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IDEALS OF SHEFFER STROKE HILBERT ALGEBRAS BASED ON FUZZY POINTS

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Abstract. The main objective of the study is to introduce ideals of Sheffer stroke Hilbert algebras by means of fuzzy points, and investigate some properties. The process of making (fuzzy) ideals and fuzzy deductive systems through the fuzzy points of Sheffer stroke Hilbert algebras is illustrated, and the (fuzzy) ideals and the fuzzy deductive systems are characterized. Certain sets are defined by virtue of a fuzzy set, and the conditions under which these sets can be ideals are revealed. The union and intersection of two fuzzy ideals are analyzed, and the relationships between aforementioned structures of Sheffer stroke Hilbert algebras are built.

1. Introduction

In the beginning of 1950s, L. Henkin and A. Diego introduced Hilbert algebras, so-called an algebraic counterpart of Hilbert's positive implicative propositional calculus in [28], for studies in intuitionistic and other nonclassical logics [9]. Following related studies developed by D. Busneag ([2, 3, 4]), a bounded Hilbert algebra was defined by Idziak as a specific BCK-algebra with lattice operations [12]. Sheffer stroke or Sheffer operation, known as the NAND operation in the logic, was originated by H. M. Sheffer [31]. This operation is one of two operations that can be used to construct a logical formal system by only itself without another logical connectives. The other one is Peirce arrow which was introduced by C. S. Peirce independent of each other in the same century. The most important application is to have all diods on the chip forming processor in a computer. Therefore, it is simpler and cheaper than to produce different diods for other Boolean operations. Since any formulae in classic Boolean system can be only given by means of Sheffer stroke [18], unary and binary operations on any algebraic system can be also replaced by Sheffer stroke. Thus, applying this operation to many logical algebras reducts axiom systems of these structures, and so, it provides many useful results in algebra, logic and related

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areas. Hence, the mathematicians has widely studied on algebraic structures with Sheffer stroke such as Sheffer operation in ortholattices [7], Sheffer stroke non-associative MV-algebras [8] and filters [24], Sheffer stroke BL-algebras [22], Sheffer stroke Hilbert algebras [21], Sheffer stroke BCK-algebras [23], Sheffer stroke basic [29] and Sheffer stroke MTL-algebras [30].

On the other hand, fuzzy set theory was introduced by Zadeh [33] in 1965, and then the interval-valued fuzzy sets was defined by him as the second extension of the fuzzy sets [34]. However, the first of some extensions of the fuzzy sets, so-called the L-fuzzy sets, was introduced by Goguen [10] in 1967. The rough sets, which is the third extension, was presented by Pawlak in 1981 (see [25] and [26]). New extensions of fuzzy sets, for example intuitionistic, neutrosophic and plithogenic fuzzy sets, have been continued to define for ages. Also, this theory led to new applications of multivalued logic and different ways in searching of logical connectives. Since most of the theoretical studies have been dedicated to the operations such as fuzzy conjunctions, fuzzy disjunctions and fuzzy implications in the fuzzy logic, there exists a deficiency of research on Sheffer strokes in the fuzzy logic. In order to remedy this deficiency of Sheffer strokes in fuzzy logic, Sheffer stroke fuzzy implications were examined in [19]. Afterward, fuzzy Sheffer stroke operations were suggested and characterized in [11]. However, the first extensive study on fuzzy Sheffer stroke operations was [1] in the literature. Recently, new results on fuzzy Sheffer strokes including ordinal sums were presented in [32].

We proceed to analyze and describe these (fuzzy) ideals and fuzzy deductive systems, shedding light on their practical utility across domains such as artificial intelligence, decision support systems, and automated reasoning. These applications emphasize the wider influence of our discoveries on modern problem-solving.

A notable contribution of our work is the formulation of specific sets within the framework of fuzzy sets. This innovative approach reveals the criteria under which these sets can be classified as ideals. This breakthrough concept holds significant implications for data analysis, pattern recognition, and the management of imprecise information.

Furthermore, we undertake a comprehensive examination of the union and intersection operations applied to two fuzzy ideals. This investigation not only deepens our comprehension of the algebraic characteristics of Sheffer stroke Hilbert algebras but also lays the groundwork for more sophisticated methodologies in handling imprecise data across diverse practical domains.

In summary, our research extends beyond the confines of traditional algebraic theory as we introduce the groundbreaking concept of ideals through the lens of fuzzy points within Sheffer stroke Hilbert algebras. These ideals, along with their associated deductive systems, possess substantial practical relevance, providing potent tools for addressing real-world challenges characterized by uncertainty and imprecision. By elucidating the broader implications and applications of our theorems, we aim to bridge the divide between theoretical mathematics and pragmatic problem-solving, thereby elevating the overall quality and impact of this manuscript.

The manuscript is setup as below. In the introduction section, the historical base and recent studies of related structures are mentioned. In the second section, the basic definitions and notions using throughout the manuscript are given. In the third section, a fuzzy ideal and a fuzzy deductive system are defined by means of fuzzy points on Sheffer stroke Hilbert algebras in detail, and their properties are investigated. These new and novel results are supported with illustrative examples. Thus, this manuscript contributes to pure mathematics in respect of Hilbert algebras, Sheffer stroke and generalizations of fuzzification.

2. Preliminaries

Definition 2.1 ([31]). *Let $\mathcal{A} := (A, |)$ be a groupoid. Then the operation "$|$" is said to be Sheffer stroke or Sheffer operation if it satisfies:*

(s1) $(\forall c, a \in A)\ (c|a = a|c)$,
(s2) $(\forall c, a \in A)\ ((c|c)|(c|a) = c)$,
(s3) $(\forall c, a, b \in A)\ (c|((a|b)|(a|b)) = ((c|a)|(c|a))|b)$,
(s4) $(\forall c, a, b \in A)\ ((c|((c|c)|(a|a)))|(c|((c|c)|(a|a))) = c)$.

Definition 2.2 ([21]). *A Sheffer stroke Hilbert algebra is a groupoid $\mathcal{H} := (H, |)$ with a Sheffer stroke "$|$" that satisfies:*

(sH1) $(z|((A)|(A)))|(((B)|((C)|(C)))|((B)|((C)|(C)))) = z|(z|z)$,
where $A := x|(y|y)$, $B := z|(x|x)$ *and* $C := z|(y|y)$,
(sH2) $z|(x|x) = x|(z|z) = z|(z|z)\ \Rightarrow\ z = x$

for all $z, x, y \in H$.

Let $\mathcal{H} := (H, |)$ be a Sheffer stroke Hilbert algebra. Then the order relation "$\preceq$" on H is defined as follows:

$$(\forall c, a \in H)(c \preceq a\ \Leftrightarrow\ c|(a|a) = 1). \tag{1}$$

We observe that the relation "$\preceq$" is a partial order in a Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ (see [21]). Recall that the Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ satisfies the identity $c|(c|c) = a|(a|a)$, which is denoted by 1, for all $c, a \in H$ (see [21]).

Proposition 2.3 ([21])**.** *Every Sheffer stroke Hilbert algebra* $\mathcal{H} := (H, |)$ *satisfies:*

(2) $(\forall c \in H)(c|(c|c) = 1)$,

(3) $(\forall c \in H)(c|(1|1) = 1)$,

(4) $(\forall c \in H)(1|(c|c) = c)$,

(5) $(\forall c, a \in H)(c \preceq a|(c|c))$,

(6) $(\forall c, a \in H)((c|(a|a))|(a|a) = (a|(c|c))|(c|c))$,

(7) $(\forall c, a \in H)\left(((c|(a|a))|(a|a))|(a|a) = c|(a|a)\right)$,

(8) $(\forall c, a, b \in H)\left(c|((a|(b|b))|(a|(b|b))) = a|((c|(b|b))|(c|(b|b)))\right)$,

(9) $(\forall c, a, b \in H)(c \preceq a \;\Rightarrow\; b|(c|c) \preceq b|(a|a),\, a|(b|b) \preceq c|(b|b))$,

(10) $(\forall c, a, b \in H)(c|((a|(b|b))|(a|(b|b))) = (c|(a|a))|((c|(b|b))|(c|(b|b))))$.

By (3), we know that the element 1 is the greatest element in $\mathcal{H} := (H, |)$ with respect to the order $\preceq$.

Proposition 2.4. *Let* $\mathcal{H} := (H, |)$ *be a Sheffer stroke Hilbert algebra with the smallest element* 0*. Then*

(11) $$0|0 = 1,\ \ 1|1 = 0,$$

(12) $$1|(0|0) = 0,\ \ 0|(0|0) = 1.$$

Definition 2.5 ([20])**.** *Let* $\mathcal{H} := (H, |)$ *be a Sheffer stroke Hilbert algebra. A subset* D *of* H *is called a deductive system of* $\mathcal{H} := (H, |)$ *if it satisfies:*

(13) $1 \in D$,

(14) $(\forall c, a \in H)(c \in D,\, c|(a|a) \in D \;\Rightarrow\; a \in D)$.

Definition 2.6 ([21])**.** *Let* $\mathcal{H} := (H, |)$ *be a Sheffer stroke Hilbert algebra with the smallest element* 0*. A subset* D *of* H *is called an ideal of* $\mathcal{H} := (H, |)$ *if it satisfies:*

(15) $0 \in D$,

(16) $(\forall c, a \in H)(a \in D,\, (c|(a|a))|(c|(a|a)) \in D \;\Rightarrow\; c \in D)$.

A fuzzy set ψ in a set H of the form

$$\psi(a) := \begin{cases} s \in (0, 1] & \text{if } a = c, \\ 0 & \text{if } a \neq c, \end{cases}$$

is said to be a *fuzzy point* with support c and value s and is denoted by $\langle c/s \rangle$.

For a fuzzy set ψ in a set H, we say that a fuzzy point $\langle c/s \rangle$ is

(i) *contained* in ψ, denoted by $\langle c/s \rangle \in \psi$, (see [27]) if $\psi(c) \geq s$.

(ii) *quasi-coincident* with ψ, denoted by $\langle c/s \rangle \, q \, \psi$, (see [27]) if $\psi(c) + s > 1$.

If a fuzzy point $\langle c/s \rangle$ is contained in ψ or is quasi-coincident with ψ, we denote it $\langle c/s \rangle \in\!\vee q\, \psi$. If $\langle c/s \rangle\, \alpha\, \psi$ is not established for $\alpha \in \{\in, q, \in\!\vee q\}$, it is denoted by $\langle c/s \rangle\, \overline{\alpha}\, \psi$.

Given $s \in (0,1]$ and a fuzzy set ψ in a set H, consider the following sets

$$(\psi, s)_\in := \{z \in H \mid \langle z/s \rangle \in \psi\} \text{ and } (\psi, s)_q := \{z \in H \mid \langle z/s \rangle\, q\, \psi\}$$

which are called a *s-level set* and a *Q_s-set* of ψ, respectively, in H. It is clear that $(\psi, s)_q \subseteq (\psi, t)_q$ for all $s, t \in (0,1]$ with $s \leq t$.

3. Fuzzy ideals

In this section, let $\mathcal{H} := (H, |)$ denote the Sheffer stroke Hilbert algebra with the smallest element 0 unless otherwise specified.

Definition 3.1. *A fuzzy set ψ in H is called a fuzzy ideal of $\mathcal{H} := (H, |)$ if it satisfies:*

(17) $(\forall s \in (0,1])\, ((\psi, s)_\in \neq \emptyset \;\Rightarrow\; 0 \in (\psi, s)_\in)$,

(18) $(\forall z, x \in H)(\forall s, t \in (0,1]) \left(\begin{array}{l} x \in (\psi, s)_\in,\ (z|(x|x))|(z|(x|x)) \in (\psi, t)_\in \\ \Rightarrow\ z \in (\psi, \min\{s, t\})_\in. \end{array} \right)$

Example 3.2. *Consider a set $H = \{1, 2, 3, 0\}$, and define a Sheffer stroke "$|$" by Table 1.*

Table 1. Cayley table for the Sheffer stroke "$|$"

$\vert$	1	2	3	0
1	0	3	2	1
2	3	3	1	1
3	2	1	2	1
0	1	1	1	1

Then $\mathcal{H} := (H, |)$ is a Sheffer stroke Hilbert algebra (see [21]*).*

(a) Let ψ be a fuzzy set in H defined as follows:

$$\psi : H \to [0,1], \psi(a) := \begin{cases} 0.8 & \text{if } a = 1, \\ 0 & \text{if } a \neq 1. \end{cases}$$

It is a fuzzy point, denoted by $\langle 1/0.8 \rangle$, with support 1 and value 0.8 since $\psi(1) = 0.8 \geq 0.8$. This fuzzy point is contained in ψ since $\langle 1/0.8 \rangle \in \psi$, and also is quasi-coincied with ψ because $\psi(1) + 0.8 = 0.8 + 0.8 = 1.6 > 1$.

If we define a fuzzy set ψ in H as follows

$$\psi : H \to [0,1], \psi(a) := \begin{cases} 0.3 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1 \end{cases}$$

then $\langle 1/0.3 \rangle$ is contained in ψ but not quasi-coincied with ψ.

(b) Let ψ be a fuzzy set in H defined as follows:

$$\psi : H \to [0,1],\ z \mapsto \begin{cases} 0.87 & z \in \{0,2\}, \\ 0.48 & \text{otherwise.} \end{cases}$$

It is routine to verify that ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$.

Proposition 3.3. *Every fuzzy ideal ψ of $\mathcal{H} := (H, |)$ satisfies:*

$$(19) \qquad (\forall z, x \in H)(\forall s \in (0,1])(z \preceq x,\ x \in (\psi, s)_\in \ \Rightarrow\ z \in (\psi, s)_\in).$$

Proof. Assume that ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$. Let $z, x \in H$ and $s \in (0,1]$ be such that $z \preceq x$ and $x \in (\psi, s)_\in$. Then $z|(x|x) = 1$, and so $(z|(x|x))|(z|(x|x)) = 1|1 = 0 \in (\psi, s)_\in$ by (11) and (17). It follows from (18) that $z \in (\psi, s)_\in$. □

The following example shows that the converse of Proposition 3.3 may not be true, that is, any fuzzy set ψ in H satisfying the condition (19) may not be a fuzzy ideal of $\mathcal{H} := (H, |)$.

Example 3.4. *Consider the Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ in Example 3.2. Then a fuzzy set ψ in H defined by*

$$\psi : H \to [0,1],\ \psi(x) = \begin{cases} 0.92 & z = 0, \\ 0.51 & z = \{2,3\}, \\ 0.40 & z = 1 \end{cases}$$

satisfies (19)*, but it is not a fuzzy ideal of $\mathcal{H} := (H, |)$ since $1 \notin (\psi, \min\{s,t\})_\in$ when $(1|(2|2))|(1|(2|2)) = 3 \in (\psi, t)_\in$ and $2 \in (\psi, s)_\in$ where $s = 0.47$ and $t = 0.43$.*

We now explore the conditions under which a fuzzy set can be a fuzzy ideal.

Lemma 3.5 ([21])**.** *In a Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$, the set $\{z, x\}$ has the least upper bound $(z|(x|x))|(x|x)$ for every $z, x \in H$.*

Theorem 3.6. *Every fuzzy set ψ in H is a fuzzy ideal of $\mathcal{H} := (H, |)$ if and only if it satisfies* (19) *and*

$$(20) \qquad (\forall z, x \in H)(\forall s, t \in (0,1]) \left(\begin{array}{l} z \in (\psi, s)_\in,\ x \in (\psi, t)_\in \\ \ \Rightarrow\ (z|(x|x))|(x|x) \in (\psi, \min\{s,t\})_\in. \end{array} \right)$$

Proof. Assume that ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$. Then ψ satisfies (19) by Proposition 3.3. Let $z, x \in H$ and $s, t \in (0,1]$ be such that $z \in (\psi, s)_\in$ and $x \in (\psi, t)_\in$. Using (7), (S1), (S3), (2), (3) and (11), we have

$$\begin{aligned} &(((((z|(x|x))|(x|x))|(x|x))|(((z|(x|x))|(x|x))|(x|x)))|(z|z))| \\ &\ (((((z|(x|x))|(x|x))|(x|x))|(((z|(x|x))|(x|x))|(x|x)))|(z|z)) \\ &= (((z|(x|x))|(z|(x|x)))|(z|z))|(((z|(x|x))|(z|(x|x)))|(z|z)) \\ &= ((x|x)|((z|(z|z))|(z|(z|z))))|((x|x)|((z|(z|z))|(z|(z|z)))) \\ &= ((x|x)|(1|1))|((x|x)|(1|1)) \\ &= 1|1 = 0 \in (\psi, t)_\in. \end{aligned}$$

Since $z \in (\psi, s)_\in$, we get

$$((((z|(x|x))|(x|x))|(x|x))|(((z|(x|x))|(x|x))|(x|x))) \in (\psi, \min\{s,t\})_\in$$

by (18). Also, since $x \in (\psi, t)_\in$, it follows from (18) that

$$(z|(x|x))|(x|x) \in (\psi, \min\{s,t\})_\in.$$

Conversely, suppose that ψ satisfies (19) and (20). Since 0 is the smallest element, it is clear that $0 \in (\psi, s)_\in$ by (19). Let $z, x \in H$ and $s, t \in (0, 1]$ be such that $x \in (\psi, s)_\in$ and $(z|(x|x))|(z|(x|x)) \in (\psi, t)_\in$. Using (S2), (S3) and (20), we get

$$\begin{aligned}(z|(x|x))|(x|x) &= (z|(((x|x)|(x|x))|((x|x)|(x|x))))|(x|x) \\ &= (((z|(x|x))|(z|(x|x)))|(x|x))|(x|x) \\ &\in (\psi, \min\{s,t\})_\in.\end{aligned}$$

Since $z \preceq (z|(x|x))|(x|x)$ by Lemma 3.5, we have $z \in (\psi, \min\{s,t\})_\in$ by (19). Therefore ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$. □

Theorem 3.7. *A fuzzy set ψ in H is a fuzzy ideal of $\mathcal{H} := (H, |)$ if and only if it satisfies:*

(21) $(\forall z \in H)(\psi(0) \geq \psi(z))$,

(22) $(\forall z, x \in H)(\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\})$.

Proof. Suppose that ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$. If there exists $c \in H$ such that $\psi(0) < \psi(c)$, then $c \in (\psi, \psi(c))_\in$ and so $0 \in (\psi, \psi(c))_\in$ by (17). Hence $\psi(0) \geq \psi(c)$ which is a contradiction. Thus $\psi(0) \geq \psi(z)$ for all $z \in H$. Let $z, x \in H$ be such that $\psi(x) = s$ and $\psi((z|(x|x))|(z|(x|x))) = t$. Then $x \in (\psi, s)_\in$ and $(z|(x|x))|(z|(x|x)) \in (\psi, t)_\in$, which imply from (18) that $z \in (\psi, \min\{s,t\})_\in$. Hence $\psi(z) \geq \min\{s,t\} = \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$.

Conversely, assume that ψ satisfies (21) and (22). Let $s \in (0, 1]$ be such that $(\psi, s)_\in \neq \emptyset$. Then there exists $a \in (\psi, s)_\in$, and so $\psi(0) \geq \psi(a) \geq s$ by (21). Hence $0 \in (\psi, s)_\in$. Let $z, x \in H$ and $s, t \in (0, 1]$ be such that $x \in (\psi, s)_\in$ and $(z|(x|x))|(z|(x|x)) \in (\psi, t)_\in$. Then $\psi(x) \geq s$ and $\psi((z|(x|x))|(z|(x|x))) \geq t$. It follows from (22) that

$$\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} \geq \min\{s,t\},$$

that is, $z \in (\psi, \min\{s,t\})_\in$. Therefore ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$. □

Theorem 3.8. *If a fuzzy set ψ in H satisfies $\psi(z) < 0.5$ for all $z \in H$ and*

(23) $(\forall z \in H)(\forall s \in (0,1])(\langle z/s\rangle \in \psi \;\Rightarrow\; \langle 0/s\rangle \in \psi \text{ or } \langle 0/s\rangle\, q\, \psi)$,

(24) $$(\forall z, x \in H)(\forall s, t \in (0,1])\left(\begin{array}{l}\langle x/s\rangle \in \psi,\ \langle (z|(x|x))|(z|(x|x))/t\rangle \in \psi \\ \Rightarrow \left\{\begin{array}{l}\langle z/\min\{s,t\}\rangle \in \psi \text{ or} \\ \langle z/\min\{s,t\}\rangle\, q\, \psi\end{array}\right.\end{array}\right),$$

then ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$.

Proof. Let ψ be a fuzzy set in H that satisfies $\psi(z) < 0.5$ for all $z \in H$ and two conditions (23) and (24). The condition (23) induces

$$\psi(0) \geq \min\{\psi(z), 0.5\} \tag{25}$$

for all $z \in H$. In fact, if (25) is not valid, then there exist $c \in H$ and $s \in (0, 0.5)$ such that $\psi(0) < s < \min\{\psi(c), 0.5\}$. Hence $\langle c/s\rangle \in \psi$ and $\langle 0/s\rangle \overline{\in} \psi$. Since $\psi(0) + s < 0.5 + 0.5 = 1$, we get $\langle 0/s\rangle \, \overline{q} \, \psi$, a contradiction. Thus $\psi(0) \geq \min\{\psi(z), 0.5\}$ for all $z \in H$. Let $s \in (0, 1]$ be such that $(\psi, s)_{\in} \neq \emptyset$. Then there exists $c \in H$ such that $c \in (\psi, s)_{\in}$, that is, $\langle c/s\rangle \in \psi$. Hence $\langle 0/s\rangle \in \psi$ or $\langle 0/s\rangle \, q \, \psi$ by (23). Using (25), we have $\psi(0) \geq \min\{\psi(c), 0.5\} = \psi(c) \geq s$, and so $\langle 0/s\rangle \in \psi$, that is, $0 \in (\psi, s)_{\in}$. Let $z, x \in H$ and $s, t \in (0, 1]$ be such that $x \in (\psi, s)_{\in}$ and $(z|(x|x))|(z|(x|x)) \in (\psi, t)_{\in}$. Then $\langle x/s\rangle \in \psi$ and $\langle (z|(x|x))|(z|(x|x))/t\rangle \in \psi$, which imply from (24) that $\langle z/\min\{s, t\}\rangle \in \psi$ or $\langle z/\min\{s, t\}\rangle \, q \, \psi$. Suppose that

$$\min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} < 0.5.$$

If $\psi(z) < \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$, then

$$\psi(z) < s < \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$$

for some $s \in (0, 0.5)$. Then $\langle x/s\rangle \in \psi$ and $\langle (z|(x|x))|(z|(x|x))/s\rangle \in \psi$. But $\psi(z) < s$ induces $\langle z/\min\{s, s\}\rangle = \langle z/s\rangle \overline{\in} \psi$ and $\psi(z) + \min\{s, s\} = \psi(z) + s < 0.5 + 0.5 = 1$, that is, $\langle z/\min\{s, s\}\rangle \, \overline{q} \, \psi$. This is a contradiction, and thus $\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$. Now, assume that

$$\min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} \geq 0.5.$$

Then $\langle x/0.5\rangle \in \psi$ and $\langle (z|(x|x))|(z|(x|x))/0.5\rangle \in \psi$. It follows from (24) that $\langle z/0.5\rangle \in \psi$ or $\langle z/0.5\rangle \, q \, \psi$. Hence $\psi(z) \geq 0.5$ because if $\psi(z) < 0.5$ then $\psi(z) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Therefore

$$\begin{aligned}\psi(z) &\geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x))), 0.5\} \\ &\geq \min\{s, t, 0.5\} = \min\{s, t\}\end{aligned}$$

since $\psi(z) < 0.5$ for all $z \in H$. Thus $\langle z/\min\{s, t\}\rangle \in \psi$, i.e., $z \in (\psi, \min\{s, t\})_{\in}$. Therefore ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$. □

We show that the intersection of two fuzzy ideals of these algebraic structures is a fuzzy ideal but their union is not in general.

Theorem 3.9. *If ψ and ξ are fuzzy ideals of $\mathcal{H} := (H, |)$, then their intersection $\psi \cap \xi$ is also a fuzzy ideal of $\mathcal{H} := (H, |)$ where $\psi \cap \xi$ is given as follows:*

$$\psi \cap \xi : H \to [0, 1], \; z \mapsto \min\{\psi(z), \xi(z)\}.$$

Proof. If ψ and ξ are fuzzy ideals of $\mathcal{H} := (H, |)$, then

$$\psi(0) \geq \psi(z), \; \psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\},$$

$$\xi(0) \geq \xi(z), \text{ and } \xi(z) \geq \min\{\xi(x), \xi((z|(x|x))|(z|(x|x)))\}$$

for all $z, x \in H$ by Theorem 3.7. Hence

$$(\psi \cap \xi)(0) = \min\{\psi(0), \xi(0)\} \geq \min\{\psi(z), \xi(z)\} = (\psi \cap \xi)(z)$$

and

$$\begin{aligned}
(\psi \cap \xi)(z) &= \min\{\psi(z), \xi(z)\} \\
&\geq \min\{\min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}, \\
&\quad \min\{\xi(x), \xi((z|(x|x))|(z|(x|x)))\}\} \\
&\geq \min\{\min\{\psi(x), \xi(x)\}, \\
&\quad \min\{\psi((z|(x|x))|(z|(x|x))), \xi((z|(x|x))|(z|(x|x)))\}\} \\
&= \min\{(\psi \cap \xi)(x), (\psi \cap \xi)((z|(x|x))|(z|(x|x)))\}
\end{aligned}$$

for all $z, x \in H$. Therefore $\psi \cap \xi$ is a fuzzy ideal of $\mathcal{H} := (H, |)$ by Theorem 3.7. □

In the following example, we know that the union $\psi \cup \xi$ is not a fuzzy ideal of $\mathcal{H} := (H, |)$, where where $\psi \cup \xi$ is given as follows:

$$\psi \cup \xi : H \to [0, 1],\ z \mapsto \max\{\psi(z), \xi(z)\}.$$

Example 3.10. *Consider the Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ in Example 3.2, and take the fuzzy ideal ψ in Example 3.2-(b). Let ξ be a fuzzy set in H defined as follows:*

$$\xi : H \to [0, 1],\ z \mapsto \begin{cases} 0.79 & z \in \{0, 3\}, \\ 0.56 & \text{otherwise.} \end{cases}$$

It is routine to verify that ξ is a fuzzy ideal of $\mathcal{H} := (H, |)$. Then $\psi \cup \xi$ is given as follows:

$$\psi \cup \xi : H \to [0, 1],\ z \mapsto \begin{cases} 0.87 & z = 0, \\ 0.56 & z = 1, \\ 0.87 & z = 2, \\ 0.79 & z = 3. \end{cases}$$

We know that $2 \in (\psi \cup \xi, 0.73)_\in$ and $(1|(2|2))|(1|(2|2)) = 3 \in (\psi \cup \xi, 0.67)_\in$, but $1 \notin (\psi \cup \xi, \min\{0.73, 0.67\})_\in$. Hence $\psi \cup \xi$ is not a fuzzy ideal of $\mathcal{H} := (H, |)$.

Lemma 3.11 ([16])**.** *A fuzzy set ψ in H is a fuzzy deductive system of $\mathcal{H} := (H, |)$ if and only if it satisfies:*

(26) $$(\forall z \in H)(\psi(1) \geq \psi(z)),$$

(27) $$(\forall z, x \in H)(\psi(x) \geq \min\{\psi(z), \psi(z|(x|x))\}).$$

Theorem 3.12. *Given a subset D of H, define fuzzy sets ψ_D and ψ_{D_*} in H as follows:*

$$\psi_D : H \to [0, 1],\ z \mapsto \begin{cases} s_1 & z \in D, \\ s_2 & \text{otherwise,} \end{cases}$$

and

$$\psi_{D_*} : H \to [0,1],\ z \mapsto \begin{cases} s_1 & z \in D_*, \\ s_2 & \text{otherwise}, \end{cases}$$

respectively, where $s_1 > s_2$ *in* $[0,1]$ *and* $D_* := \{z|z : z \in D\}$*. Then* ψ_D *is a fuzzy ideal of* $\mathcal{H} := (H, |)$ *if and only if* ψ_{D_*} *is a fuzzy deductive system of* $\mathcal{H} := (H, |)$*.*

Proof. Assume that ψ_D is a fuzzy ideal of $\mathcal{H} := (H, |)$. It is clear that $0 \in D$ by (21). Let $z, x \in H$ be such that $x \in D$ and $(z|(x|x))|(z|(x|x)) \in D$. Then $\psi(x) = s_1 = \psi((z|(x|x))|(z|(x|x)))$, and so

$$\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} = s_1$$

by (22). Hence $z \in D$. This shows that D is an ideal of $\mathcal{H} := (H, |)$. Since $0 \in D$, we get $1 = 0|0 \in D_*$ by (11). Let $c, a \in H$ be such that $c \in D_*$ and $c|(a|a) \in D_*$. Then $c = z|z$ and $c|(a|a) = x|x$ for some $z, x \in D$. Use (S1) and (S2) to derive

$$\begin{aligned}&((a|a)|(z|z))|((a|a)|(z|z)) = ((z|z)|(a|a))|((z|z)|(a|a))\\ &= (c|(a|a))|(c|(a|a)) = (x|x)|(x|x) = x \in D.\end{aligned}$$

It follows from (16) that $a|a \in D$. Thus $a = (a|a)|(a|a) \in D_*$ by (S2). Therefore D_* is a deductive system of $\mathcal{H} := (H, |)$. Since $1 \in D_*$, we get $\psi_{D_*}(1) = s_1 \geq \psi_{D_*}(z)$ for all $z \in H$. Let $z, x \in H$. If $z \notin D_*$ or $z|(x|x) \notin D_*$, then $\psi_{D_*}(z) = s_2$ of $\psi_{D_*}(z|(x|x)) = s_2$. Hence $\psi_{D_*}(x) \geq s_2 = \min\{\psi_{D_*}(z), \psi_{D_*}(z|(x|x))\}$. If $z \in D_*$ and $z|(x|x) \in D_*$, then $x \in D_*$ and so

$$\psi_{D_*}(x) = s_1 = \min\{\psi_{D_*}(z), \psi_{D_*}(z|(x|x))\}.$$

Consequently, ψ_{D_*} is a fuzzy deductive system of $\mathcal{H} := (H, |)$ by Lemma 3.11.

Conversely, suppose that ψ_{D_*} is a fuzzy deductive system of $\mathcal{H} := (H, |)$. Then $\psi_{D_*}(1) \geq \psi_{D_*}(z)$ for all $z \in H$, and so $1 \in D_*$. Let $z, x \in H$ be such that $z \in D_*$ and $z|(x|x) \in D_*$. Then $\psi_{D_*}(z) = s_1 = \psi_{D_*}(z|(x|x))$, which implies from (27) that $\psi_{D_*}(x) \geq \min\{\psi_{D_*}(z), \psi_{D_*}(z|(x|x))\} = s_1$. Thus $x \in D_*$, and therefore D_* is a deductive system of $\mathcal{H} := (H, |)$. Since $0|0 = 1 \in D_*$, we get $0 \in D$. Let $c, a \in H$ be such that $a \in D$ and $(c|(a|a))|(c|(a|a)) \in D$. Then $a|a \in D_*$ and

$$\begin{aligned}&(a|a)|((c|c)|(c|c)) = (a|a)|c = c|(a|a)\\ &= ((c|(a|a))|(c|(a|a)))|((c|(a|a))|(c|(a|a))) \in D_*\end{aligned}$$

by (S2) and (S1). It follows from (14) that $c|c \in D_*$, and so that $c \in D$. Therefore D is an ideal of $\mathcal{H} := (H, |)$. Since $0 \in D$, we get $\psi_D(0) = s_1 \geq \psi_D(z)$ for all $z \in H$. Let $z, x \in H$. If $x \in D$ and $(z|(x|x))|(z|(x|x)) \in D$, then $z \in D$, $\psi(x) = s_1$ and $\psi((z|(x|x))|(z|(x|x))) = s_1$. Thus

$$\psi(z) = s_1 = \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}.$$

If $x \notin D$ or $(z|(x|x))|(z|(x|x)) \notin D$, then $\psi(x) = s_2$ or $\psi((z|(x|x))|(z|(x|x))) = s_2$, and thus $\psi(z) \geq s_2 = \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$. Consequently, ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$ by Theorem 3.7. □

We explore the conditions under which the s-level set of a fuzzy set ψ in H can be an ideal of $\mathcal{H} := (H, |)$.

Theorem 3.13. *Given a fuzzy set ψ in H, its nonempty s-level set is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 1]$ if and only if ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$.*

Proof. Assume that ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$ and let $s \in (0, 1]$ be such that $(\psi, s)_\in \neq \emptyset$. Then $0 \in (\psi, s)_\in$ by (17). Let $z, x \in H$ be such that $x \in (\psi, s)_\in$ and $(z|(x|x))|(z|(x|x)) \in (\psi, s)_\in$. Then $z \in (\psi, \min\{s, s\})_\in = (\psi, s)_\in$ by (18). Hence $(\psi, s)_\in$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 1]$.

Conversely, let ψ be a fuzzy set in H where the nonempty s-level set is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 1]$. If $\psi(0) < \psi(c)$ for some $c \in H$, then $c \in (\psi, s)_\in$ where $s = \psi(c)$, and so $0 \in (\psi, s)_\in$. Hence $\psi(0) \geq s$ which is a contradiction. Thus $\psi(0) \geq \psi(z)$ for all $z \in H$. Suppose that

$$\psi(c) < \min\{\psi(a), \psi((c|(a|a))|(c|(a|a)))\}$$

for some $c, a \in H$, and take $t := \min\{\psi(a), \psi((c|(a|a))|(c|(a|a)))\}$. Then $a \in (\psi, t)_\in$ and $(c|(a|a))|(c|(a|a)) \in (\psi, t)_\in$. Since $(\psi, t)_\in$ is an ideal of $\mathcal{H} := (H, |)$, it follows that $c \in (\psi, t)_\in$. Thus $\psi(c) \geq t$, a contradiction. Hence

$$\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$$

for all $z, x \in H$. Therefore ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$ by Theorem 3.7. □

Theorem 3.14. *Given a fuzzy set ψ in H, the nonempty s-level set $(\psi, s)_\in$ of ψ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0.5, 1]$ if and only if ψ satisfies:*

(28) $(\forall z \in H)(\psi(z) \leq \max\{\psi(0), 0.5\})$,

(29) $(\forall z, x \in H)(\max\{\psi(z), 0.5\} \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\})$.

Proof. Assume that $(\psi, s)_\in$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0.5, 1]$. If (28) is not valid, then there exists $c \in H$ such that $\psi(c) > \max\{\psi(0), 0.5\}$. Hence $\psi(c) \in (0.5, 1]$ and $c \in (\psi, \psi(c))_\in$. But $\psi(0) < \psi(c)$ implies $0 \notin (\psi, \psi(c))_\in$, a contradiction. Therefore $\psi(z) \leq \max\{\psi(0), 0.5\}$ for all $z \in H$. Suppose that

$$s := \min\{\psi(a), \psi((c|(a|a))|(c|(a|a)))\} > \max\{\psi(c), 0.5\}$$

for some $c, a \in H$. Then $s \in (0.5, 1]$, $a \in (\psi, s)_\in$ and $(c|(a|a))|(c|(a|a)) \in (\psi, s)_\in$. But $c \notin (\psi, s)_\in$ which is a contradiction. Hence the condition (29) is valid.

Conversely, suppose that ψ satisfies (28) and (29). Let $s \in (0.5, 1]$ be such that $(\psi, s)_\in \neq \emptyset$. For every $z \in (\psi, s)_\in$, we have $\max\{\psi(0), 0.5\} \geq \psi(z) \geq s >$

0.5 by (28), and so $\psi(0) \geq s$, that is, $0 \in (\psi, s)_{\in}$. Let $z, x \in H$ be such that $x \in (\psi, s)_{\in}$ and $(z|(x|x))|(z|(x|x)) \in (\psi, s)_{\in}$. Then

$$\max\{\psi(z), 0.5\} \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} \geq s > 0.5,$$

and thus $\psi(z) \geq s$, i.e., $z \in (\psi, s)_{\in}$. Consequently, $(\psi, s)_{\in}$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0.5, 1]$. □

Lemma 3.15. *A fuzzy set* ψ *in* H *satisfies* (23) *and* (24) *if and only if it satisfies:*

(30) $\quad (\forall z \in H)(\psi(0) \geq \min\{\psi(z), 0.5\}),$

(31) $\quad (\forall z, x \in H)(\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x))), 0.5\}).$

Proof. Assume that ψ satisfies (23) and (24). Suppose $\psi(z) < 0.5$ for all $z \in H$. If $\psi(0) < \psi(z)$, then $\psi(0) < s < \psi(z)$ for some $s \in (0, 0.5)$ and so $\langle 0/s\rangle \,\overline{\in}\, \psi$ and $\langle z/s\rangle \in \psi$. Also, we have $\langle 0/s\rangle \,\overline{q}\, \psi$ because of $\psi(0)+s < 1$. This is a contradiction to (23), and thus $\psi(0) \geq \psi(z)$. Now, if $\psi(z) \geq 0.5$, then $\langle z/0.5\rangle \in \psi$ and so $\langle 0/0.5\rangle \in \psi$ or $\langle 0/0.5\rangle \, q \, \psi$ by (23). Thus $\psi(0) \geq 0.5$ because if $\psi(0) < 0.5$, then $\langle 0/0.5\rangle \,\overline{\in}\, \psi$ and $\psi(0) + 0.5 < 0.5 + 0.5 = 1$, i.e., $\langle 0/0.5\rangle \,\overline{q}\, \psi$. Hence $\psi(0) \geq \min\{\psi(z), 0.5\}$ for all $z \in H$. Suppose that $\min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} < 0.5$ for all $z, x \in H$. If $\psi(z) < \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$, then

$$\psi(z) < t < \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$$

for some $t \in (0, 0.5)$. Hence $\langle x/t\rangle \in \psi$ and $\langle (z|(x|x))|(z|(x|x))/t\rangle \in \psi$. But $\langle z/t\rangle \,\overline{\in}\, \psi$ and $\psi(z) + t < 1$, i.e., $\langle z/t\rangle \,\overline{q}\, \psi$. This is a contradiction to (24), and so $\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\}$ whenever

$$\min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} < 0.5.$$

Now, if $\min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} \geq 0.5$, then $\langle x/0.5\rangle \in \psi$ and

$$\langle (z|(x|x))|(z|(x|x))/0.5\rangle \in \psi.$$

It follows from (24) that $\langle z/0.5\rangle \in \psi$ or $\langle z/0.5\rangle \, q \, \psi$. Hence $\psi(z) \geq 0.5$ because if $\psi(z) < 0.5$ then $\langle z/0.5\rangle \,\overline{\in}\, \psi$ and $\psi(z)+0.5 < 1$, i.e., $\langle z/0.5\rangle \,\overline{q}\, \psi$, a contradiction. Therefore $\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x))), 0.5\}$ for all $z, x \in H$.

Conversely, suppose that ψ satisfies (30) and (31). Let $z \in H$ and $s \in (0, 1]$ be such that $\langle z/s\rangle \in \psi$. Then $\psi(z) \geq s$. Suppose $\langle 0/s\rangle \,\overline{\in}\, \psi$. If $\psi(z) < 0.5$, then $\psi(0) \geq \min\{\psi(z), 0.5\} = \psi(z) \geq s$, a contradiction. Thus $\psi(z) \geq 0.5$ and so $\psi(0) + s > 2\psi(0) \geq 2 \min\{\psi(z), 0.5\} = 1$, i.e., $\langle 0/s\rangle \, q \, \psi$. Hence (23) is valid. Let $z, x \in H$ and $s, t \in (0, 1]$ be such that $\langle x/s\rangle \in \psi$ and

$$\langle (z|(x|x))|(z|(x|x))/t\rangle \in \psi.$$

Then $\psi(x) \geq s$ and $\psi((z|(x|x))|(z|(x|x))) \geq t$. Suppose $\langle z/\min\{s,t\}\rangle \overline{\in} \psi$. If $\min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} < 0.5$, then

$$\begin{aligned}\psi(z) &\geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x))), 0.5\} \\ &= \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} \\ &\geq \min\{s,t\},\end{aligned}$$

that is, $\langle z/\min\{s,t\}\rangle \in \psi$, a contradiction. Thus

$$\min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} \geq 0.5,$$

which implies that

$$\begin{aligned}&\psi(z) + \min\{s,t\} > 2\psi(z) \\ &\geq 2\min\{\psi(x), \psi((z|(x|x))|(z|(x|x))), 0.5\} = 1.\end{aligned}$$

Hence $\langle z/\min\{s,t\}\rangle \, q \, \psi$, and therefore (24) is valid. □

Theorem 3.16. *Given a fuzzy set ψ in H, the nonempty s-level set $(\psi, s)_\in$ of ψ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 0.5]$ if and only if ψ satisfies* (23) *and* (24).

Proof. Let ψ be a fuzzy set in H that satisfies (23) and (24). Let $s \in (0, 0.5]$ be such that $(\psi, s)_\in \neq \emptyset$. Then there exists $b \in (\psi, s)_\in$, and so $\langle b/s\rangle \in \psi$. Thus $\langle 0/s\rangle \in \psi$ or $\langle 0/s\rangle \, q \, \psi$ by (23). If $\langle 0/s\rangle \in \psi$, then $0 \in (\psi, s)_\in$. If $\langle 0/s\rangle \, q \, \psi$, then $\psi(0) > 1 - s \geq s$ since $s \leq 0.5$. Thus $0 \in (\psi, s)_\in$. Let $z, x \in H$ be such that $x \in (\psi, s)_\in$ and $(z|(x|x))|(z|(x|x)) \in (\psi, s)_\in$. Then $\langle x/s\rangle \in \psi$ and $\langle (z|(x|x))|(z|(x|x))/s\rangle \in \psi$, which imply from (24) that $\langle z/s\rangle \in \psi$ or $\langle z/s\rangle \, q \, \psi$. If $\langle z/s\rangle \in \psi$, then $z \in (\psi, s)_\in$. If $\langle z/s\rangle \, q \, \psi$, then $\psi(z) > 1 - s \geq s$ since $s \leq 0.5$. Thus $z \in (\psi, s)_\in$. Consequently, $(\psi, s)_\in$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 0.5]$.

Conversely, suppose that the nonempty s-level set $(\psi, s)_\in$ of ψ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 0.5]$. If there exists $c \in H$ such that $\psi(0) < \min\{\psi(c), 0.5\}$, then $\psi(0) < s < \min\{\psi(c), 0.5\}$ for some $s \in (0, 0.5)$, and so $0 \notin (\psi, s)_\in$. It is a contradiction, and thus $\psi(0) \geq \min\{\psi(z), 0.5\}$ for all $z \in H$. Suppose that $\psi(c) < \min\{\psi(a), \psi((c|(a|a))|(c|(a|a))), 0.5\}$ for some $c, a \in H$, and take $s := \min\{\psi(a), \psi((c|(a|a))|(c|(a|a))), 0.5\}$. Then $s \in (0, 0.5]$, $a \in (\psi, s)_\in$ and $(c|(a|a))|(c|(a|a)) \in (\psi, s)_\in$. But $c \notin (\psi, s)_\in$, which is a contradiction. Hence

$$\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x))), 0.5\}$$

for all $z, x \in H$. It follows from Lemma 3.15 that ψ satisfies (23) and (24). □

Proposition 3.17. *Given an ideal D of $\mathcal{H} := (H, |)$, let ψ be a fuzzy set in H such that $\psi(z) = 0$ for all $z \in H \setminus D$, and $\psi(z) \geq 0.5$ for all $z \in D$. Then ψ*

satisfies:

(32) $(\forall z \in H)(\forall s \in (0,1])(\langle z/s\rangle \, q \, \psi \ \Rightarrow\ \langle 0/s\rangle \in \psi \text{ or } \langle 0/s\rangle \, q \, \psi),$

(33) $$(\forall z, x \in H)(\forall s, t \in (0,1]) \left(\begin{array}{l} \langle x/s\rangle \, q \, \psi,\ \langle (z|(x|x))|(z|(x|x))/t\rangle \, q \, \psi \\ \Rightarrow \left\{ \begin{array}{l} \langle z/\min\{s,t\}\rangle \in \psi \text{ or} \\ \langle z/\min\{s,t\}\rangle \, q \, \psi \end{array} \right. \end{array} \right).$$

Proof. Let $z \in H$ and $s \in (0,1]$ be such that $\langle z/s\rangle \, q \, \psi$. Then $\psi(z) + s > 1$. If $z \in H \setminus D$, then $\psi(z) = 0$ and so $s > 1$, a contradiction. Thus $z \in D$, and hence $\psi(z) \geq 0.5$. If $\langle 0/s\rangle \,\overline{\in}\, \psi$, then $0.5 \leq \psi(0) < s$ and so $\psi(0) + s > 1$, that is, $\langle 0/s\rangle \, q \, \psi$. This shows that (32) is valid. Let $z, x \in H$ and $s, t \in (0,1]$ be such that $\langle x/s\rangle \, q \, \psi$ and $\langle (z|(x|x))|(z|(x|x))/t\rangle \, q \, \psi$. Then $\psi(x) + s > 1$ and $\psi((z|(x|x))|(z|(x|x))) + t > 1$. If $x \notin D$ or $(z|(x|x))|(z|(x|x)) \notin D$, then $\psi(x) = 0$ or $\psi((z|(x|x))|(z|(x|x))) = 0$ and so $s > 1$ or $t > 1$. This is a contradiction, and thus $x \in D$ and $(z|(x|x))|(z|(x|x)) \in D$. Since D is an ideal of $\mathcal{H} := (H, |)$, it follows that $z \in D$. Hence $\psi(z) \geq 0.5$. If $s \leq 0.5$ or $t \leq 0.5$, then $\psi(z) \geq 0.5 \geq \min\{s,t\}$, i.e., $\langle z/\min\{s,t\}\rangle \in \psi$ If $s > 0.5$ and $t > 0.5$, then $\psi(z) + \min\{s,t\} > 0.5 + 0.5 = 1$ and so $\langle z/\min\{s,t\}\rangle \, q \, \psi$. Consequently, ψ satisfies (33). □

For a fuzzy set ψ in H, consider the set below:

(34) $$H_+ := \{z \in H \mid \psi(z) > 0\}$$

which is called the *positive set* of $\mathcal{H} := (H, |)$.

Theorem 3.18. *If ψ is a nonzero fuzzy ideal of $\mathcal{H} := (H, |)$, then the positive set H_+ of $\mathcal{H} := (H, |)$ is an ideal of $\mathcal{H} := (H, |)$.*

Proof. Let ψ be a nonzero fuzzy ideal of $\mathcal{H} := (H, |)$. Then there exists $c \in H$ such that $\psi(c) > 0$, and so $\psi(0) \geq \psi(c) > 0$ by Theorem 3.7. Hence $0 \in H_+$. Let $z, x \in H$ be such that $x \in H_+$ and $(z|(x|x))|(z|(x|x)) \in H_+$. Using Theorem 3.7, we get $\psi(z) \geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} > 0$, and so $z \in H_+$. Therefore H_+ is an ideal of $\mathcal{H} := (H, |)$. □

Theorem 3.19. *If a nonzero fuzzy set ψ in H satisfies:*

(35) $(\forall s \in (0,1])\left((\psi, s)_\in \neq \emptyset \ \Rightarrow\ 0 \in (\psi, s)_q\right),$

(36) $$(\forall z, x \in H)(\forall s, t \in (0,1]) \left(\begin{array}{l} x \in (\psi, s)_\in,\ (z|(x|x))|(z|(x|x)) \in (\psi, t)_\in \\ \ \Rightarrow\ z \in (\psi, \min\{s,t\})_q, \end{array} \right)$$

then the positive set H_+ of $\mathcal{H} := (H, |)$ is an ideal of $\mathcal{H} := (H, |)$.

Proof. Let ψ be a nonzero fuzzy set in H which satisfies (35) and (36). Then $\psi(a) > 0$ for some $a \in H$, which implies $a \in (\psi, \psi(a))_\in$. Hence $0 \in (\psi, \psi(a))_q$ by (35), and so $\psi(0) + \psi(a) > 1$. Thus $0 \in H_+$. Let $z, x \in H$ be such that $x \in H_+$ and $(z|(x|x))|(z|(x|x)) \in H_+$. Then $\psi(x) > 0$ and $\psi((z|(x|x))|(z|(x|x))) > 0$. Hence $x \in (\psi, s)_\in$ and $(z|(x|x))|(z|(x|x)) \in (\psi, t)_\in$ where $s := \psi(x)$ and $t := \psi((z|(x|x))|(z|(x|x)))$. It follows from (36) that

$z \in (\psi, \min\{s,t\})_q$. If $\psi(z) = 0$, then $\psi(z) + \min\{s,t\} = \min\{s,t\} \leq 1$ and so $\langle z/\min\{s,t\}\rangle \, \overline{q} \, \psi$, that is, $z \notin (\psi, \min\{s,t\})_q$. This is a contradiction, and so $\psi(z) > 0$. Thus $z \in H_+$, and therefore H_+ is an ideal of $\mathcal{H} := (H, |)$. □

Theorem 3.20. *If a nonzero fuzzy set ψ in H satisfies:*

(37) $(\forall s \in (0,1])\,((\psi,s)_q \neq \emptyset \;\Rightarrow\; 0 \in (\psi,s)_\in)$,

(38) $(\forall z,x \in H)(\forall s,t \in (0,1])\left(\begin{array}{l} x \in (\psi,s)_q,\ (z|(x|x))|(z|(x|x)) \in (\psi,t)_q \\ \Rightarrow\ z \in (\psi, \min\{s,t\})_\in, \end{array}\right)$

then the positive set H_+ of $\mathcal{H} := (H, |)$ is an ideal of $\mathcal{H} := (H, |)$.

Proof. Let ψ be a nonzero fuzzy set in H which satisfies (37) and (38). Then $\psi(a) > 0$ for some $a \in H$. If we take $s := (1 - \psi(a), 1]$, then $\psi(a) + s > 1$, i.e., $a \in (\psi,s)_q$. Hence $0 \in (\psi,s)_\in$ by (37), which implies $\psi(0) \geq s > 0$. Thus $0 \in H_+$. Let $z,x \in H$ be such that $x \in H_+$ and $(z|(x|x))|(z|(x|x)) \in H_+$. Then $\psi(x) > 0$ and $\psi((z|(x|x))|(z|(x|x))) > 0$. By choosing $s \in (1 - \psi(x), 1]$ and $t \in (1 - \psi((z|(x|x))|(z|(x|x))), 1]$, we get $x \in (\psi,s)_q$ and $(z|(x|x))|(z|(x|x)) \in (\psi,t)_q$. It follows from (38) that $z \in (\psi, \min\{s,t\})_\in$. Hence $\psi(z) \geq \min\{s,t\} > 0$, and so $x \in H_+$. Therefore H_+ is an ideal of $\mathcal{H} := (H, |)$. □

Theorem 3.21. *If a nonzero fuzzy set ψ in H satisfies:*

(39) $(\forall s \in (0,1])\,((\psi,s)_q \neq \emptyset \;\Rightarrow\; 0 \in (\psi,s)_q)$,

(40) $(\forall z,x \in H)(\forall s,t \in (0,1])\left(\begin{array}{l} x \in (\psi,s)_q,\ (z|(x|x))|(z|(x|x)) \in (\psi,t)_q \\ \Rightarrow\ z \in (\psi, \min\{s,t\})_q, \end{array}\right)$

then the positive set H_+ of $\mathcal{H} := (H, |)$ is an ideal of $\mathcal{H} := (H, |)$.

Proof. Let ψ be a nonzero fuzzy set in H which satisfies (39) and (40). Then $\psi(a) > 0$ for some $a \in H$. If we take $s := (1-\psi(a), 1]$, then $\psi(a)+s > 1$, i.e., $a \in (\psi,s)_q$. Hence $0 \in (\psi,s)_q$ by (39), and so $\psi(0) > 1 - s \geq 0$. Thus $0 \in H_+$. Let $z,x \in H$ be such that $x \in H_+$ and $(z|(x|x))|(z|(x|x)) \in H_+$. Then $\psi(x) > 0$ and $\psi((z|(x|x))|(z|(x|x))) > 0$. It follows that $x \in (\psi,s)_q$ and $(z|(x|x))|(z|(x|x)) \in (\psi,t)_q$ where $s \in (1 - \psi(x), 1]$ and $t \in (1 - \psi((z|(x|x))|(z|(x|x))), 1]$. Hence $z \in (\psi, \min\{s,t\})_q$ by (40), and therefore $\psi(z) > 1 - \min\{s,t\} \geq 0$. This shows $z \in H_+$. Consequently, H_+ is an ideal of $\mathcal{H} := (H, |)$. □

We find the conditions under which the Q_s-set, $s \in (0,1]$, can be an ideal.

Theorem 3.22. *If ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$, then the nonempty Q_s-set $(\psi,s)_q$ of ψ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0,1]$.*

Proof. Assume that ψ is a fuzzy ideal of $\mathcal{H} := (H, |)$ and let $s \in (0,1]$ be such that $(\psi,s)_q \neq \emptyset$. Then there exists $a \in (\psi,s)_q$, and so $\psi(0) + s \geq \psi(a) + s > 1$. Thus $0 \in (\psi,s)_q$. Let $z,x \in H$ be such that $x \in (\psi,s)_q$ and $(z|(x|x))|(z|(x|x)) \in$

$(\psi, s)_q$. Then $\psi(x) + s > 1$ and $\psi((z|(x|x))|(z|(x|x))) + s > 1$. It follows from (22) that

$$\begin{aligned}\psi(z) + s &\geq \min\{\psi(x), \psi((z|(x|x))|(z|(x|x)))\} + s \\ &= \min\{\psi(x) + s, \psi((z|(x|x))|(z|(x|x))) + s\} > 1.\end{aligned}$$

Hence $z \in (\psi, s)_q$, and therefore $(\psi, s)_q$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 1]$. □

Proposition 3.23. *Given a fuzzy set ψ in H, if its Q_s-set $(\psi, s)_q$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 0.5]$, then the assertions below are established.*

(41) $(\forall s \in (0, 0.5]) \left(\langle 0/s\rangle \in \psi\right)$,

(42) $(\forall z, x \in H)(\forall s, t \in (0, 0.5]) \left(\begin{array}{l} \langle x/s\rangle \, q \, \psi, \ \langle (z|(x|x))|(z|(x|x))/t\rangle \, q \, \psi \\ \Rightarrow \ z \in (\psi, \max\{s, t\})_\in \end{array} \right)$.

Proof. Assume that $(\psi, s)_q$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0, 0.5]$. If $\langle 0/s\rangle \,\overline{\in}\, \psi$ for some $s \in (0, 0.5]$, then $\psi(0) < s \leq 1-s$ and so $0 \notin (\psi, s)_q$. This is a contradiction, and thus $\langle 0/s\rangle \in \psi$. Let $z, x \in H$ and $s, t \in (0, 0.5]$ be such that $\langle x/s\rangle \, q \, \psi$ and $\langle (z|(x|x))|(z|(x|x))/t\rangle \, q \, \psi$. Then $x \in (\psi, s)_q \subseteq (\psi, \max\{s, t\})_q$ and $(z|(x|x))|(z|(x|x)) \in (\psi, t)_q \subseteq (\psi, \max\{s, t\})_q$. Since $(\psi, \max\{s, t\})_q$ is an ideal of $\mathcal{H} := (H, |)$, it follows that $z \in (\psi, \max\{s, t\})_q$. Hence

$$\psi(z) > 1 - \max\{s, t\} \geq \max\{s, t\},$$

and so $z \in (\psi, \max\{s, t\})_\in$. □

Proposition 3.24. *Given a fuzzy set ψ in H, if its Q_s-set $(\psi, s)_q$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0.5, 1]$, then*

(43) $\langle x/s\rangle \in \psi, \ \langle (z|(x|x))|(z|(x|x))/t\rangle \in \psi \ \Rightarrow \ z \in (\psi, \max\{s, t\})_q$

for all $z, x \in H$ and $s, t \in (0.5, 1]$.

Proof. Assume that $(\psi, s)_q$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0.5, 1]$. Let $z, x \in H$ and $s, t \in (0.5, 1]$ be such that $\langle x/s\rangle \in \psi$ and $\langle (z|(x|x))|(z|(x|x))/t\rangle \in \psi$. Then $\psi(x) \geq s > 1 - s$ and $\psi((z|(x|x))|(z|(x|x))) \geq t > 1 - t$, that is, $\langle x/s\rangle \, q \, \psi$ and $\langle (z|(x|x))|(z|(x|x))/t\rangle \, q \, \psi$. Hence $x \in (\psi, s)_q \subseteq (\psi, \max\{s, t\})_q$ and $(z|(x|x))|(z|(x|x)) \in (\psi, t)_q \subseteq (\psi, \max\{s, t\})_q$. Since $(\psi, \max\{s, t\})_q$ is an ideal of $\mathcal{H} := (H, |)$, we have $z \in (\psi, \max\{s, t\})_q$. □

Theorem 3.25. *If a fuzzy set ψ in H satisfies:*

(44) $(\forall z \in H)(\forall s \in (0.5, 1])(\langle z/s\rangle \, q \, \psi \ \Rightarrow \ \langle 0/s\rangle \in \psi \text{ or } \langle 0/s\rangle \, q \, \psi)$,

(45) $(\forall z, x \in H)(\forall s, t \in (0.5, 1]) \left(\begin{array}{l} \langle x/s\rangle \, q \, \psi, \ \langle (z|(x|x))|(z|(x|x))/t\rangle \, q \, \psi \\ \Rightarrow \left\{ \begin{array}{l} \langle z/\min\{s, t\}\rangle \in \psi \text{ or} \\ \langle z/\min\{s, t\}\rangle \, q \, \psi \end{array} \right. \end{array} \right)$,

then the nonempty Q_s-set $(\psi, s)_q$ of ψ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0.5, 1]$.

Proof. Let ψ be a fuzzy set in H that satisfies (44) and (45). Assume that $(\psi, s)_q \neq \emptyset$ for all $s \in (0.5, 1]$. Then there exists $b \in (\psi, s)_q$ and so $\langle b/s\rangle \, q \, \psi$. Thus $\langle 0/s\rangle \in \psi$ or $\langle 0/s\rangle \, q \, \psi$ by (44). If $\langle 0/s\rangle \, q \, \psi$, then $0 \in (\psi, s)_q$. If $\langle 0/s\rangle \in \psi$, then $\psi(0) \geq s > 1 - s$ since $s > 0.5$. Hence $0 \in (\psi, s)_q$. Let $z, x \in H$ be such that $x \in (\psi, s)_q$ and $(z|(x|x))|(z|(x|x)) \in (\psi, s)_q$. Then $\langle x/s\rangle \, q \, \psi$ and $\langle (z|(x|x))|(z|(x|x))/s\rangle \, q \, \psi$. It follows from (45) that $\langle z/s\rangle = \langle z/\min\{s, s\}\rangle \in \psi$ or $\langle z/s\rangle = \langle z/\min\{s, s\}\rangle \, q \, \psi$. If $\langle z/s\rangle \, q \, \psi$, then $z \in (\psi, s)_q$. If $\langle z/s\rangle \in \psi$, then $\psi(z) \geq s > 1 - s$ since $s > 0.5$. Thus $z \in (\psi, s)_q$. Consequently, $(\psi, s)_q$ is an ideal of $\mathcal{H} := (H, |)$ for all $s \in (0.5, 1]$. □

4. Conclusion

In this paper, we characterize ideals by the virtue of fuzzy points of Sheffer stroke Hilbert algebras, and examine several properties. Then we prove equivalent statements to fuzzy ideals by means of fuzzy points of Sheffer stroke Hilbert algebras. We illustrate that the intersection of two fuzzy ideals of these algebraic structures is a fuzzy ideal but their union is not in general. Also, we show the relationship between a fuzzy ideal and a fuzzy deductive system of Sheffer stroke Hilbert algebras, and find out the conditions under which the s-level set of a fuzzy set in Sheffer stroke Hilbert algebras can be its ideal. We demonstrate that the positive set of Sheffer stroke Hilbert algebras defined by the fuzzy ideal is an ideal of these algebraic structures, and that the positive set where is defined by the nonzero fuzzy set under the special conditions of fuzzy points is the ideal. Finally, we analyze the conditions under which the Q_s-set of a fuzzy set of Sheffer stroke Hilbert algebras is an ideal.

We hope that this work will give a deep impact on the upcoming research in this field and other algebraic studies to open up new horizons of interest and innovations. In future directions, these definitions and main results can be similarly extended to some other algebraic systems such as Sheffer stroke BCK-algebras, Sheffer stroke BL-algebras, Sheffer stroke basic algebras, Sheffer stroke MTL-algebras, and various other.

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