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APPLICATION OF NEW CONTRACTIVE CONDITION IN INTEGRAL EQUATION

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Abstract. In this paper, first we establish a unique common fixed point theorem satisfying new contractive condition on partially ordered non-Archimedean fuzzy metric spaces and give an example to support our result. By using the result established in the first section of the manuscript, we formulate a unique common coupled fixed point theorem and also give an example to validate our result. In the end, we study the existence of solution of integral equation to verify our hypothesis. These results generalize, improve and fuzzify several well-known results in the existing literature.

1. Introduction

George and Veeramani [6] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [13] with the help of continuous t-norm and defined the Hausdorff topology of fuzzy metric spaces. In [11], Istratescu introduced the concept of non-Archimedean fuzzy metric space.

In [7], Guo and Lakshmikantham introduced the notion of coupled fixed point for single-valued mappings. Using this notion, Gnana-Bhaskar and Lakshmikantham [3] established some coupled fixed point theorems by defining mixed monotone property. After that, Lakshmikantham and Ciric [14] extended the notion of mixed monotone property to mixed $\beta-$monotone property and established coupled coincidence point results using a pair of commutative mappings, which generalized the results of Gnana-Bhaskar and Lakshmikantham [3]. For more details one can consult [1, 2, 5, 8, 9, 12, 15, 16].

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This manuscript splits into three sections. In first section, we establish a unique common fixed point theorem satisfying new contractive condition on partially ordered non-Archimedean fuzzy metric spaces and give an example to support our result. In the second section, some multidimensional common fixed point results are derived from our main results and also give an example to validate our result. In the last section of this manuscript, we give an application to integral equation to show the fruitfulness of the obtained results. We generalize, extend, improve and fuzzify the results of Gnana-Bhaskar and Lakshmikantham [3], Lakshmikantham and Ciric [14] and several well-known results of the existing literature.

2. Preliminaries

Definition 2.1 ([17]). A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0, 1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$.

A few examples of continuous t-norm are

$$a * b = ab,\ a * b = \min\{a,\ b\} \text{ and } a * b = \max\{a + b - 1,\ 0\}.$$

Definition 2.2 ([6]). The 3-tuple $(X, M, *)$ is called *fuzzy metric space* if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for each $\varepsilon, \delta, \tau \in X$ and $t, s > 0$,

(FM_1) $M(\varepsilon, \delta, t) > 0$,
(FM_2) $M(\varepsilon, \delta, t) = 1$ iff $\varepsilon = \delta$,
(FM_3) $M(\varepsilon, \delta, t) = M(\delta, \varepsilon, t)$,
(FM_4) $M(\varepsilon, \tau, t + s) \geq M(\varepsilon, \delta, t) * M(\delta, \tau, s)$,
(FM_5) $M(\varepsilon, \delta, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

Remark 2.1. If in the above definition (FM_4) is replaced by

$$(NAFM_4)\ M(\varepsilon,\ \tau,\ \max\{t,\ s\}) \geq M(\varepsilon,\ \delta,\ t) * M(\delta,\ \tau,\ s),$$

or equivalently,

$$(NAFM_4)\ M(\varepsilon,\ \tau,\ t) \geq M(\varepsilon,\ \delta,\ t) * M(\delta,\ \tau,\ t).$$

Then $(X, M, *)$ is called a *non-Archimedean fuzzy metric space* [11]. It is easy to check that $(NAFM_4)$ implies (FM_4), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Example 2.1 ([6])**.** Let $X = [0, 1]$ equipped with the usual metric $d : X \times X \to [0, +\infty)$. Define t-norm by $a * b = ab$ and

$$M(\varepsilon,\ \delta,\ t) = \frac{t}{t + d(\varepsilon,\ \delta)} \text{ for all } \varepsilon,\ \delta \in X \text{ and } t > 0.$$

Then $(X, M, *)$ is a fuzzy metric space but not a non-Archimedean fuzzy metric space.

Remark 2.2 ([6])**.** In fuzzy metric space $(X, M, *)$, $M(\varepsilon, \delta, \cdot)$ is non-decreasing for all $\varepsilon, \delta \in X$.

Definition 2.3 ([6])**.** Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{\varepsilon_n\}_n$ in X is called *Cauchy* if for each $\eta \in (0, 1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that

$$M(\varepsilon_n,\ \varepsilon_m,\ t) > 1 - \eta \text{ whenever } n \geq m \geq n_0.$$

We say that $(X, M, *)$ is *complete* if every Cauchy sequence is convergent, that is, if there exists $\delta \in X$ such that $\lim_{n\to\infty} M(\varepsilon_n, \delta, t) = 1$, for all $t > 0$.

Definition 2.4 ([3])**.** Let $\alpha : X^2 \to X$ be a given mapping. An element $(\varepsilon, \delta) \in X^2$ is called a *coupled fixed point* of α if $\alpha(\varepsilon, \delta) = \varepsilon$ and $\alpha(\delta, \varepsilon) = \delta$.

Definition 2.5 ([3])**.** Let $(X, \preceq)$ be a partially ordered set and $\alpha : X^2 \to X$ be a given mapping. We say that α has the *mixed monotone property* if for all $\varepsilon, \delta \in X$, we have

$$\begin{aligned} &\varepsilon_1,\ \varepsilon_2 \in X,\ \varepsilon_1 \preceq \varepsilon_2 \implies \alpha(\varepsilon_1,\ \delta) \preceq \alpha(\varepsilon_2,\ \delta), \\ &\delta_1,\ \delta_2 \in X,\ \delta_1 \preceq \delta_2 \implies \alpha(\varepsilon,\ \delta_1) \succeq \alpha(\varepsilon,\ \delta_2). \end{aligned}$$

Definition 2.6 ([14])**.** Let $\alpha : X^2 \to X$ and $\beta : X \to X$ be given mappings. An element $(\varepsilon, \delta) \in X^2$ is called a *coupled coincidence point* of the mappings α and β if $\alpha(\varepsilon, \delta) = \beta\varepsilon$ and $\alpha(\delta, \varepsilon) = \beta\delta$.

Definition 2.7 ([14])**.** Let $\alpha : X^2 \to X$ and $\beta : X \to X$ be given mappings. An element $(\varepsilon, \delta) \in X^2$ is called a *common coupled fixed point* of the mappings α and β if $\varepsilon = \alpha(\varepsilon, \delta) = \beta\varepsilon$ and $\delta = \alpha(\delta, \varepsilon) = \beta\delta$.

Definition 2.8 ([14])**.** The mappings $\alpha : X^2 \to X$ and $\beta : X \to X$ are said to be *commutative* if $\beta\alpha(\varepsilon, \delta) = \alpha(\beta\varepsilon, \beta\delta)$, for all $(\varepsilon, \delta) \in X^2$.

Definition 2.9 ([14])**.** Let $(X, \preceq)$ be a partially ordered set. Suppose $\alpha : X^2 \to X$ and $\beta : X \to X$ are given mappings. We say that α has the *mixed $\beta-$monotone property* if for all $\varepsilon, \delta \in X$, we have

$$\varepsilon_1,\ \varepsilon_2 \in X,\ \beta\varepsilon_1 \preceq \beta\varepsilon_2 \implies \alpha(\varepsilon_1,\ \delta) \preceq \alpha(\varepsilon_2,\ \delta),$$
$$\delta_1,\ \delta_2 \in X,\ \beta\delta_1 \preceq \beta\delta_2 \implies \alpha(\varepsilon,\ \delta_1) \succeq \alpha(\varepsilon,\ \delta_2).$$

If β is the identity mapping on X, then α satisfies the mixed monotone property.

Definition 2.10 ([3, 5])**.** A partially ordered metric space $(X, d, \preceq)$ is a metric space (X, d) provided with a partial order $\preceq$. A partially ordered metric space $(X, d, \preceq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for every sequence $\{\varepsilon_n\} \subseteq \varepsilon$ such that $\{\varepsilon_n\} \to \varepsilon$ and $\varepsilon_n \preceq \varepsilon_{n+1}$ (respectively, $\varepsilon_n \succeq \varepsilon_{n+1}$) for all $n \geq 0$, we have $\varepsilon_n \preceq \varepsilon$ (respectively, $\varepsilon_n \succeq \varepsilon$) for all $n \geq 0$. $(X, d, \preceq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular. Let α, $\beta : X \to X$ be two mappings. We say that α is $(\beta, \preceq)-$*non-decreasing* if $\alpha\varepsilon \preceq \alpha\delta$ for all $\varepsilon, \delta \in X$ such that $\beta\varepsilon \preceq \beta\delta$. If β is the identity mapping on X, we say that α is $\preceq -$*non-decreasing*. If α is $(\beta, \preceq)-$non-decreasing and $\beta\varepsilon = \beta\delta$, then $\alpha\varepsilon = \alpha\delta$. Moreover, $\varepsilon, \delta \in X$ are comparable, that is, $\varepsilon \preceq \delta$ or $\delta \preceq \varepsilon$, then we will write $\varepsilon \asymp \delta$.

Definition 2.11 ([4])**.** Let $(X, M, *)$ be an partially ordered fuzzy metric space. Two mappings $\alpha, \beta : X \to X$ are said to be *compatible* if $\lim\limits_{n\to\infty} M(\beta\alpha\varepsilon_n, \alpha\beta\varepsilon_n, t) = 1$, provided that $\{\varepsilon_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} \alpha\varepsilon_n = \lim\limits_{n\to\infty} \beta\varepsilon_n = \varepsilon \in X$.

Definition 2.12 ([10])**.** Let $\alpha : X^2 \to X$ and $\beta : X \to X$ be two mappings. We say that the pair $\{\alpha, \beta\}$ is *compatible* if

$$\begin{aligned}\lim_{n\to\infty} M(\alpha(\beta\varepsilon_n,\ \beta\delta_n),\ \beta(\alpha(\varepsilon_n,\ \delta_n),\ \alpha(\delta_n,\ \varepsilon_n)),\ t) &= 1,\\ \lim_{n\to\infty} M(\alpha(\beta\delta_n,\ \beta\varepsilon_n),\ \beta(\alpha(\delta_n,\ \varepsilon_n),\ \alpha(\varepsilon_n,\ \delta_n)),\ t) &= 1,\end{aligned}$$

whenever (ε_n) and (δ_n) are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} \beta\varepsilon_n &= \lim_{n\to\infty} \alpha(\varepsilon_n,\ \delta_n) = \varepsilon \in X,\\ \lim_{n\to\infty} \beta\delta_n &= \lim_{n\to\infty} \alpha(\delta_n,\ \varepsilon_n) = \delta \in X.\end{aligned}$$

Definition 2.13 ([10])**.** Let X be a non-empty set. Mappings $\alpha : X^2 \to X$ and $\beta : X \to X$ are called *weakly compatible* if

$$\alpha(\varepsilon,\ \delta) = \beta\varepsilon \text{ and } \alpha(\delta,\ \varepsilon) = \beta\delta,$$

implies that

$$\beta(\alpha(\varepsilon,\ \delta),\ \alpha(\delta,\ \varepsilon)) = \alpha(\beta\varepsilon,\ \beta\delta)\ \text{ and } \beta(\alpha(\delta,\ \varepsilon),\ \alpha(\varepsilon,\ \delta)) = \alpha(\beta\delta,\ \beta\varepsilon),$$

for all $\varepsilon, \delta \in X$.

3. Fixed Point Results

Throughout the paper, X denotes a non-empty set, β represents a self-mapping on X and $\beta(\varepsilon)$ is abbreviated by $\beta\varepsilon$ where $\varepsilon \in X$.

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying
(i_φ) φ is non-decreasing,
(ii_φ) $\varphi(t) < t$ for all $t > 0$,
(iii_φ) $\lim_{r\to t+} \varphi(r) < t$ for all $t > 0$.

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfies
(i_ψ) ψ is continuous and non-decreasing,
(ii_ψ) $\psi(t) < t$ for all $t > 0$.

Note that, by (i_ψ) and (ii_ψ) we have that $\psi(t) = 0$ if and only if $t = 0$.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and there be a non-Archimedean fuzzy metric M such that $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Suppose α and β are two self mappings on X satisfying*

(i) α is $(\beta, \preceq)$-non-decreasing and $\alpha(X) \subseteq \beta(X)$,

(ii) there exists $\varepsilon_0 \in X$ such that $\beta\varepsilon_0 \preceq \alpha\varepsilon_0$,

(iii) there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that

(3.1)
$$\left(\frac{1}{M(\alpha\varepsilon,\ \alpha\delta,\ t)} - 1\right)$$
$$\leq \varphi\left(\frac{1}{M(\beta\varepsilon,\ \beta\delta,\ t)} - 1\right) + \psi\left(\frac{1}{P(\varepsilon,\ \delta)} - 1\right),\ \textit{for all } \varepsilon,\ \delta \in X \textit{ with } \beta\varepsilon \preceq \beta\delta,$$

where

$$P(\varepsilon,\ \delta) = \max\left\{ \begin{array}{l} M(\alpha\varepsilon,\ \beta\varepsilon,\ t),\ M(\alpha\delta,\ \beta\delta,\ t), \\ M(\alpha\delta,\ \beta\varepsilon,\ t),\ M(\alpha\varepsilon,\ \beta\delta,\ t) \end{array} \right\}. \tag{3.2}$$

Also assume that one of the following conditions holds.

(a) (X, M) is complete, α and β are continuous and the pair (α, β) is compatible,

(b) $(\beta(X), M)$ is complete and $(X, M, \preceq)$ is non-decreasing-regular,

(c) (X, M) is complete, β is continuous and monotone non-decreasing, the pair (α, β) is compatible and $(X, M, \preceq)$ is non-decreasing-regular.

Then α and β have a coincidence point. Furthermore, if (*d*) *for every* ε, $\delta \in X$, *there exists* $\theta \in X$ *such that* $\alpha\theta \asymp \alpha\varepsilon$ *and* $\alpha\theta \asymp \alpha\delta$, *and also the pair* (α, β) *is weakly compatible.*

Then α *and* β *have a unique common fixed point.*

Proof. Let $\varepsilon_0 \in X$ be arbitrary. By (i), $\alpha(X) \subseteq \beta(X)$, there exists $\varepsilon_1 \in X$ such that $\beta\varepsilon_1 = \alpha\varepsilon_0$. Now by (ii), $\beta\varepsilon_0 \preceq \alpha\varepsilon_0 = \beta\varepsilon_1$. Since α is $(\beta, \preceq)$ non-decreasing, $\alpha\varepsilon_0 \preceq \alpha\varepsilon_1$. Again $\alpha\varepsilon_1 \in \alpha(X) \subseteq \beta(X)$, there exists $\varepsilon_2 \in X$ such that $\beta\varepsilon_2 = \alpha\varepsilon_1$. Then $\beta\varepsilon_1 = \alpha\varepsilon_0 \preceq \alpha\varepsilon_1 = \beta\varepsilon_2$. Since α is $(\beta, \preceq)-$non-decreasing, $\alpha\varepsilon_1 \preceq \alpha\varepsilon_2$. Proceeding in the similar manner, we get a sequence $\{\varepsilon_n\}_{n\geq 0}$ such that $\{\beta\varepsilon_n\}$ is $\preceq$-non-decreasing, $\beta\varepsilon_{n+1} = \alpha\varepsilon_n \preceq \alpha\varepsilon_{n+1} = \beta\varepsilon_{n+2}$ and

$$\beta\varepsilon_{n+1} = \alpha\varepsilon_n, \text{ for all } n \geq 0. \tag{3.3}$$

Let

$$\zeta_n = \left(\frac{1}{M(\beta\varepsilon_n,\ \beta\varepsilon_{n+1},\ t)} - 1\right), \text{ for all } n \geq 0.$$

Since $\beta\varepsilon_n \preceq \beta\varepsilon_{n+1}$, therefore by using contractive condition (3.1), we have

$$\begin{aligned}
&\left(\frac{1}{M(\beta\varepsilon_{n+1},\ \beta\varepsilon_{n+2},\ t)} - 1\right) \\
&= \left(\frac{1}{M(\alpha\varepsilon_n,\ \alpha\varepsilon_{n+1},\ t)} - 1\right) \\
&\leq \varphi\left(\frac{1}{M(\beta\varepsilon_n,\ \beta\varepsilon_{n+1},\ t)} - 1\right) + \psi\left(\frac{1}{P(\varepsilon_n,\ \varepsilon_{n+1})} - 1\right).
\end{aligned}$$

Thus, by the fact that $\psi(0) = 0$, we get

$$\left(\frac{1}{M(\beta\varepsilon_{n+1},\ \beta\varepsilon_{n+2},\ t)} - 1\right) \leq \varphi\left(\frac{1}{M(\beta\varepsilon_n,\ \beta\varepsilon_{n+1},\ t)} - 1\right), \tag{3.4}$$

which, by (ii_φ), implies

$$\frac{1}{M(\beta\varepsilon_{n+1},\ \beta\varepsilon_{n+2},\ t)} - 1 < \frac{1}{M(\beta\varepsilon_n,\ \beta\varepsilon_{n+1},\ t)} - 1, \text{ that is, } \zeta_{n+1} < \zeta_n.$$

This shows that the sequence $\{\zeta_n\}_{n\geq 0}$ is a decreasing sequence of positive numbers. Then there exists $\zeta \geq 0$ such that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \left(\frac{1}{M(\beta\varepsilon_n,\ \beta\varepsilon_{n+1},\ t)} - 1\right) = \zeta. \tag{3.5}$$

Suppose that $\zeta > 0$. Letting $n \to \infty$ in (3.4), by using (3.5) and (iii_φ), we get

$$\zeta \leq \lim_{n\to\infty} \varphi(\zeta_n) = \lim_{\zeta_n\to\zeta+} \varphi(\zeta_n) < \zeta,$$

which is a contradiction. Hence

$$\lim_{n\to\infty}\zeta_n = \lim_{n\to\infty}\left(\frac{1}{M(\beta\varepsilon_n,\ \beta\varepsilon_{n+1},\ t)} - 1\right) = 0,$$

or

$$\lim_{n\to\infty} M(\beta\varepsilon_n,\ \beta\varepsilon_{n+1},\ t) = 1. \tag{3.6}$$

We now claim that $\{\beta\varepsilon_n\}_{n\geq 0}$ is a Cauchy sequence in X. Suppose, to the contrary, that it is not a Cauchy sequence. Then there exists an $\eta > 0$ for which we can find subsequences $\{\beta\varepsilon_{n(k)}\}$, $\{\beta\varepsilon_{m(k)}\}$ of $\{\beta\varepsilon_n\}_{n\geq 0}$ such that

$$M(\beta\varepsilon_{n(k)},\ \beta\varepsilon_{m(k)},\ t) \leq 1-\eta,\ k = 1,\ 2,\ ... \tag{3.7}$$

We can choose $n(k)$ to be the smallest positive integer satisfying (3.7), then

$$M(\beta\varepsilon_{n(k)-1},\ \beta\varepsilon_{m(k)},\ t) > 1-\eta. \tag{3.8}$$

By (3.7), (3.8) and $(NAFM_4)$, we have

$$\begin{aligned}1-\eta \geq \omega_k &= M(\beta\varepsilon_{n(k)},\ \beta\varepsilon_{m(k)},\ t)\\ &\geq M(\beta\varepsilon_{n(k)},\ \beta\varepsilon_{n(k)-1},\ t) * M(\beta\varepsilon_{n(k)-1},\ \beta\varepsilon_{m(k)},\ t),\\ &> M(\beta\varepsilon_{n(k)},\ \beta\varepsilon_{n(k)-1},\ t) * (1-\eta).\end{aligned}$$

Letting $k\to\infty$ in the above inequality and using (3.6), we get

$$\lim_{k\to\infty}\omega_k = \lim_{k\to\infty} M(\beta\varepsilon_{n(k)},\ \beta\varepsilon_{m(k)},\ t) = 1-\eta. \tag{3.9}$$

By $(NAFM_4)$, we have

$$\begin{aligned}\omega_k &= M(\beta\varepsilon_{n(k)},\ \beta\varepsilon_{m(k)},\ t)\\ &\geq M(\beta\varepsilon_{n(k)},\ \beta\varepsilon_{n(k)+1},\ t) * M(\beta\varepsilon_{n(k)+1},\ \beta\varepsilon_{m(k)+1},\ t) * M(\beta\varepsilon_{m(k)+1},\ \beta\varepsilon_{m(k)},\ t).\end{aligned}$$

Letting $k\to\infty$ in the above inequalities, using (3.6) and (3.9), we have

$$\lim_{k\to\infty} M(\beta\varepsilon_{n(k)+1},\ \beta\varepsilon_{m(k)+1},\ t) = 1-\eta. \tag{3.10}$$

Since $n(k) > m(k)$, $\beta\varepsilon_{n(k)} \succeq \beta\varepsilon_{m(k)}$, by using contractive condition (3.1),

$$\begin{aligned}&\left(\frac{1}{M(\beta\varepsilon_{n(k)+1},\ \beta\varepsilon_{m(k)+1},\ t)} - 1\right)\\ &= \left(\frac{1}{M(\alpha\varepsilon_{n(k)},\ \alpha\varepsilon_{m(k)},\ t)} - 1\right)\\ &\leq \varphi\left(\frac{1}{M(\beta\varepsilon_{n(k)},\ \beta\varepsilon_{m(k)},\ t)} - 1\right) + \psi\left(\frac{1}{P(\varepsilon_{n(k)},\ \varepsilon_{m(k)})} - 1\right).\end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using (3.9), (3.10), (iii_φ) and by the fact that $\psi(0) = 0$, we get

$$\frac{\varepsilon}{1-\varepsilon} \leq \lim_{k\to\infty} \varphi\left(\frac{1}{\omega_k} - 1\right) + 0 \leq \lim_{r_k\to\varepsilon+} \varphi\left(\frac{1}{\omega_k} - 1\right) < \frac{\varepsilon}{1-\varepsilon},$$

which is a contradiction. This proves that $\{\beta\varepsilon_n\}_{n\geq 0}$ is a Cauchy sequence in X.

Suppose that (a) holds, that is, $(X,\ M)$ is complete, α and β are continuous and $(\alpha,\ \beta)$ is compatible. Since $(X,\ M)$ is complete, there exists $\varepsilon \in X$ such that $\{\beta\varepsilon_n\} \to \varepsilon$. It follows from (3.3) that $\{\alpha\varepsilon_n\} \to \varepsilon$. Since α and β are continuous, $\{\beta\alpha\varepsilon_n\} \to \beta\varepsilon$ and $\{\alpha\alpha\varepsilon_n\} \to \alpha\varepsilon$. Since the pair $(\alpha,\ \beta)$ is compatible,

$$M(\alpha\varepsilon,\ \beta\varepsilon,\ t) = \lim_{n\to\infty} M(\alpha\alpha\varepsilon_{n+1},\ \beta\alpha\varepsilon_n,\ t) = \lim_{n\to\infty} M(\alpha\beta\varepsilon_n,\ \beta\alpha\varepsilon_n,\ t) = 1,$$

that is, ε is a coincidence point of α and β.

Suppose now (b) holds, that is, $(\beta(X),\ M)$ is complete and $(X,\ M,\ \preceq)$ is non-decreasing-regular. Since $\{\beta\varepsilon_n\}$ is a Cauchy sequence in the complete space $(\beta(X),\ M)$, there exists $\delta \in \beta(X)$ such that $\{\beta\varepsilon_n\} \to \delta$. Let $\varepsilon \in X$ be any point such that $\delta = \beta\varepsilon$, then $\{\beta\varepsilon_n\} \to \beta\varepsilon$. Since $(X,\ M,\ \preceq)$ is non-decreasing-regular and $\{\beta\varepsilon_n\}$ is $\preceq$-non-decreasing which converges to $\beta\varepsilon$, we obtain that $\beta\varepsilon_n \preceq \beta\varepsilon$, for all $n \geq 0$. Applying the contractive condition (3.1), we have

$$\begin{aligned}
&\left(\frac{1}{M(\beta\varepsilon_{n+1},\ \alpha\varepsilon,\ t)} - 1\right) \\
&= \left(\frac{1}{M(\alpha\varepsilon_n,\ \alpha\varepsilon,\ t)} - 1\right) \\
&\leq \varphi\left(\frac{1}{M(\beta\varepsilon_n,\ \beta\varepsilon,\ t)} - 1\right) + \psi\left(\frac{1}{P(\varepsilon_n,\ \varepsilon)} - 1\right).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and by using the continuity of ψ, $\{\beta\varepsilon_n\} \to \beta\varepsilon$ and (iii_φ), we get

$$\frac{1}{M(\beta\varepsilon,\ \alpha\varepsilon,\ t)} - 1 \leq \lim_{t\to 0+} \varphi(t) + 0 = 0 + 0 = 0.$$

It follows that $M(\beta\varepsilon,\ \alpha\varepsilon,\ t) = 1$, that is, ε is a coincidence point of α and β.

Suppose now that (c) holds, that is, $(X,\ M)$ is complete, β is continuous and monotone-non-decreasing, the pair $(\alpha,\ \beta)$ is compatible and $(X,\ M,\ \preceq)$ is non-decreasing-regular. Since $(X,\ M)$ is complete, there exists $\varepsilon \in X$ such that $\{\beta\varepsilon_n\} \to \varepsilon$, which, by (3.3), implies $\{\alpha\varepsilon_n\} \to \varepsilon$. Since β is continuous, $\{\beta\beta\varepsilon_n\} \to \beta\varepsilon$ and the pair $(\alpha,\ \beta)$ is compatible,

$$\lim_{n\to\infty} M(\beta\beta\varepsilon_{n+1},\ \alpha\beta\varepsilon_n,\ t) = \lim_{n\to\infty} M(\beta\alpha\varepsilon_n,\ \alpha\beta\varepsilon_n,\ t) = 1,$$

implies that $\{\alpha\beta\varepsilon_n\} \to \beta\varepsilon$.

Also, since $(X, d, \preceq)$ is non-decreasing-regular and $\{\beta\varepsilon_n\}$ is $\preceq$ −non-decreasing which converges to ε, we find that $\beta\varepsilon_n \preceq \varepsilon$, which, by the monotonicity of β, implies $\beta\beta\varepsilon_n \preceq \beta\varepsilon$. Using the contractive condition (3.1), we get

$$\frac{1}{M(\alpha\beta\varepsilon_n,\ \alpha\varepsilon,\ t)} - 1 \leq \varphi\left(\frac{1}{M(\beta\beta\varepsilon_n,\ \beta\varepsilon,\ t)} - 1\right) + \psi\left(\frac{1}{P(\beta\varepsilon_n,\ \varepsilon)} - 1\right).$$

Taking limit as $n \to \infty$, using the continuity of ψ, $\{\beta\beta\varepsilon_n\} \to \beta\varepsilon$, $\{\alpha\beta\varepsilon_n\} \to \beta\varepsilon$ and (iii_φ), we get

$$\frac{1}{M(\beta\varepsilon,\ \alpha\varepsilon,\ t)} - 1 \leq \lim_{t \to 0+} \varphi(t) + 0 = 0 + 0 = 0.$$

It follows that $M(\beta\varepsilon, \alpha\varepsilon, t) = 1$, that is, ε is a coincidence point of α and β.

Thus the set of coincidence points of α and β is non-empty. Let ε and δ be two coincidence points of α and β, that is, $\alpha\varepsilon = \beta\varepsilon$ and $\alpha\delta = \beta\delta$. Now, we claim that $\beta\varepsilon = \beta\delta$. By the assumption, there exists $\theta \in X$ such that $\alpha\theta$ is comparable with $\alpha\varepsilon$ and $\alpha\delta$. Put $\theta_0 = \theta$ and choose $\theta_1 \in \varepsilon$ so that $\beta\theta_0 = \alpha\theta_1$. Then, one can inductively get the sequence $\{\beta\theta_n\}$ where $\beta\theta_{n+1} = \alpha\theta_n$, for all $n \geq 0$. Hence $\alpha\varepsilon = \beta\varepsilon$ and $\alpha\theta = \alpha\theta_0 = \beta\theta_1$ are comparable. One can easily get that $\beta\theta_n \preceq \beta\varepsilon$, for all $n \geq 0$.

Let

$$\xi_n = \left(\frac{1}{M(\beta\theta_n,\ \beta\varepsilon,\ t)} - 1\right), \text{ for all } n \geq 0.$$

Since $\beta\theta_n \preceq \beta\varepsilon$, using the contractive condition (3.1) and (3.3), we have

$$\begin{aligned}\left(\frac{1}{M(\beta\theta_{n+1},\ \beta\varepsilon,\ t)} - 1\right) &= \left(\frac{1}{M(\alpha\theta_n,\ \alpha\varepsilon,\ t)} - 1\right)\\ &\leq \varphi\left(\frac{1}{M(\beta\theta_n,\ \beta\varepsilon,\ t)} - 1\right) + \psi\left(\frac{1}{P(\theta_n,\ \varepsilon)} - 1\right),\end{aligned}$$

Thus, by the fact that $\psi(0) = 0$, we get

$$\left(\frac{1}{M(\beta\theta_{n+1},\ \beta\varepsilon,\ t)} - 1\right) \leq \varphi\left(\frac{1}{M(\beta\theta_n,\ \beta\varepsilon,\ t)} - 1\right), \tag{3.11}$$

which, by (ii_φ), implies

$$\frac{1}{M(\beta\theta_{n+1},\ \beta\varepsilon,\ t)} - 1 < \frac{1}{M(\beta\theta_n,\ \beta\varepsilon,\ t)} - 1, \text{ that is, } \xi_{n+1} < \xi_n.$$

This shows that the sequence $\{\xi_n\}_{n\geq 0}$ is a decreasing sequence of positive numbers, then there exists $\xi \geq 0$ such that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty}\left(\frac{1}{M(\beta\theta_n,\ \beta\varepsilon,\ t)} - 1\right) = \xi. \tag{3.12}$$

Now, suppose that $\xi > 0$. On taking limit $n \to \infty$ in (3.11), using (3.12) and (iii_φ), we get

$$\xi \le \lim_{n\to\infty} \varphi(\xi_n) = \lim_{\xi_n \to \xi+} \varphi(\xi_n) < \xi,$$

which is a contradiction. Hence

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \left(\frac{1}{M(\beta\theta_n,\ \beta\varepsilon,\ t)} - 1 \right) = 0, \tag{3.13}$$

that is,

$$\lim_{n\to\infty} M(\beta\theta_n,\ \beta\varepsilon,\ t) = 1. \tag{3.14}$$

Similarly, one can obtain that

$$\lim_{n\to\infty} M(\beta\theta_n,\ \beta\delta,\ t) = 1. \tag{3.15}$$

Hence, by (3.14) and (3.15), we get

$$\beta\varepsilon = \beta\delta. \tag{3.16}$$

Since $\alpha\varepsilon = \beta\varepsilon$, by weak compatibility of α and β, $\alpha\beta\varepsilon = \beta\alpha\varepsilon = \beta\beta\varepsilon$. Let $\theta = \beta\varepsilon$, then $\alpha\theta = \beta\theta$, that is, θ is a coincidence point of α and β. Then from (3.16) with $\delta = \theta$, it follows that $\beta\varepsilon = \beta\theta$, that is, $\theta = \alpha\theta = \beta\theta$. Hence θ is a common fixed point of α and β. To prove the uniqueness, assume that π is another common fixed point of α and β. Then by (3.16) we have $\pi = \beta\pi = \beta\theta = \theta$, that is, the common fixed point of α and β is unique. □

If we put $\psi(t) = 0$ in the Theorem 3.1, we get the following result:

Corollary 3.2. *Let $(X, \preceq)$ be a partially ordered set and there be a non-Archimedean fuzzy metric M such that $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Suppose α and β are two self mappings on X satisfying (i) and (ii) of Theorem 3.1 and there exists $\varphi \in \Phi$ such that*

$$\left(\frac{1}{M(\alpha\varepsilon,\ \alpha\delta,\ t)} - 1 \right) \le \varphi \left(\frac{1}{M(\beta\varepsilon,\ \beta\delta,\ t)} - 1 \right), \text{ for all } \varepsilon,\ \delta \in X \text{ with } \beta\varepsilon \preceq \beta\delta.$$

Also assume that one of the conditions $(a)-(c)$ of Theorem 3.1 holds, then α and β have a coincidence point. Furthermore, if condition (d) of Theorem 3.1 holds, then α and β have a unique common fixed point.

If we put $\varphi(t) = kt$ where $0 < k < 1$, for all $t \ge 0$ in Corollary 3.2, then we get the following result:

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set and there be a non-Archimedean fuzzy metric M such that $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Suppose α and β are two self mappings on X satisfying (i) and (ii) of Theorem 3.1 and there exists $k < 1$ such that*

$$\left(\frac{1}{M(\alpha\varepsilon,\ \alpha\delta,\ t)} - 1\right) \leq k\left(\frac{1}{M(\beta\varepsilon,\ \beta\delta,\ t)} - 1\right),\ \textit{for all } \varepsilon,\ \delta \in X \textit{ with } \beta\varepsilon \preceq \beta\delta.$$

Also assume that one of the conditions $(a) - (c)$ of Theorem 3.1 holds, then α and β have a coincidence point. Furthermore, if condition (d) of Theorem 3.1 holds, then α and β have a unique common fixed point.

Put $\beta = I$ (the identity mapping) in Corollary 3.3, we get the following result:

Corollary 3.4. Let $(X, \preceq)$ *be a partially ordered set and there be a non-Archimedean fuzzy metric M such that $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Suppose α be a self mapping on X satisfying*

(i) α is $\preceq$-non-decreasing,

(ii) there exists $\varepsilon_0 \in X$ such that $\varepsilon_0 \preceq \alpha\varepsilon_0$,

(iii) there exists some $k < 1$ such that

$$\left(\frac{1}{M(\alpha\varepsilon,\ \alpha\delta,\ t)} - 1\right) \leq k\left(\frac{1}{M(\varepsilon,\ \delta,\ t)} - 1\right),\ \textit{for all } \varepsilon,\ \delta \in X \textit{ with } \varepsilon \preceq \delta.$$

Then α has a fixed point.

Example 3.1. Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$ with the natural ordering of real numbers $\leq$ and $*$ is defined by $x * y = xy$, for all $x, y \in [0, 1]$. Define

$$M(\varepsilon,\ \delta,\ t) = \frac{t}{t + d(\varepsilon,\ \delta)},\ \text{for all } \varepsilon,\ \delta \in X \text{ and } t > 0.$$

Clearly $(X, M, *)$ is a complete non-Archimedean fuzzy metric space. Let $\alpha, \beta : X \to X$ be defined as

$$\alpha\varepsilon = \frac{\varepsilon^2}{2} \text{ and } \beta\varepsilon = \varepsilon^2,\ \text{for all } \varepsilon \in X.$$

One can easily see that the contractive condition of Theorem 3.1 is satisfied with $\varphi(t) = t/2$ and $\psi(t) = t/4$, for all $t \geq 0$. Furthermore, all the other conditions of Theorem 3.1 are also satisfied and 0 is a unique common fixed point of α and β.

4. Coupled Fixed Point Results

Next, we deduce the two dimensional version of Theorem 3.1. For the partially ordered non-Archimedean fuzzy metric space $(X,\ M,\ \preceq)$, let us consider the partially ordered fuzzy metric space $(X^2,\ M_2,\ \sqsubseteq)$, where $M_2 : X^2 \times X^2 \times [0,\ \infty) \to [0,\ 1]$ defined by

$$M_2(V_1,\ V_2,\ t) = \min\{M(\varepsilon_1,\ \varepsilon_2,\ t),\ M(\delta_1,\ \delta_2,\ t)\},$$

for all $V_1 = (\varepsilon_1,\ \delta_1),\ V_2 = (\varepsilon_2,\ \delta_2) \in X^2$. It is easy to check that M_2 is a non-Archimedean fuzzy metric on X^2 and $\sqsubseteq$ was introduced in

$$V_1 \sqsubseteq V_2 \Leftrightarrow \varepsilon_1 \succeq \varepsilon_2 \text{ and } \delta_1 \preceq \delta_2,\ \text{ for all } V_1 = (\varepsilon_1,\ \delta_1),\ V_2 = (\varepsilon_2,\ \delta_2) \in X^2.$$

Let $F : X^2 \to X$ and $G : X \to X$ be two mappings. Define $\Theta_F,\ \Theta_G : X^2 \to X^2$ as follows:

$$\Theta_F(V) = (F(\varepsilon,\ \delta),\ F(\delta,\ \varepsilon)) \text{ and } \Theta_G(V) = (G\varepsilon,\ G\delta),\ V = (\varepsilon,\ \delta) \in X^2.$$

Under these conditions, the following properties hold.

Lemma 4.1. *Let $(X,\ \preceq)$ be a partially ordered set and there be a non-Archimedean fuzzy metric M such that $(X,\ M,\ *)$ be a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$, $G : X \to X$ and $\Theta_F,\ \Theta_G : X^2 \to X^2$ are mappings. Then*

(1) $(X,\ M)$ is complete if and only if $(X^2,\ M_2)$ is complete.

(2) If $(X,\ M,\ \preceq)$ is regular, then $(X^2,\ M_2,\ \sqsubseteq)$ is also regular.

(3) If F is $M-$continuous, then Θ_F is M_2-continuous.

(4) F has the mixed monotone property with respect to $\preceq$ if and only if Θ_F is $\sqsubseteq$-non-decreasing.

(5) F has the mixed $G-$monotone property with respect to $\preceq$ if and only if then Θ_F is $(\Theta_G,\ \sqsubseteq)$-non-decreasing.

(6) If there exist two elements $\varepsilon_0,\ \delta_0 \in X$ with $G\varepsilon_0 \preceq F(\varepsilon_0,\ \delta_0)$ and $G\delta_0 \succeq F(\delta_0,\ \varepsilon_0)$, then there exists a point $V_0 = (\varepsilon_0,\ \delta_0) \in X^2$ such that $\Theta_G(V_0) \sqsubseteq \Theta_F(V_0)$.

(7) If $F(X^2) \subseteq G(X)$, then $\Theta_F(X^2) \subseteq \Theta_G(X^2)$.

(8) If F and G are commuting in $(X,\ M,\ \preceq)$, then Θ_F and Θ_G are also commuting in $(X^2,\ M_2,\ \sqsubseteq)$.

(9) If F and G are compatible in $(X,\ M,\ \preceq)$, then Θ_F and Θ_G are also compatible in $(X^2,\ M_2,\ \sqsubseteq)$.

(10) If F and G are weak compatible in $(X,\ M,\ \preceq)$, then Θ_F and Θ_G are also weak compatible in $(X^2,\ M_2,\ \sqsubseteq)$.

(11) A point $(\varepsilon,\ \delta) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of Θ_F and Θ_G.

(12) A point $(\varepsilon,\ \delta) \in X^2$ is a coupled fixed point of F if and only if it is a fixed point of Θ_F.

Theorem 4.1. *Let $(X,\ \preceq)$ be a partially ordered set and there be a non-Archimedean fuzzy metric M such that $(X,\ M,\ *)$ be a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ and $G : X \to X$ are two mappings such that F has mixed $G-$monotone property with respect to $\preceq$ on X for which there exists $\varphi \in \Phi$ satisfying*

$$\begin{aligned} &\left(\frac{1}{M(F(\varepsilon_1,\ \delta_1),\ F(\varepsilon_2,\ \delta_2),\ t)} - 1\right) \\ &\leq \varphi\left(\frac{1}{\min\{M(G\varepsilon_1,\ G\varepsilon_2,\ t),\ M(G\delta_1,\ G\delta_2,\ t)\}} - 1\right), \end{aligned} \tag{4.1}$$

for all $\varepsilon_1,\ \delta_1,\ \varepsilon_2,\ \delta_2 \in X$, where $G\varepsilon_1 \preceq G\varepsilon_2$ and $G\delta_1 \succeq G\delta_2$. Suppose that $F(X^2) \subseteq G(X)$, G is continuous and monotone non-decreasing and the pair $\{F,\ G\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) $(X,\ M,\ \preceq)$ is regular.

Suppose there exists two elements $\varepsilon_0,\ \delta_0 \in X$ with

$$G\varepsilon_0 \preceq F(\varepsilon_0,\ \delta_0) \text{ and } G\delta_0 \succeq F(\delta_0,\ \varepsilon_0).$$

Then F and G have a coupled coincidence point. Furthermore, suppose that for every $(\varepsilon,\ \delta)$, $(\varepsilon^*, \delta^*) \in X^2$, there exists a point $(\varepsilon',\ \delta') \in X^2$ such that $(F(\varepsilon',\ \delta'),$ $F(\delta,\ \varepsilon)) \asymp (F(\varepsilon,\ \delta),\ F(\delta,\ \varepsilon))$ and $(F(\varepsilon',\ \delta'),\ F(\delta,\ \varepsilon)) \asymp (F(\varepsilon^*,\ \delta^*),\ F(\delta^*,\ \varepsilon^*))$, and also the pair $(F,\ G)$ is weakly compatible. Then F and G have a unique common coupled fixed point.

Proof. Let $V_1 = (\varepsilon_1,\ \delta_1)$ and $V_2 = (\varepsilon_2,\ \delta_2) \in X^2$ with $\Theta_G(V_1) \sqsubseteq \Theta_G(V_2)$. Then $G\varepsilon_1 \preceq G\varepsilon_2$ and $G\delta_1 \succeq G\delta_2$ and so by using (4.1), we have

$$\left(\frac{1}{M(F(\varepsilon_1,\ \delta_1), F(\varepsilon_2,\ \delta_2),\ t)} - 1\right) \leq \varphi\left(\frac{1}{\min\{M(G\varepsilon_1,\ G\varepsilon_2,\ t),\ M(G\delta_1,\ G\delta_2,\ t)\}} - 1\right).$$

Furthermore taking into account that $G\delta_1 \succeq G\delta_2$ and $G\varepsilon_1 \preceq G\varepsilon_2$, (4.1) also guarantees that

$$\left(\frac{1}{M(F(\delta_1,\ \varepsilon_1),\ F(\delta_2,\ \varepsilon_2),\ t)} - 1\right) \leq \varphi\left(\frac{1}{\min\{M(G\varepsilon_1,\ G\varepsilon_2,\ t),\ M(G\delta_1,\ G\delta_2,\ t)\}} - 1\right).$$

Combining them, we get

$$\max\left\{\begin{array}{c}\left(\dfrac{1}{M(F(\varepsilon_1,\ \delta_1),\ F(\varepsilon_2,\ \delta_2),\ t)} - 1\right), \\ \left(\dfrac{1}{M(F(\delta_1,\ \varepsilon_1),\ F(\delta_2,\ \varepsilon_2),\ t)} - 1\right)\end{array}\right\} \leq \varphi\left(\frac{1}{\min\{M(G\varepsilon_1,\ G\varepsilon_2,\ t),\ M(G\delta_1,\ G\delta_2,\ t)\}} - 1\right). \tag{4.2}$$

Thus, it follows from (4.2) that

$$\begin{aligned}
&\left(\frac{1}{M_2(\Theta_F(V_1),\ \Theta_F(V_2),\ t)} - 1\right) \\
&= \left(\frac{1}{\min\{M(F(\varepsilon_1,\ \delta_1),\ F(\varepsilon_2,\ \delta_2),\ t),\ M(F(\delta_1,\ \varepsilon_1),\ F(\delta_2,\ \varepsilon_2),\ t)\}} - 1\right) \\
&= \left(\max\left\{\left(\frac{1}{M(F(\varepsilon_1,\ \delta_1),\ F(\varepsilon_2,\ \delta_2),\ t)} - 1\right),\right.\right. \\
&\qquad\qquad \left.\left.\left(\frac{1}{M(F(\delta_2,\ \varepsilon_2),\ F(\delta_1,\ \varepsilon_1),\ t)} - 1\right)\right\}\right) \\
&\leq \varphi\left(\frac{1}{\min\{M(G\varepsilon_1,\ G\varepsilon_2,\ t),\ M(G\delta_1,\ G\delta_2,\ t)\}} - 1\right) \\
&\leq \varphi\left(\frac{1}{M_2(\Theta_G(V_1),\ \Theta_G(V_2),\ t)} - 1\right).
\end{aligned}$$

□

It is only require to apply Corollary 3.2 with $\alpha = \Theta_F$ and $\beta = \Theta_G$ in the partially ordered metric space $(X^2,\ M_2,\ \sqsubseteq)$ taking into account of all items of Lemma 4.1.

Corollary 4.2. *Let $(X, \preceq)$ be a partially ordered set and there be a non-Archimedean fuzzy metric M such that $(X,\ M,\ *)$ be a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ has mixed monotone property with respect to $\preceq$ and there exists $\varphi \in \Phi$ satisfying*

$$\left(\frac{1}{M(F(\varepsilon_1,\ \delta_1),\ F(\varepsilon_2,\ \delta_2),\ t)} - 1\right) \leq \varphi\left(\frac{1}{\min\{M(\varepsilon_1,\ \varepsilon_2,\ t),\ M(\delta_1,\ \delta_2,\ t)\}} - 1\right),$$

for all $\varepsilon_1,\ \delta_1,\ \varepsilon_2,\ \delta_2 \in X$, with $\varepsilon_1 \preceq \varepsilon_2$ and $\delta_1 \succeq \delta_2$. Also suppose that either

(*a*) *F is continuous or*

(*b*) *$(X,\ M,\ \preceq)$ is regular.*

Suppose there exists two elements $\varepsilon_0,\ \delta_0 \in X$ *with*

$$\varepsilon_0 \preceq F(\varepsilon_0,\ \delta_0) \text{ and } \delta_0 \succeq F(\delta_0,\ \varepsilon_0).$$

Then F *has a coupled fixed point.*

In a similar way, we may state the results analogous to Corollary 3.3 for Theorem 4.1 and Corollary 4.2.

Example 4.1. Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x,\ y) = \max\{x,\ y\}$ and $d(x,\ x) = 0$ for all $x,\ y \in X$ with the natural ordering of real numbers $\leq$ and $*$ is defined by $x * y = xy$, for all $x,\ y \in [0, 1]$. Define

$$M(\varepsilon,\ \delta,\ t) = \frac{t}{t + d(\varepsilon,\ \delta)}, \text{ for all } \varepsilon,\ \delta \in X \text{ and } t > 0.$$

Clearly $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Let $F : X \times X \to X$ and $G : X \to X$ be defined as

$$F(\varepsilon,\ \delta) = \frac{\varepsilon^2 + \delta^2}{6} \text{ and } G\varepsilon = \varepsilon^2, \text{ for all } \varepsilon,\ \delta \in X.$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \begin{cases} \dfrac{t}{3}, & \text{for } t \neq 1, \\ \dfrac{3}{4}, & \text{for } t = 1. \end{cases}$$

Now, for all $\varepsilon_1,\ \delta_1,\ \varepsilon_2,\ \delta_2 \in X$, we have

Case (a). If $\varepsilon_1^2 + \delta_1^2 = \varepsilon_2^2 + \delta_2^2$, then

$$\begin{aligned} d(F(\varepsilon_1, \delta_1),\ F(\varepsilon_2, \delta_2)) &= \frac{\varepsilon_1^2 + \delta_1^2}{6} \\ &\leq \frac{1}{6}\max\{\varepsilon_1^2,\ \varepsilon_2^2\} + \frac{1}{6}\max\{\delta_1^2,\ \delta_2^2\} \\ &\leq \frac{1}{6}d(G\varepsilon_1,\ G\varepsilon_2) + \frac{1}{6}d(G\delta_1,\ G\delta_2) \\ &\leq \frac{1}{3}\max\left\{d(G\varepsilon_1,\ G\varepsilon_2),\ d(G\delta_1,\ G\delta_2)\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} &\left(\frac{1}{M(F(\varepsilon_1,\ \delta_1),\ F(\varepsilon_2,\ \delta_2),\ t)} - 1\right) \\ &\leq \varphi\left(\frac{1}{\min\{M(G\varepsilon_1,\ G\varepsilon_2,\ t),\ M(G\delta_1,\ G\delta_2,\ t)\}} - 1\right). \end{aligned}$$

Case (b). If $\varepsilon_1^2+\delta_1^2 \neq \varepsilon_2^2+\delta_2^2$ with $\varepsilon_1^2+\delta_1^2 < \varepsilon_2^2+\delta_2^2$, then

$$\begin{aligned}
d(F(\varepsilon_1,\delta_1),\ F(\varepsilon_2,\delta_2)) &= \frac{\varepsilon_2^2+\delta_2^2}{6} \\
&\leq \frac{1}{6}\max\{\varepsilon_1^2,\ \varepsilon_2^2\} + \frac{1}{6}\max\{\delta_1^2,\ \delta_2^2\} \\
&\leq \frac{1}{6}d(G\varepsilon_1,\ G\varepsilon_2) + \frac{1}{6}d(G\delta_1,\ G\delta_2) \\
&\leq \frac{1}{3}\max\left\{d(G\varepsilon_1,\ G\varepsilon_2),\ d(G\delta_1,\ G\delta_2)\right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left(\frac{1}{M(F(\varepsilon_1,\ \delta_1),\ F(\varepsilon_2,\ \delta_2),\ t)} - 1\right) \\
&\leq \varphi\left(\frac{1}{\min\{M(G\varepsilon_1,\ G\varepsilon_2,\ t),\ M(G\delta_1,\ G\delta_2,\ t)\}} - 1\right).
\end{aligned}$$

Similarly, we obtain the same result for $\varepsilon_2^2+\delta_2^2 < \varepsilon_1^2+\delta_1^2$. Thus the contractive condition (4.1) is satisfied for all $\varepsilon_1,\ \delta_1,\ \varepsilon_2,\ \delta_2 \in X$. In addition, all the other conditions of Theorem 4.1 are satisfied and $z = (0,\ 0)$ is a coincidence point of F and G.

5. APPLICATIONS

In this section, we give an application to integral equation of our results. Consider the integral equation

$$\varepsilon(t) = \int_0^T K(t,\ s,\ \varepsilon(s))ds + h(t),\ t \in [0,\ T], \tag{5.1}$$

where $T > 0$. We introduce the following space:

$$C[0,\ T] = \{\varepsilon : [0,\ T] \to \mathbb{R} : \varepsilon \text{ is continuous on } [0,\ T]\},$$

equipped with the metric

$$d(\varepsilon,\ \delta) = \sup_{t\in[0,\ T]} |\varepsilon(t) - \delta(t)|\,,\ \text{for all } \varepsilon,\ \delta \in C[0,\ T].$$

It is clear that $(C[0,\ T],\ M,\ *)$ is a complete non-Archimedean fuzzy metric space with respect to the fuzzy metric

$$M(\varepsilon,\ \delta,\ t) = \frac{t}{t+d(\varepsilon,\ \delta)},\ \text{for all } \varepsilon,\ \delta \in C[0,\ T] \text{ and } t > 0,$$

with $*$ is defined by $a * b = ab$, for all $a, b \in I$. Furthermore, $C[0, T]$ can be equipped with the partial order $\preceq$ as follows: for $\varepsilon, \delta \in C[0, T]$,

$$\varepsilon \preceq \delta \Longleftrightarrow \varepsilon(t) \leq \delta(t), \text{ for each } t \in [0, T].$$

Now, we state the main result of this section.

Theorem 5.1. *We assume that the following hypotheses hold:*

(i) $K : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$ *and* $h : \mathbb{R} \to \mathbb{R}$ *are continuous,*

(ii) for all $s, t, \varepsilon, \delta \in C[0, T]$ *with* $\delta \preceq \varepsilon$, *we have*

$$K(t, s, \delta(s)) \leq K(t, s, \varepsilon(s)),$$

(iii) there exists a continuous function $\mathcal{G} : [0, T] \times [0, T] \to [0, +\infty)$:

$$|K(t, s, \varepsilon) - K(t, s, \delta)| \leq \mathcal{G}(t, s) \cdot \frac{|\varepsilon - \delta|}{2},$$

for all $s, t \in C[0, T]$ *and* $\varepsilon, \delta \in \mathbb{R}$ *with* $\varepsilon \succeq \delta$,

(iv) $\sup\limits_{t \in [0, T]} \int_0^T \mathcal{G}(t, s)^2 ds \leq \frac{1}{T}$.

Then the integral (5.1) *has a solution* $\varepsilon_0 \in C[0, T]$.

Proof. Define $\alpha : C[0, T] \to C[0, T]$ by

$$\alpha\varepsilon(t) = \int_0^T K(t, s, \varepsilon(s)) ds + h(t), \text{ for all } t \in [0, T] \text{ and } \varepsilon \in C[0, T].$$

First we shall prove that α is non-decreasing. Suppose that $\delta \preceq \varepsilon$. From (ii), for all $s, t \in [0, T]$, we have $K(t, s, \delta(s)) \leq K(t, s, \varepsilon(s))$. Thus, we get,

$$\alpha\delta(t) = \int_0^T K(t, s, \delta(s)) ds + h(t) \leq \int_0^T K(t, s, \varepsilon(s)) ds + h(t) = \alpha\varepsilon(t).$$

Now, for all $\varepsilon, \delta \in C[0, T]$ with $\delta \preceq \varepsilon$, due to (iii) and by using Cauchy-Schwarz inequality, we get

$$\begin{aligned} d(\alpha\varepsilon, \alpha\delta) &= \sup_{t \in [0, T]} |\alpha\varepsilon(t) - \alpha\delta(t)| \\ &\leq \int_0^T |K(t, s, \varepsilon(s)) - K(t, s, \delta(s))| \, ds \end{aligned}$$

$$\leq \int_0^T \mathcal{G}(t,\ s) \cdot \frac{|\varepsilon(s) - \delta(s)|}{2} ds$$
$$\leq \left(\int_0^T \mathcal{G}(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|\varepsilon(s) - \delta(s)|}{2} \right)^2 ds \right)^{\frac{1}{2}}.$$

Thus

$$|\alpha\varepsilon(t) - \alpha\delta(t)| \leq \left(\int_0^T \mathcal{G}(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|\varepsilon(s) - \delta(s)|}{2} \right)^2 ds \right)^{\frac{1}{2}}. \tag{5.2}$$

Using (iv), we estimate the first and second integral in (5.2) as follows:

$$\left(\int_0^T \mathcal{G}(t,\ s)^2 ds \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{T}}, \tag{5.3}$$

$$\left(\int_0^T \left(\frac{|\varepsilon(s) - \delta(s)|}{2} \right)^2 ds \right)^{\frac{1}{2}} \leq \sqrt{T} \cdot \frac{d(\varepsilon,\ \delta)}{2}. \tag{5.4}$$

Equations (5.2), (5.3) and (5.4) together conclude that

$$d(\alpha\varepsilon,\ \alpha\delta) \leq \frac{1}{2} d(\varepsilon,\ \delta).$$

It follows that

$$\frac{1}{M(\alpha\varepsilon,\ \alpha\delta,\ t)} - 1 \leq \frac{1}{2} \left(\frac{1}{M(\varepsilon,\ \delta,\ t)} - 1 \right),$$

for all $\varepsilon,\ \delta \in C[0,\ T]$ with $\delta \preceq \varepsilon$. Thus the contractive condition of Corollary 3.4 satisfied with $k = 1/2 \in (0,\ 1)$. Hence, all hypotheses of Corollary 3.4 are satisfied. Thus, α has a fixed point $\varepsilon_0 \in C[0,\ T]$ which is a solution of (5.1). □

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