) > 0$, so we obtain that minima will occur at $t = 0$. Hence, we have $g(t) \geqslant g(0) = \alpha + \gamma((3-\alpha)/2)$ and so

$$\operatorname{Re}\psi(r,s) \geqslant \alpha+\gamma\left(\frac{3-\alpha}{2}\right).$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and so $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. □

Theorem 3.10. *Let $1 < \alpha < 3$ and $1/(\alpha-3)(\alpha+1) < \gamma < 0$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}\left(\frac{\gamma}{Q_{ST}^2(z)}+\frac{Q_{ST}(z)}{Q_{CV}(z)}\right) < \gamma\alpha^2+\frac{2}{3-\alpha}$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r,s) = \gamma/r^2 + r/s$. Then for r, s satisfying (5), we have $\operatorname{Re}\psi(r,s) = \gamma(\alpha^2 - r_0^2) + r/s$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w < \gamma\alpha^2 + 2/(3-\alpha)\}$. Then for $1 < \alpha < 3$ and $1/(\alpha-3)(\alpha+1) < \gamma < 0$, the function ψ satisfies

$$\operatorname{Re}\psi(r,s) = \gamma(\alpha^2 - r_0^2) + \frac{r}{s} \geqslant \gamma(\alpha^2 - r_0^2) + \frac{2(1-\alpha)}{(3-\alpha)(1-\alpha) + r_0^2},$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(r_0) = \gamma(\alpha^2 - r_0^2) + 2(1-\alpha)/((3-\alpha)(1-\alpha) + r_0^2).$$

By double derivative test we obtain that minima will occur at $r_0 = 0$. Hence, we have

$$\operatorname{Re}\psi(r,s) \geqslant \gamma\alpha^2 + \frac{2}{3-\alpha}.$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and therefore $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. □

Theorem 3.11. *Let $\alpha > 1$ and $\gamma > 0$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}\left(\frac{\gamma}{Q_{CV}} + \frac{Q_{ST}}{Q_{CV}}\right) < \frac{2(\alpha\gamma+1)}{3-\alpha}$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r,s) = \gamma/s + r/s = ((\gamma/r)+1)r/s$. Then for r, s satisfying (5), we have $\operatorname{Re}\psi(r,s) = (\alpha\gamma+1)r/s$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w \leqslant 2(\alpha\gamma+1)/(3-\alpha)\}$. Then for $\alpha > 1$ and $\gamma > 0$, function ψ satisfies

$$\operatorname{Re}\psi(r,s) = (\alpha\gamma+1)\frac{r}{s} \geqslant \frac{\alpha\gamma+1}{(3-\alpha)/2 + r_0^2/2(1-\alpha)} = \frac{2(1-\alpha)(1+\alpha\gamma)}{(3-\alpha)(1-\alpha) + r_0^2},$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(r_0) = 2(1-\alpha)(1+\alpha\gamma)/((3-\alpha)(1-\alpha) + r_0^2).$$

By double derivative test we get that minima will occur at $r_0 = 0$. Hence

$$\operatorname{Re}\psi(r,s) \geqslant \frac{2(1+\alpha\gamma)}{3-\alpha}.$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and so $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows from the Theorem 2.9. □

Theorem 3.12. *Let $\alpha > 1$ and $\gamma < 3/2\alpha$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}\left(\frac{\gamma}{Q_{ST}} + Q_{CV}\right) > \alpha\gamma + \frac{3-\alpha}{2\alpha}$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\psi(r,s) = \gamma/r + s = \gamma/r + sr/r$. Then for r, s satisfying (5), we have $\operatorname{Re}\psi(r,s) = \alpha\gamma + s/r(\alpha/(\alpha^2 + r_0^2))$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re}\Omega > \alpha\gamma + (3-\alpha)/2\alpha\}$. Then for $\alpha > 1$ and $\gamma < 3/2\alpha$, function ψ satisfies

$$\operatorname{Re}\psi(r,s) = \alpha\gamma + \left(\frac{\alpha}{\alpha^2 + r_0^2}\right)\frac{s}{r} \leqslant \alpha\gamma + \left(\frac{\alpha}{\alpha^2 + r_0^2}\right)\left(\frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}\right),$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(r_0) = \alpha\gamma + (\alpha/(\alpha^2 + r_0^2))((3-\alpha)/2 + r_0^2/2(1-\alpha)).$$

By double derivative test we obtain that maxima will exist at $r_0 = 0$. Hence, we get

$$\operatorname{Re}\psi(r,s) \leqslant \alpha\gamma + \frac{3-\alpha}{2\alpha}.$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and therefore $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. □

Theorem 3.13. *Let $\alpha > 1$ and $\gamma > 0$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}(Q_{ST} + \gamma Q_{CV}) > \frac{2 + \gamma(3-\alpha)}{2\alpha}$$

for all $z \in \mathbb{D}$, then the function f belongs to the class $\mathcal{M}_{\mathcal{R}}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by

$$\psi(r,s) = r + \gamma s = r(1 + \gamma s/r).$$

Then for r, s satisfying (5), we have $\operatorname{Re}\psi(r,s) = (\alpha/(\alpha^2 + r_0^2))(1 + \gamma s/r)$. Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > 2 + \gamma(3-\alpha)/2\alpha\}$. Then for $\alpha > 1$ and $\gamma > 0$ the function ψ satisfies

$$\begin{aligned}\operatorname{Re}\psi(r,s) &= \frac{\alpha}{\alpha^2 + r_0^2}\left(1 + \frac{\gamma s}{r}\right)\\ &\leqslant \frac{\alpha}{\alpha^2 + r_0^2}\left(1 + \gamma\left(\frac{3-\alpha}{2} + \frac{r_0^2}{2(1-\alpha)}\right)\right)\\ &= \frac{\alpha}{2(1-\alpha)}\left(\frac{2(1-\alpha) + \gamma(3-\alpha)(1-\alpha) + \gamma r_0^2}{\alpha^2 + r_0^2}\right),\end{aligned}$$

whenever $r_0 \in \mathbb{R}$. Let the function $g : [0,\infty) \to \mathbb{R}$ be defined by $g(t) = \alpha(2(1-\alpha) + \gamma(3-\alpha)(1-\alpha) + \gamma t)/2(1-\alpha)(\alpha^2 + t)$. The function g is a decreasing function of t, so the maxima will occur at $t = 0$. Hence, we get that $g(t) \leqslant g(0) = (2 + \gamma(3-\alpha))/2\alpha$ and therefore

$$\operatorname{Re}\psi(r,s) \leqslant \frac{2 + \gamma(3-\alpha)}{2\alpha}.$$

Thus, it follows that $\psi(r,s) \notin \Omega$, for all r, s satisfying (5) and so $\psi \in \mathcal{M}_{\mathcal{R}}(\Omega)$. Since, $\psi(Q_{ST}, Q_{CV}) \in \Omega$, the result now follows by an application of Theorem 2.9. □

4. Further Sufficient Conditions

Let $\mathcal{M}_R[\alpha]$ be the class of all analytic functions satisfying the inequality $|zf'(z)/f(z) - 1| < \alpha - 1$ or equivalently the subordination $f(z)/(zf'(z)) \prec 1+(\alpha-1)z$. In this section, we obtain some sufficient condition for the functions to be in the class $\mathcal{M}_R[\alpha]$. Since $\operatorname{Re}(1+(\alpha-1)z) < \alpha$, it follows that $\mathcal{M}_R[\alpha] \subset \mathcal{M}_R(\alpha)$ and therefore the sufficient conditions obtained here also sufficient for functions to be starlike of reciprocal order $\alpha > 1$.

Theorem 4.1. *Let $\alpha > 1$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality $|(Q_{CV}/Q_{ST}) - 1| < \alpha - 1$ for all $z \in \mathbb{D}$, then the function f is in the class $\mathcal{M}_R[\alpha]$.*

Proof. Let $w : \mathbb{D} \to \mathbb{C}$ be defined by

$$w(z) = \frac{1}{\alpha - 1}\left(\frac{f(z)}{zf'(z)} - 1\right). \tag{9}$$

Then, the function w is analytic in $\mathbb{D}$ and $w(0) = 0$. To prove that the function f is in the class $\mathcal{M}_R[\alpha]$, it is enough to show that the function w satisfy the hypothesis of Schwarz lemma, namely, $|w(z)| < 1$. From equation (9), we have

$$Q_{ST} := \frac{zf'(z)}{f(z)} = \frac{1}{(\alpha-1)w(z)+1}. \tag{10}$$

A further calculation gives

$$Q_{CV} := 1 + \frac{zf''(z)}{f'(z)} = \frac{1-(\alpha-1)zw'(z)}{(\alpha-1)w(z)+1}. \tag{11}$$

Using equations (10) and (11), we get

$$\frac{Q_{CV}}{Q_{ST}} - 1 = -(\alpha-1)zw'(z).$$

Let $\psi(r,s) = (s/r) - 1$ and $\Omega = \{w \in \mathbb{C} : |w| < \alpha - 1\}$. The function ψ satisfies

$$|\psi(e^{i\theta}, Ke^{i\theta})| = |-(1-\alpha)Ke^{i\theta}| = (\alpha-1)K \geqslant \alpha - 1,$$

whenever $K \geqslant 1$ and $\alpha > 1$. Thus, $\psi \in \mathcal{B}_n(\Omega)$. Hence, by Theorem 2.7, we obtain the desired result. □

Theorem 4.2. *Let $\alpha > 1$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality $|Q_{CV} - 1| < 2(\alpha-1)/\alpha$ for all $z \in \mathbb{D}$, then the function f is in the class $\mathcal{M}_R[\alpha]$.*

Proof. Let $w : \mathbb{D} \to \mathbb{C}$ be defined by (9). Let $\psi(r,s) = s-1$ and the domain $\Omega = \{w \in \mathbb{C} : |w| < 2(\alpha-1)/\alpha\}$. Then the function ψ satisfies

$$\begin{aligned}\left|\psi(e^{i\theta}, Ke^{i\theta})\right| &= \left|\frac{1-(\alpha-1)Ke^{i\theta}}{(\alpha-1)e^{i\theta}+1} - 1\right| \\ &= \left|\frac{(1-\alpha)(Ke^{i\theta}+e^{i\theta})}{(\alpha-1)e^{i\theta}+1}\right| \\ &\geqslant \frac{(\alpha-1)(K+1)}{\alpha} \\ &= \frac{2(\alpha-1)}{\alpha},\end{aligned}$$

whenever $K \geqslant 1$ and $\alpha > 1$. Thus, $\psi \in \mathcal{B}_n(\Omega)$. Hence, by Theorem 2.7, we obtain the desired result. □

Theorem 4.3. *Let $\alpha > 1$. If the locally univalent function $f \in \mathcal{A}$ satisfies the inequality $|Q_{ST} - 1| < (\alpha-1)/\alpha$ for all $z \in \mathbb{D}$, then the function f is in the class $\mathcal{M}_R[\alpha]$.*

Proof. Let $w : \mathbb{D} \to \mathbb{C}$ be defined by (9). Let $\psi(r,s) = r-1$ and the domain $\Omega = \{w \in \mathbb{C} : |w| < (\alpha-1)/\alpha\}$. From the equation (10), we have

$$Q_{ST} - 1 = \frac{1}{(\alpha-1)w(z)+1} - 1 = \frac{(1-\alpha)w(z)}{(\alpha-1)w(z)+1}.$$

The function ψ satisfies

$$\psi(e^{i\theta}, Ke^{i\theta}) = \frac{(1-\alpha)e^{i\theta}}{(\alpha-1)e^{i\theta}+1} \geqslant \frac{\alpha-1}{\alpha}.$$

whenever $K \geqslant 1$ and $\alpha > 1$. Thus, $\psi \in \mathcal{B}_n(\Omega)$. Hence, by Theorem 2.7, we obtain the desired result. □

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