Honam Mathematical J. **46** (2024), No. 1, pp. 114–119
https://doi.org/10.5831/HMJ.2024.46.1.114

VECTORIAL LINEAR CONNECTIONS WITH PARALLEL TORSION

Hwajeong Kim

Abstract. In this article, we consider a connection of vectorial type in some sense which is not a metric connection. We will then discuss when this connection $\tilde{\nabla}^*$ has parallel torsion T, that is, $\tilde{\nabla}^* T = 0$.

1. Introduction

On a Riemannian manifold M equipped a metric tensor g, we consider a connection ∇ satisfying some basic linearity which actually enables the parallel transport between tangent spaces. In particular, a connection which preserves the metric tensor g, that is, $\nabla g = 0$ is called a metric connection. The Levi-Civita connection, denoted by ∇^g, is then the well known unique metric connection without torsion.

Recently, in developing general theories, a metric connection whose torsion is not necessarily zero is considered. Actually, a connection with total skew-symmetric torsion has become a tool in recent years for holonomy theory in mathematics and superstring theory in physics, as in the papers by physicists such as Strominger, Luest, Theisen from the 1980's, and Alexandro and Schnee.

We can require a further condition that the torsion T be parallel with respect to the connection ∇, that is, $\nabla T = 0$. Indeed, there are many examples of parallel torsions which have interesting geometric structures, so these parallel torsions are considered natural generalizations of the Levi-Civita connection. ([1, 3, 10, 12]).

A metric connection ∇^V of vectorial type, which is determined by a fixed vector field V, can be expressed as follows:

$$\nabla^V_X Y = \nabla^g_X Y + g(X,Y)V - g(V,Y)X.$$

It is known that the condition $\nabla V = 0$ guarantees that this vectorial metric connection has parallel torsion. Furthermore, the converse holds in the case of the so-called canonical connection.

Received June 30, 2023. Accepted August 16, 2023.
2020 Mathematics Subject Classification. Primary 53A35, 53B05, 53B15, 53C05, 53C80.
Key words and phrases. metric connection, parallel torsion, dual connection, Weyl connection.
This work was financially supported by Hannam University 2023

In this article, we will consider another type of vectorial connection, that is, the dual connection of the Weyl connection $\tilde{\nabla}^*$, defined by

$$\tilde{\nabla}^*_X Y = \nabla^g_X Y + g(X,S)Y - g(Y,S)X + g(X,Y)S$$

for a fixed vector field S.

This connection $\tilde{\nabla}^*$ is not a metric connection and has non-zero torsion. It turns out that the parallel condition $\tilde{\nabla}^* S = 0$ does not give a parallel torsion, differently from the case of the vectorial metric connection.

In the main Theorem 3.1, we will then find an equivalent condition of the vector field S whose associated connection $\tilde{\nabla}^*$ has parallel torsion.

2. Preliminaries

Given a Riemannian manifold (M,g), let $\Gamma(M)$ denote the set of sections of the tangent bundle TM. Then a linear connection ∇ can be considered as a map

$$\nabla : \Gamma(M) \otimes \Gamma(M) \to \Gamma(M).$$

The torsion T of a connection ∇ is then a $(2,1)$-tensor field defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

where $[X,Y]$ is the Lie-bracket.

A connection ∇ satisfying $\nabla g = 0$ is called a metric connection which gives isometries between tangent spaces by parallel transport, that is,

$$V(g(X,Y)) = g(\nabla_V X, Y) + g(X, \nabla_V Y), \tag{1}$$

for any tangent vector fields $V, X, Y \in \Gamma(M)$. Here we recall that for $(2,0)$-tensor field g, we have:

$$(\nabla_V g)(X,Y) = V(g(X,Y)) - g(\nabla_V X, Y) - g(X, \nabla_V Y).$$

The Levi-Civita connection, denoted by ∇^g, is the unique metric connection with $T = 0$.

The difference of a linear connection ∇ with the Levi-Civita connection ∇^g is a $(2,1)$-tensor field denoted by A, that is,

$$\nabla_X Y = \nabla^g_X Y + A(X,Y).$$

Then some geometric properties of the connection ∇ are induced from the symmetric or antisymmetric properties of A.

Using the difference tensor $A(X,Y)$, the types of the torsion tensors of metric connections are classified algebraically. For details, we refer to [6, 12].

We now consider the dual connection ∇^* of ∇. This notion of dual connections is introduced by Norden, Nagaoka, and Amari ([2, 9, 11]).

Definition 2.1 (dual connections). *For a linear connection ∇, the dual connection ∇^* of ∇ with respect to g is defined by the identity for any vector fields X, Y, Z:*

$$Z\langle X, Y\rangle_g = \langle \nabla_Z X, Y\rangle_g + \langle X, \nabla^*_Z Y\rangle_g.$$

Using the expression of the difference between a linear connection ∇ and the Levi-Civita connection ∇^g, we have

$$\nabla_X Y = \nabla^g_X Y + A(X, Y),$$

where A is a $(2,1)$-tensor field. And the notation A will also be used for the $(3,0)$-tensor defined by

$$A(X, Y, Z) = \langle A(X, Y), Z\rangle.$$

Furthermore, let

$$\nabla_X Y = \nabla^g + A(X, Y), \tag{2}$$

$$\nabla^*_X Y = \nabla^g + A^*(X, Y). \tag{3}$$

We can then easily check that a linear connection has a unique dual connection as follows.

Remark 2.2. *For a linear connection ∇ and its dual connection ∇^* as above (2), (3), it holds*

$$A(Z, X, Y) + A^*(Z, Y, X) = 0. \tag{4}$$

3. Vectorial connections

Given a metric connection $\nabla = \nabla^g + A$, it is easily checked from (1) that ∇ is a metric connection if and only if the $(3,0)$-tensor field A is in $\mathcal{A}^m$, where

$$\mathcal{A}^m = TM \otimes \Lambda^2 TM = \{A \in \otimes^3 TM | A(X, Y, Z) = -A(X, Z, Y)\}. \tag{5}$$

Furthermore, it is known that the space $\mathcal{A}^m$ splits into the sum of three irreducible components under the action of $O(n)$, that is,

$$\mathcal{A}^m = TM \oplus \Lambda^3(TM) \oplus \mathcal{A}', \tag{6}$$

for some subspace $\mathcal{A}'$ as the third irreducible representation([6, 12]).

In particular, the type corresponding to the first component of (6) is expressed by

$$A(X, Y) = g(X, Y)V - g(V, Y)X, \tag{7}$$

where V is a fixed vector field on M. It is known that for $\nabla V = 0$, the torsion of the connection ∇ is parallel, that is, $\nabla T = 0$.

We now consider another type of vectorial connection, the so-called Weyl connection. This Weyl connection ∇^w is the unique connection which preserves a given conformal structure, known to be constructed using a fixed vector field S, that is,

$$\nabla^w_X Y = \nabla^g_X Y - g(X,S)Y - g(Y,S)X + g(X,Y)S.$$

For details, we refer to [7].

From (4), the dual connection of Weyl connection, denoted by $\tilde{\nabla}^*$, is then expressed by

$$\tilde{\nabla}^* = \nabla^g + A \tag{8}$$

with the $(3,0)$-tensor field A, as follows:

$$A(X,Y,Z) = g(X,S)g(Y,Z) - g(Y,S)g(X,Z) + g(X,Y)g(S,Z). \tag{9}$$

We will now discuss when the connection $\tilde{\nabla}^*$ has parallel torsion.

Theorem 3.1. *The connection $\tilde{\nabla}^*$ as above (8), the dual connection of the Weyl connection, has parallel torsion if and only if the vector field S satisfies*

$$g(\nabla^g_W S, X) = 6g(W,S)g(X,S) + g(S,S)g(W,X) \tag{10}$$

for all $W, X, Y, Z \in \Gamma(M)$.

Proof. The torsion of $\tilde{\nabla}^*$, denoted by T, is then

$$\begin{aligned} T(X,Y,Z) &= A(X,Y,Z) - A(Y,X,Z) \\ &= g(X,2S)g(Y,Z) + g(Y,-2S)g(X,Z). \end{aligned} \tag{11}$$

For $X, Y, Z, U \in \Gamma(M)$, it holds

$$\begin{aligned} & W[g(X,U)g(Y,Z)] \\ &= W[g(X,U)]g(Y,Z) + g(X,U)W[g(Y,Z)] \\ &= \{g(\nabla^g_W X,U) + g(X,\nabla^g_W U)\}g(Y,Z) \\ &\qquad + g(X,U)\{g(\nabla^g_W Y,Z) + g(Y,\nabla^g_W Z)\} \\ &= \{g(\tilde{\nabla}^*_W X,U) - A(W,X,U) + g(X,\tilde{\nabla}^*_W U) - A(W,U,X)\}g(Y,Z) \\ &\qquad + g(X,U)\{g(\tilde{\nabla}^*_W Y,Z) - A(W,Y,Z) + g(Y,\tilde{\nabla}^*_W Z) - A(W,Z,Y)\}. \end{aligned}$$

So, we obtain

$$\begin{aligned} & W[T(X,Y,Z)] \\ &= W[g(X,2S)g(Y,Z) - g(Y,2S)g(X,Z)] \\ &= \{g(\tilde{\nabla}^*_W X,2S) - A(W,X,2S) + g(X,\tilde{\nabla}^*_W 2S) - A(W,V,X)\}g(Y,Z) \\ &\qquad + g(X,2S)\{g(\tilde{\nabla}^*_W Y,Z) - A(W,Y,Z) + g(Y,\tilde{\nabla}^*_W Z) - A(W,Z,Y)\} \\ &\quad - \{g(\tilde{\nabla}^*_W Y,2S) - A(W,Y,2S) + g(Y,\tilde{\nabla}^*_W 2S) - A(W,2S,Y)\}g(X,Z) \\ &\qquad - g(Y,2S)\{g(\tilde{\nabla}^*_W X,Z) - A(W,X,Z) + g(X,\tilde{\nabla}^*_W Z) - A(W,Z,X)\}. \end{aligned}$$

Now from (11) we have

$$\begin{aligned} & T(\tilde{\nabla}^*_W X, Y, Z) + T(X, \tilde{\nabla}^*_W Y, Z) + T(X, Y, \tilde{\nabla}^*_W Z) \\ = \ & g(\tilde{\nabla}^*_W X, 2S) g(Y, Z) - g(Y, 2S) g(\tilde{\nabla}^*_W X, Z) \\ & + g(X, 2S) g(\tilde{\nabla}^*_W Y, Z) - g(\tilde{\nabla}^*_W Y, 2S) g(X, Z) \\ & + g(X, 2S) g(Y, \tilde{\nabla}^*_W Z) - g(Y, 2S) g(X, \tilde{\nabla}^*_W Z). \end{aligned}$$

The condition $\tilde{\nabla}^* T = 0$ means

$$W(T(X, Y, Z)) = T(\tilde{\nabla}^*_W X, Y, Z) + T(X, \tilde{\nabla}^*_W Y, Z) + T(X, Y, \tilde{\nabla}^*_W Z).$$

So, comparing the above equations we obtain

$$\begin{aligned} & \{g(X, \tilde{\nabla}^*_W 2S) - A(W, X, 2S) - A(W, 2S, X)\} g(Y, Z) \\ & \qquad\qquad - \{A(W, Y, Z) + A(W, Z, Y)\} g(X, 2S) \\ = \ & \{g(Y, \tilde{\nabla}^*_W 2S) - A(W, Y, 2S) - A(W, 2S, Y)\} g(X, Z) \\ & \qquad\qquad - \{A(W, X, Z) + A(W, Z, X)\} g(Y, 2S). \end{aligned}$$

By (9), the condition $\tilde{\nabla}^* T = 0$ is equivalent to that the following holds:

$$\begin{aligned} & \{g(X, \tilde{\nabla}^*_W S) - 8g(W, S) g(X, S)\} g(Y, Z) \\ & \qquad = \{g(Y, \tilde{\nabla}^*_W S) - 8g(W, S) g(Y, S)\} g(X, Z), \end{aligned} \tag{12}$$

for all $W, X, Y, Z \in \Gamma(M)$.

If we choose $Z = Y$ of length 1 which is perpendicular to X, (12) simplifies in the form:

$$g(X, \tilde{\nabla}^*_W S) = 8g(W, S) g(X, S), \qquad \text{for all } W, X \in \Gamma(M). \tag{13}$$

We note here that the property (13) also implies (12).

Finally, since $g(\tilde{\nabla}^*_X Y, Z) = g(\nabla^g_X Y, Z) + A(X, Y, Z)$ with

$$A(X, Y, Z) = g(X, S) g(Y, Z) - g(Y, S) g(X, Z) + g(X, Y) g(S, Z),$$

the equation (13) gives

$$g(\nabla^g_W S, X) = 6g(W, S) g(X, S) + g(S, S) g(W, X).$$

□

Remark 3.2. *(i) The connection $\tilde{\nabla}^*$ is non-metric because the difference tensor $A = \tilde{\nabla}^* - \nabla^g$ is not in $\mathcal{A}^m$ which is defined by (5).*

(ii) In local coordinates system for an orthonormal basis $\{e_i\}$, (10) implies differential equations for S:

$$g(\nabla^g_{e_i} S, e_j) = 6g(e_i, S) g(e_j, S) + g(S, S) g(e_i, e_j).$$

(iii) Contrary to the case of metric connection of vectorial type, $\tilde{\nabla}^*$-parallel S does not give parallel torsion for the $\tilde{\nabla}^*$. This is because, by (12), for parallel torsion of $\tilde{\nabla}^*$ it must hold that for all X, Y, Z

$$g(W, S) g(X, S) g(Y, Z) = 8g(W, S) g(Y, S) g(X, Z),$$

but this cannot be satisfied for $W = Y = S$, $X \perp S$ *and* $X = Z$.

References

[1] I. Agricola, J. Becker-Bender, and H. Kim *Twistor eigenvalue estimates for generalized Dirac operators with torsions*, Adv. Math. **243** (2013), 296–329.

[2] S. Amari, *Information Geometry and its Applications*, Applied Mathematical Sciences. Springer Japan, 2016.

[3] W. Ambrose and I. M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669.

[4] A. M. Blaga and A. Nannicini, *Conformal-projective transformations on statistical and semi-Weyl manifolds with torsion*, preprint (2022); arXiv.2209.00689vl.

[5] O. Calin and C. Udriste, *Geometric Modeling in Probability and Statistics*, Springer, 2013.

[6] E. Cartan, *Sur les varietes a connexion affine et la theorie de la relativite generalisee*, Ann. Ec. Norm. Sup. **42** (1925), 17–88; English transl. by Magnon and A. Ashtekar, *On manifolds with an affine conneciton and the theory of general relativity*, Napoli: Bibliopolis, 1986.

[7] G. B. Folland, *Weyl Manifolds*, J. Diff. Geometry **4** (1970), 145–153.

[8] H. Nagaoka and S. Amari, *Differential geometry of smooth families of probability distributions*, Technical Report (METR) 82-7, Dept. of Math. Eng. and Instr. Phys., Univ. of Tokyo, 1982.

[9] A. P. Norden, *On pairs of conjugate parallel displacements in multidimensional spaces*, In Doklady Akademii nauk SSSR **49** (1945), 1345–1347.

[10] N. Schoemann, *Almost Hermitian structures with parallel torsion*, J. Geom. Phys. **57** (2007), 2187–2212.

[11] U. Simon, in *Affine Differential Geometry*, Handbook of Differential Geometry, vol. I (ed. by F. Dillen, L. Verstraelen), 905–961, Elsevier Science, Amaterdam, 2000.

[12] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes Series, Vol. **83**, 1983.

Hwajeong Kim
Department of Mathematics, Hannam University,
Daejeon 306-791, Korea.
E-mail: hwajkim@hnu.ac.kr