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SYMMETRY OF SPECIAL COMPOSITION OPERATORS ON THE HARDY SPACE

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Abstract. We consider a special orthonormal basis for the Hardy space of the unit disc to compute the matrix representations of the composition operators with respect to the basis particulary associated to two symbols which are the inverse and the origin symmetry of the Riemann self map in the unit disc, and then we find a certain symmetry of the matrices.

1. Introduction

Given a set S and a function $\varphi : S \to S$, the composition operator C_φ on a Banach space $\mathcal{H}$ on S with symbol φ is defined by

$$C_\varphi(f) = f \circ \varphi \text{ for } f \in \mathcal{H}.$$

One of important research areas related to composition operators is how the properties of the operator relate to those of the symbol. On the other hand, when the Banach space $\mathcal{H}$ is a Hilbert space, the composition operator is heavily dependent on the matrix representation of the operator with respect to the orthonormal bases for the function space. However, except for very simple cases, computation of the matrix of a composition operator in an infinite dimensional Hilbert space turned out to be very complicated and difficult, so it has rarely been formulated so far.

Suppose now that the base set is the unit disc U in the complex plane and the function space is the Hardy space $H^2(bU)$. In this category the composition operator C_φ with a holomorphic self map φ on U becomes a bounded operator on $H^2(bU)$. In particular, when φ is a Riemann map which maps the unit disc into itself, it characterizes the automorphisms up to rotations, and the associated composition operator becomes complex symmetric and vice versa with certain conditions. See [2], [7] for these references. So, from this point of view, it seems very important to formulate the matrices of the composition operators with the symbols of the Riemann map or more general its automorphisms.

In this paper, we consider a special orthonormal basis for the Hardy space which is a generalization of monomials $z^n, n = 1, 2,$ to compute the matrices

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of the composition operators with respect to the basis with the two associated symbols which are the inverse and the origin symmetry of the Riemann self map in the unit disc.

2. Some preliminaries

From now on, we denote U by the unit disc in the complex plane and we fix a point a in U, unless otherwise specified. Let $L^2(bU)$ be the space of square integrable functions on the boundary bU of the unit disc U with the usual inner product $< u, v >= \int_{bU} u\overline{v}\, ds$, where ds is the differential element of arc length on bU. And let $H^2(bU)$ be the classical Hardy space which is the space of holomorphic functions on U with L^2-boundary values in bU.

For a holomorphic self map φ of U, the composition operator C_φ with the symbol φ is defined by

$$C_\varphi(v) = v \circ \varphi \text{ for } v \in H^2(bU).$$

It is well known that C_φ is a bounded linear operator on the Hardy space $H^2(bU)$. See [6] for details. For the case $\varphi = f_a$ of the Riemann map

$$f_a(z) = \frac{a - z}{1 - \overline{a}z},$$

the associated composition operator C_{f_a} gives an important classification, such as being the only symbol that makes C_{f_a} a complex symmetric operator under suitable conditions. See [2] for this matter. On the other hand, the author formulated in [5] the matrix of the operator C_{f_a} explicitly.

In this paper, we more extendibly consider two symbols f_{-a} and $-f_a$ which are the inverse f_a and the symmetry $-f_a$ about the origin of the Riemann map, and find useful formulae of the matrices of the corresponding operators.

Let $a \in U$ be fixed. For a positive integer m, we define the function v_m by

$$v_m(z) := \frac{\sqrt{1 - |a|^2}}{\sqrt{2\pi}} \frac{(z - a)^{m-1}}{(1 - \overline{a}z)^m} = \frac{\sqrt{1 - |a|^2}}{\sqrt{2\pi}} f_a^{m-1}(z) \frac{1}{1 - \overline{a}z}$$

which is holomorphic in a neighborhood of $\overline{U}$. It is well known that the class of $v_m, m = 1, 2, 3, \cdots$ forms an orthonormal basis for the Hardy space $h^2(bU)$. See [1] for general cases. The author also proved that the set $\{v_m \,|\, m = 0, \pm 1, \pm 2, \cdots\}$ forms an orthonormal basis for $L^2(bU)$. See [3] and [4] for more details.

Observe that since the basis is orthonormal, for positive integers l and m, the (l, m)-th entry of the matrix $[C_\varphi]$ of the operator C_φ with symbol φ with respect to the basis $\{v_m \,|\, m = 1, 2, \cdots\}$ is obtained by $[C_\varphi]_{lm} =< C_\varphi(v_m), v_l >$. For reference in the next section, we lists several properties whose proofs are all trivial.

Suppose that z is on the boundary of the unit disc. Then the following identities hold.

$$\overline{\left(\frac{1-\overline{a}z}{z-a}\right)} = \frac{z-a}{1-\overline{a}z}. \tag{1}$$

$$\overline{\left(\frac{1}{1-\overline{a}z}\right)} = \frac{z}{z-a}. \tag{2}$$

$$z\,ds = -i\,dz. \tag{3}$$

3. Necessary lemmas

In this section, we compute higher-order derivatives of functions dealt with in the next section.

Lemma 3.1. *For integers m and k with $m \geq 1$ and $k \geq 0$,*

$$\left(z^{m-1}\right)^{(k)}(a) = \left[\chi_{\{0\}}(k) + \chi_{[1,m)}(k)\cdot(m-1)(m-2)\cdots(m-k)\right]a^{m-k-1},$$

where the function χ_X associated to the set X is the characteristic function defined $\chi_X(z) = 1$ for $z \in X$ and $\chi_X(z) = 0$ for $z \notin X$.

Proof. The proof is trivial. $\square$

Lemma 3.2. *For a positive integer l, define*

$$g_l(z) = (1-\overline{a}z)^l,\ z \in U.$$

Then for a nonnegative integer k,

$$g_l^{(k)}(a) = (-1)^k\frac{l!}{(l-k)!}\overline{a}^k(1-|a|^2)^{l-k}.$$

Proof. For $k = 0$, it is obvious. Since $g_l^{(1)}(z) = l(1-\overline{a}z)^{l-1}(-\overline{a})$ and $g_l^{(2)}(z) = l(l-1)(1-\overline{a}z)^{l-2}(-\overline{a})^2$, it is easy to see from mathematical induction on the order of derivative that

$$g_l^{(k)}(z) = l(l-1)\cdots(l-k+1)(1-\overline{a}z)^{l-k}(-\overline{a})^k,$$

which proves Lemma 3.2. $\square$

Lemma 3.3. *For integers l and k with $0 \leq k \leq l-1$,*

$$\begin{aligned}
&\left[(1+\overline{a}z)(1-\overline{a}z)^{l-1}\right]^{(k)}\Big|_{z=a} \\
&= \frac{(-1)^{l-1}l!}{(l-k)!}\overline{a}^l a^{l-k} + \frac{(l-1)!\left[\chi_{[1,\infty)}(k)\cdot(-1)^{k-1}k + (-1)^k(l-k)\right]}{(l-k)!}\overline{a}^k \\
&+ \sum_{j=1}^{l-1-k}\frac{(l-1)!(l-2j)}{j!(l-k-j)!}\overline{a}^{l-j}a^{l-k-j}.
\end{aligned} \tag{4}$$

Proof. We use the Leibniz product rule of differentiation and Lemma 3.2.

$$
\begin{aligned}
&\left[(1+\overline{a}z)(1-\overline{a}z)^{l-1}\right]^{(k)}\Big|_{z=a} \\
&= \sum_{j=0}^{k} \binom{k}{j} (1+\overline{a}z)^{(k-j)} \left[(1-\overline{a}z)^{l-1}\right]^{(j)}\Big|_{z=a} \\
&= \binom{k}{0} (1+\overline{a}z)^{(k)}\Big|_{z=a} (1-|a|^2)^{l-1} \\
&+ \binom{k}{1} (1+\overline{a}z)^{(k-1)} \left[(1-\overline{a}z)^{l-1}\right]^{(1)}\Big|_{z=a} \\
&+ \cdots \\
&+ \binom{k}{k-1} (1+\overline{a}z)^{(1)} \left[(1-\overline{a}z)^{l-1}\right]^{(k-1)}\Big|_{z=a} \\
&+ \binom{k}{k} (1+\overline{a}z)^{(0)} \left[(1-\overline{a}z)^{l-1}\right]^{(k)}\Big|_{z=a} \\
&= k\overline{a} \left[(1-\overline{a}z)^{l-1}\right]^{(k-1)}\Big|_{z=a} + (1+|a|^2) \left[(1-\overline{a}z)^{l-1}\right]^{(k)}\Big|_{z=a} \\
&= k\overline{a}(-1)^{k-1}\frac{(l-1)!}{(l-k)!}\overline{a}^{k-1}(1-|a|^2)^{l-k} \cdot \chi_{[1,\infty)}(k) \\
&\qquad\qquad + (1+|a|^2)(-1)^k \frac{(l-1)!}{(l-1-k)!}\overline{a}^k(1-|a|^2)^{l-1-k}.
\end{aligned}
$$

Using the binomial formula, the above identity is equal to

$$
\begin{aligned}
&k\overline{a}(-1)^{k-1}\frac{(l-1)!}{(l-k)!}\overline{a}^{k-1}\chi_{[1,\infty)}(k)\sum_{j=0}^{l-k}\binom{l-k}{j}(-1)^{l-k-j}\overline{a}^{l-k-j}a^{l-k-j} \\
&+ (1+\overline{a}a)(-1)^k \frac{(l-1)!}{(l-1-k)!}\overline{a}^k \sum_{p=0}^{l-k-1}\binom{l-k-1}{p}(-1)^{l-1-k-p}\overline{a}^{l-1-k-p}a^{l-1-k-p} \\
&= (l-1)!\chi_{[1,\infty)}(k)\sum_{j=0}^{l-k}\frac{(-1)^{l-1-j}k}{j!(l-k-j)!}\overline{a}^{l-j}a^{l-k-j} \\
&\qquad + (l-1)!\sum_{p=0}^{l-k-1}\frac{(-1)^{l-1-p}}{p!(l-1-k-p)!}\overline{a}^{l-1-p}a^{l-1-k-p} \\
&\qquad + (l-1)!\sum_{q=0}^{l-k-1}\frac{(-1)^{l-1-q}}{q!(l-1-k-q)!}\overline{a}^{l-q}a^{l-k-q}.
\end{aligned}
$$

We extract the first and the last terms from the first summation, the first terms from the second and third summations and then simplify them to get the above identity equal to

$$
\begin{aligned}
&(l-1)!\chi_{[1,\infty)}(k)\frac{(-1)^{l-1}k}{0!(l-k)!}\overline{a}^l a^{l-k} + \chi_{[1,\infty)}(k)\frac{(l-1)!(-1)^{k-1}k}{(l-k)!0!}\overline{a}^k a^0 \\
&\qquad + (l-1)!\chi_{[1,\infty)}(k)\sum_{j=1}^{l-k-1}\frac{(-1)^{l-1-j}k}{j!(l-k-j)!}\overline{a}^{l-j}a^{l-k-j} \\
&+ \frac{(l-1)!(-1)^k}{(l-k-1)!0!}\overline{a}^k a^0 + (l-1)!\sum_{p=0}^{l-k-2}\frac{(-1)^{l-1-p}}{p!(l-1-k-p)!}\overline{a}^{l-1-p}a^{l-1-k-p} \\
&+ (l-1)!\frac{(-1)^{l-1}}{0!(l-1-k)!}\overline{a}^l a^{l-k} + (l-1)!\sum_{q=1}^{l-k-1}\frac{(-1)^{l-1-q}}{q!(l-1-k-q)!}\overline{a}^{l-q}a^{l-k-q} \\
&= (l-1)!\left[\frac{(-1)^{l-1}k\chi_{[1,\infty)}(k)}{(l-k)!} + \frac{(-1)^{l-1}}{(l-1-k)!}\right]\overline{a}^l a^{l-k} \\
&\qquad + (l-1)!\left[\frac{(-1)^{k-1}k\chi_{[1,\infty)}(k)}{(l-k)!} + \frac{(-1)^k}{(l-1-k)!}\right]\overline{a}^k \\
&+ (l-1)!\sum_{j=1}^{l-1-k} \\
&\left[\frac{(-1)^{l-1-j}k\chi_{[1,\infty)}(k)}{j!(l-k-j)!} + \frac{(-1)^{l-j}}{(j-1)!(l-k-j)!} + \frac{(-1)^{l-1-j}}{j!(l-1-k-j)!}\right]\overline{a}^{l-j}a^{l-k-j}.
\end{aligned}
$$

Observing that

$$(-1)^{l-1}k\chi_{[1,\infty)}(k) + (-1)^{l-1}(l-k) = (-1)^{l-1}(k\chi_{[1,\infty)}(k) + l - k) = (-1)^{l-1}l$$

and

$$
\begin{aligned}
&(-1)^{l-1-j}k\chi_{[1,\infty)}(k) + (-1)^{l-j}j + (-1)^{l-1-j}(l-k-j) \\
&= (-1)^{l-1-j}\left[\chi_{[1,\infty)}(k)k + (-1)j + l - k - j\right] = l - 2j,
\end{aligned}
$$

the above identity equals the formula (4) in Lemma 3.3 and hence we are done. □

4. Computation of the matrix of the composition operator with the symbol of inverse of the Riemann map

In this section, we consider the function

$$f_{-a}(z) = \frac{z+a}{1+\overline{a}z}$$

which is in fact the inverse of the Riemann mapping function f_a, and compute the matrix of the composition operator $C_{f_{-a}}$ with the symbol f_{-a}. For two positive integers l and m, the orthonormality of the functions $v_m, m = 1, 2, \cdots$

yields the computation of the matrix as follows. It then follows from the equations (1) and (2) that

$$
\begin{aligned}
&[C_{f_{-a}}]_{lm} = < v_m \circ f_{-a}, v_l > \\
&= \frac{1-|a|^2}{2\pi} \int_{bU} \left[(f_a^{m-1} \circ f_{-a})(z)\right] \left(\frac{1}{1-\overline{a}z} \circ f_{-a}(z)\right) \overline{f_a(z)^{l-1}} \overline{\left(\frac{1}{1-\overline{a}z}\right)} \, ds_z \\
&= \frac{1-|a|^2}{2\pi} \int_{bU} z^{m-1} \frac{1+\overline{a}z}{1-|a|^2} \left(\frac{1-\overline{a}z}{z-a}\right)^{l-1} \frac{z}{z-a} \, ds_z \\
&= \frac{1}{2\pi i} \int_{bU} \frac{z^{m-1}(1+\overline{a}z)(1-\overline{a}z)^{l-1}}{(z-a)^l} \, dz.
\end{aligned}
$$

Applying the Cauchy's residue theorem, we obtain the following formula.

Theorem 4.1. *Let U be the unit disc and let a be fixed in U. Then for given two positive integers l, m, the matrix $[C_{f_{-a}}]$ of the composition operator $C_{f_{-a}}$ on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\{v_m | m = 1, 2, \cdots\}$ has (l, m)-th entry*

$$
(5) \qquad [C_{f_{-a}}]_{lm} = \frac{1}{(l-1)!} \frac{d^{l-1}}{dz^{l-1}} \left[z^{m-1}(1+\overline{a}z)(1-\overline{a}z)^{l-1}\right] \Big|_{z=a}.
$$

Now we can use Lemma 3.1, Lemma 3.3 and the Leibniz product rule to get the formula of the matrix without differentiation as follows. The identity (5) equals

$$
\begin{aligned}
&= \frac{1}{(l-1)!} \sum_{j=0}^{l-1} \binom{l-1}{j} (z^{m-1})^{(j)} \left[(1+\overline{a}z)(1-\overline{a}z)^{l-1}\right]^{(l-1-j)} \Big|_{z=a} \\
&= \frac{1}{(l-1)!} \sum_{j=0}^{l-1} \binom{l-1}{j} \cdot \\
&\left\{ \left[\chi_{\{0\}}(j) + \chi_{[1,m)}(j)(m-1)(m-2)\cdots(m-j)\right] a^{m-1-j} \right\} \cdot \left\{ \frac{(-1)^{l-1} l!}{(j+1)!} \overline{a}^l a^{j+1} \right. \\
&+ \frac{(l-1)! \left[\chi_{[1,\infty)}(l-1-j)(-1)^{l-2-j}(l-1-j) + (-1)^{l-1-j}(j+1)\right]}{(j+1)!} \overline{a}^{l-1-j} \\
&\left. + \sum_{p=1}^{j} \frac{(l-1)!(l-2p)}{p!(j+1-p)!} \overline{a}^{l-p} a^{j+1-p} \right\}.
\end{aligned}
$$

Thus we obtain the following simplified form of the matrix of the composition operator $C_{f_{-a}}$.

Theorem 4.2. *Let U be the unit disc and let a be in U. Then for given two positive integers l, m, the matrix $[C_{f_{-a}}]$ of the composition operator $C_{f_{-a}}$ on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\{v_m | m = 1, 2, \cdots\}$*

has (l, m)*-th entry*

$$
\begin{aligned}
&[C_{f_{-a}}]_{lm} \\
&= \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{(-1)^{l-1} l}{(j+1)!} \left[\chi_{\{0\}}(j) + \chi_{[1,m)}(j)(m-1)(m-2)\cdots(m-j)\right] \overline{a}^l a^m \\
&+ \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{1}{(j+1)!} \left[\chi_{\{0\}}(j) + \chi_{[1,m)}(j)(m-1)(m-2)\cdots(m-j)\right] \cdot \\
&\quad \left[\chi_{[1,\infty)}(l-1-j)(-1)^{l-2-j}(l-1-j) + (-1)^{l-1-j}(j+1)\right] \overline{a}^{l-1-j} a^{m-1-j}.
\end{aligned}
$$

5. Symmetry of the matrix of a composition operator

In this final section we compute the entry $[C_\varphi]_{lm}$ where the symbol $\varphi = -f_a$ is the symmetry $-f_a$ about the origin of the Riemann map. In particular, it turns out that the matrix $[C_{-f_a}]$ has a certain symmetry which is very interesting. As in the previously section, we get the following (l, m)-th entry of the matrix $[C_{-f_a}]$.

$$
\begin{aligned}
&[C_\varphi]_{lm} = < v_m \circ \varphi, v_l > \\
(6) \qquad &= \frac{1-|a|^2}{2\pi} \int_{bU} \left[(f_a^{m-1} \circ \varphi)(z)\right] \left(\frac{1}{1-\overline{a}z} \circ \varphi\right) \overline{f_a(z)^{l-1} \left(\frac{1}{1-\overline{a}z}\right)}\, ds_z.
\end{aligned}
$$

Observe that

$$
f_a \circ \varphi = \frac{\dfrac{a-z}{1-\overline{a}z} - a}{1 - \overline{a}\left(\dfrac{a-z}{1-\overline{a}z}\right)} = -z.
$$

It thus follows from (1) and (2) that the identity (6) is equal to

$$
\frac{1-|a|^2}{2\pi} \int_{bU} (-1)^{m-1} z^{m-1} \cdot \frac{1-\overline{a}z}{1-|a|^2} \left(\frac{1-\overline{a}z}{z-a}\right)^{l-1} \frac{z}{z-a}\, ds_z.
$$

And then by (3), the above identity equals

$$
\frac{(-1)^{m-1}}{2\pi i} \int_{bU} \frac{z^{m-1}(1-\overline{a}z)^l}{(z-a)^l}\, dz.
$$

By using the residue theorem we have proved the following proposition.

Theorem 5.1. *The matrix* $[C_{-f_a}]$ *of the composition operator* C_{-f_a} *on the Hardy space* $H^2(bU)$ *with respect to the orthonormal basis* $\{v_m | m = 1, 2, \cdots\}$ *has the* (l, m)*-th entry*

$$
(7) \qquad [C_{-f_a}]_{lm} = \frac{(-1)^{m-1}}{(l-1)!} \frac{d^{l-1}}{dz^{l-1}} \left[z^{m-1}(1-\overline{a}z)^l\right]\Big|_{z=a}.
$$

Notice that as using Lemma 3.1 and Lemma 3.2 the identity (7) is written as

$$\frac{(-1)^{m-1}}{(l-1)!}\sum_{j=0}^{l-1}\binom{l-1}{j}(z^{m-1})^{(j)}g_l^{(l-1-j)}\Big|_{z=a}$$
$$=\frac{1}{(l-1)!}\sum_{j=0}^{l-1}(-1)^{m+l-2-j}\frac{l!}{(j+1)!}\binom{l-1}{j}\cdot$$
$$\left[\chi_{\{0\}}(j)+\chi_{[1,m)}(j)(m-1)(m-2)\cdots(m-j)\right]a^{m-1-j}\overline{a}^{l-1-j}(1-|a|^2)^{j+1}.$$

We have thus obtained another form of the entries of the matrix without differentiation.

Theorem 5.2. *Let U be the unit disc and let a be in U. Then for given two positive integers l,m, the matrix $[C_{-f_a}]$ of the composition operator C_{-f_a} on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\{v_m|m=1,2,\cdots\}$ has (l,m)-th entry*

$$[C_{-f_a}]_{lm}$$
$$=\sum_{j=0}^{l-1}\frac{(-1)^{m+l-2-j}}{j!(j+1)!}l(l-1)\cdots(l-j)\cdot$$
$$\left[\chi_{\{0\}}(j)+\chi_{[1,m)}(j)(m-1)(m-2)\cdots(m-j)\right]a^{m-1-j}\overline{a}^{l-1-j}(1-|a|^2)^{j+1}.$$

Finally we are ready to prove a kind of symmetry that the matrix $[C_{-f_a}]$ has, as follows.

Theorem 5.3. *Let U be the unit disc and let a be in U. Then the matrix $[C_{-f_a}]$ of the composition operator C_{-f_a} on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\{v_m|m=1,2,\cdots\}$ has the following modified symmetry: for any positive integers l and m,*

$$l\,[C_{-f_a}]_{ml}=m\,\overline{[C_{-f_a}]_{lm}},$$

where the symbol bar means conjugation.

Proof. It is enough to assume that the number m is bigger than l. Then we have

$$m\,\overline{[C_{-f_a}]_{lm}}$$
$$=\sum_{j=0}^{l-1}\frac{(-1)^{m+l-2-j}}{j!(j+1)!}l(l-1)\cdots(l-j)m\cdot$$
$$\left[\chi_{\{0\}}(j)+\chi_{[1,m)}(j)(m-1)(m-2)\cdots(m-j)\right]a^{l-1-j}\overline{a}^{m-1-j}(1-|a|^2)^{j+1}$$
$$=(-1)^{m+l-2}lma^{l-1}\overline{a}^{m-1}(1-|a|^2)+\sum_{j=1}^{l-1}\frac{(-1)^{m+l-2-j}}{j!(j+1)!}l(l-1)\cdots(l-j)\cdot$$

$$m\chi_{[1,m)}(j)(m-1)(m-2)\cdots(m-j)a^{l-1-j}\overline{a}^{m-1-j}(1-|a|^2)^{j+1}.$$

Notice that

$$[1,m)\cap\{j\,|\,0\le j\le l-1\}=\{1,2,\cdots,l-1\}.$$

It thus follows that the above identity equals

$$(-1)^{m+l-2}lma^{l-1}\overline{a}^{m-1}(1-|a|^2)+\sum_{j=1}^{l-1}\frac{(-1)^{m+l-2-j}}{j!(j+1)!}l(l-1)\cdots(l-j)\cdot$$
$$m\chi_{[1,l)}(j)(m-1)(m-2)\cdots(m-j)a^{l-1-j}\overline{a}^{m-1-j}(1-|a|^2)^{j+1}.$$

Now observe that

$$\begin{aligned}[1,l)\cap\{j\,|\,0\le j\le m-1\}&=\{1,2,\cdots,l-1\}\cap\{0,1,\cdots,m-1\}\\&=\{1,2,\cdots,l-1\}.\end{aligned}$$

It thus follows that the above identity equals

$$\begin{aligned}&l(-1)^{m+l-2}ma^{l-1}\overline{a}^{m-1}(1-|a|^2)+\sum_{j=1}^{m-1}\frac{(-1)^{m+l-2-j}}{j!(j+1)!}l(l-1)\cdots(l-j)\cdot\\&m\chi_{[1,l)}(j)(m-1)(m-2)\cdots(m-j)a^{l-1-j}\overline{a}^{m-1-j}(1-|a|^2)^{j+1}\\&=l(-1)^{m+l-2}ma^{l-1}\overline{a}^{m-1}(1-|a|^2)+l\sum_{j=1}^{m-1}\frac{(-1)^{m+l-2-j}}{j!(j+1)!}m(m-1)\cdots(m-j)\cdot\\&\chi_{[1,l)}(j)(l-1)(l-2)\cdots(l-j)a^{l-1-j}\overline{a}^{m-1-j}(1-|a|^2)^{j+1}\\&=l\sum_{j=0}^{m-1}\frac{(-1)^{m+l-2-j}}{j!(j+1)!}m(m-1)\cdots(m-j)\cdot\\&\left[\chi_{\{0\}}(j)+\chi_{[1,l)}(j)(l-1)(l-2)\cdots(l-j)\right]a^{l-1-j}\overline{a}^{m-1-j}(1-|a|^2)^{j+1}\\&=l\,[C_{-f_a}]_{ml},\end{aligned}$$

which proves Theorem 5.3. $\square$

References

[1] Steven R. Bell, *Complexity of the classical kernel functions of potential theory*, Indiana Univ. Math. J. **44** (1995), no. 4, 1337–1369.

[2] Paul S. Bourdon and S. Waleed Noor, *Complex symmetry of invertible composition operators*, J. Math. Anal. Appl. **429** (2015), no. 1, 105–110.

[3] Y.-B. Chung, *Classification of Toeplitz operators on Hardy spaces of bounded domains in the plane*, Math. Notes **101** (2017), no. 3-4, 529–541.

[4] Y.-B. Chung, *Special orthonormal basis for L^2 functions on the unit circle*, Bull. Korean Math. Soc. **54** (2017), no. 6, 2013–2027.

[5] Y.-B. Chung, *The matrix representation of a composition operator on the Hardy space*, Bull. Korean Math. Soc. **59** (2022), no. 5, 1119–1129.

[6] Carl C. Cowen and Barbara D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[7] Sivaram K. Narayan, Daniel Sievewright, and Maria Tjani, *Complex symmetric composition operators on weighted Hardy spaces*, Proc. Amer. Math. Soc. **148** (2020), no. 5, 2117–2127.

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