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TWO SUBRAHMANYAM TYPE OF COMMON FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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Abstract. In this paper, we introduce new types of weakly Picard operators being available to a much wider class of maps, and prove common fixed point theorems of Subrahmanyam type for two these weakly Picard operators in the collection of single-valued and multi-valued mappings in complete metric spaces. Our results extend and generalize the corresponding fixed point theorems in the literature [3, 6].

1. Introduction

Suzuki [8] categorized fixed point theorems on metric spaces (X, d) into the following four types;

(1) Leader type [4] : T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$.
(2) Unnamed type : T has a unique fixed point and $\{T^n x\}$ does not necessarily converge to the fixed point for all $x \in X$.
(3) Subrahmanyam type [7] : T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point for all $x \in X$.
(4) Caristi type [1, 2] : T may have more than one fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point for all $x \in X$.

Khojasteh et. al. [3] introduced two types of fixed point theorems in the collection of multi-valued and single valued mappings and proved them, which belongs to (3). One year later, Rhoades [6] extended the results of Khojasteh et. al. [3] to two maps and to a much wider class of maps.

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Motivated by the previous works, in this paper, we establish two Subrahmanyam types of common fixed point theorems in the collection of single-valued and multi-valued mappings in metric spaces, which generalize the corresponding results of Rhoades [6].

2. Common Fixed Point Theorem for Single-valued Mappings

First of all, we prove the following lemma to obtain a common fixed point theorem.

Lemma 2.1. *Let (X, d) be a complete metric space and let S and T be self-mappings on X satisfying, for all $x, y \in X$,*

$$\varphi(d(Sx, Ty)) \leq N(x, y) \cdot \varphi(m(x, y)) - \psi(m(x, y)), \tag{2.1}$$

where

$$N(x, y) = \frac{\max\{d(x, y), d(x, Sx) + d(y, Ty), d(x, Ty) + d(y, Sx)\}}{d(x, Sx) + d(y, Ty) + 1}, \tag{2.2}$$

$$m(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\}, \tag{2.3}$$

$\varphi : [0, \infty) \to [0, \infty)$ a non-decreasing continuous function and $\psi : [0, \infty) \to [0, \infty)$ a continuous function with $\psi(t) = 0$ if and only if $t = 0$. Then each fixed point of S is a fixed point of T, and vice versa.

Proof. Let p is a fixed point of S and suppose that p is not a fixed point of T. From (2.2), we have

$$N(p, p) = \frac{\max\{d(p, p), d(p, Sp) + d(p, Tp), d(p, Tp) + d(p, Sp)\}}{d(p, Sp) + d(p, Tp) + 1} = \frac{d(p, Tp)}{d(p, Tp) + 1} < 1$$

and, from (2.3),

$$m(p, p) = \max\{d(p, p), d(p, Sp), d(p, Tp), \frac{d(p, Tp) + d(p, Sp)}{2}\} = d(p, Tp).$$

Substituting the above inequality and equality into (2.1), we get

$$\begin{aligned} \varphi(d(p, Tp)) = \varphi(d(Sp, Tp)) &\leq N(p, p) \cdot \varphi(m(p, p)) - \psi(m(p, p)) \\ &< \varphi(d(p, Tp)) - \psi(d(p, Tp)), \end{aligned}$$

which implies that $\psi(d(p, Tp)) < 0$. This contradicts the fact that the range of ψ is $[0, \infty)$. Therefore, p is a fixed point of T.

Similarly, it can be shown that, if q is a fixed point of T then it is also a fixed point of S. □

Theorem 2.2. *Assume that S and T satisfy the hypotheses of Lemma 2.1. Then*
(a) S and T have at least one common fixed point $p \in X$.
(b) For even natural number n, $\{(ST)^{\frac{n}{2}}x\}$ and $\{T(ST)^{\frac{n}{2}}x\}$ converge to a common fixed point for $x \in X$.
(c) If p and q are distinct common fixed ponits of S and T, then $d(p,q) \geq \frac{1}{2}$.

Proof. (a) Let x_0 be an arbitrary element of X and define $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n}, \;\; x_{2n+2} = Tx_{2n+1} \;\text{ for }\; n \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

Suppose that there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_{2n+1} = x_{2n+2}$. Then, from (2.4), $x_{2n+1} = x_{2n+2} = Tx_{2n+1}$, thus x_{2n+1} is a fixed point of T. By Lemma 2.1, x_{2n+1} is a fixed point of S and so it is a common fixed point of S and T.

Similarly, if there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_{2n} = x_{2n+1}$, then x_{2n} is a common fixed point of S and T.

Therefore, we assume that

$$x_n \neq x_{n+1} \;\text{ for }\; n \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

From (2.1) and (2.4), we have

$$\begin{aligned} \varphi(d(x_{2n+1}, x_{2n+2})) &= \varphi(d(Sx_{2n}, Tx_{2n+1})) \\ &\leq N(x_{2n}, x_{2n+1}) \cdot \varphi(m(x_{2n}, x_{2n+1})) - \psi(m(x_{2n}, x_{2n+1})). \end{aligned} \tag{2.6}$$

Defining $d_n := d(x_n, x_{n+1})$, from (2.2), (2.4) and the metric triangle property,

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Tx_{2n+1}) \\ &\quad + d(x_{2n+1}, Sx_{2n})\} / (d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1}) + 1) \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}) \\ &\quad + d(x_{2n+1}, x_{2n+1})\} / (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + 1) \\ &= \frac{\max\{d_{2n}, d_{2n} + d_{2n+1}, d_{2n} + d_{2n+1}\}}{d_{2n} + d_{2n+1} + 1} \\ &= \frac{d_{2n} + d_{2n+1}}{d_{2n} + d_{2n+1} + 1} := \beta_{2n}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} m(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})}{2}\} \end{aligned} \tag{2.8}$$

$$\begin{aligned} &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2}\} \\ &= \max\{d_{2n}, d_{2n+1}\}. \end{aligned}$$

Substituting (2.7) and (2.8) into (2.6), since $d_{2n} \neq 0$, we obtain the following inequality

$$\begin{aligned} \varphi(d_{2n+1}) &\leq \beta_{2n} \cdot \varphi(\max\{d_{2n}, d_{2n+1}\}) - \psi(\max\{d_{2n}, d_{2n+1}\}) \\ &< \beta_{2n} \cdot \varphi(\max\{d_{2n}, d_{2n+1}\}) \\ &= \beta_{2n} \cdot \varphi(d_{2n}). \end{aligned} \tag{2.9}$$

If $\max\{d_{2n}, d_{2n+1}\} = d_{2n+1}$ in the last formula, then the above inequality means $(1 - \beta_{2n})\varphi(d_{2n+1}) < 0$. Since $0 < \beta_{2n} < 1$, $\varphi(d_{2n+1}) < 0$, which contradicts the range of φ.

Applying the same method to $\varphi(d_{2n})$ instead of $\varphi(d_{2n+1})$ in (2.6), we have

$$\varphi(d_{2n}) < \beta_{2n-1} \cdot \varphi(d_{2n-1}). \tag{2.10}$$

Therefore, from (2.9) and (2.10), we have

$$\varphi(d_n) < \beta_{n-1} \cdot \varphi(d_{n-1}) < \varphi(d_{n-1}).$$

Since φ is non-decreasing,

$$d_n < d_{n-1} \text{ for each } n \in \mathbb{N}. \tag{2.11}$$

On the other hand, for $n \in \mathbb{N}$, $\beta_n < \beta_{n-1}$. In fact, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{x+1}$ is increasing and $d_n + d_{n+1} < d_{n-1} + d_n$ so that we have, from (2.11),

$$\frac{d_n + d_{n+1}}{d_n + d_{n+1} + 1} < \frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1},$$

and thus

$$\beta_n < \beta_{n-1} \text{ for } n \in \mathbb{N}. \tag{2.12}$$

From (2.11) and (2.12), we have

$$d_n < \beta_1 \cdot d_{n-1} < \beta_1^n \cdot d_0. \tag{2.13}$$

For any positive integers m, n with $m > n$, it follows from (2.13) that

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d_i < \sum_{i=n}^{m-1} \beta_1^i \cdot d_0 \\ &= \beta_1^n \cdot d_0 \sum_{j=0}^{m-n-1} \beta_1^j < \frac{\beta_1^n}{1-\beta_1} d_0. \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Using (2.1)-(2.3), we have

$$\begin{aligned} N(x_{2n}, p) &\leq \max\{d(x_{2n}, p), d(x_{2n}, x_{2n+1}) + d(p, Tp), d(x_{2n}, Tp) \\ &\quad + d(p, x_{2n+1})\}/(d(x_{2n}, x_{2n+1}) + d(p, Tp) + 1), \end{aligned}$$

$$m(x_{2n}, p) \leq \max\{d(x_{2n}, p), d(x_{2n}, x_{2n+1}), d(p, Tp), \frac{d(x_{2n}, Tp) + d(p, x_{2n+1})}{2}\}$$

and

$$\begin{aligned} \varphi(d(x_{2n+1}, Tp)) &= \varphi(d(Sx_{2n}, Tp)) \\ &\leq N(x_{2n}, p) \cdot \varphi(m(x_{2n}, p)) - \psi(m(x_{2n}, p)). \end{aligned}$$

Taking the limit of both sides of the above inequalty as $n \to \infty$

$$\varphi(d(p, Tp)) \leq \frac{d(p, Tp)}{d(p, Tp) + 1} \cdot \varphi(d(p, Tp)),$$

which implies that $p = Tp$. From Lemma 2.1, p is a fixed point of S.

(b) For $x \in X$, let $x_1 = Tx$. Define $x_{2n} = Sx_{2n-1}$ and $x_{2n+1} = Tx_{2n}$ for $n \in \mathbb{N}$. Then $\{(ST)^{\frac{n}{2}}x\}$ and $\{T(ST)^{\frac{n}{2}}x\}$ converge to common fixed point of S and T.

(c) Suppose that p and q are distinct common fixed points of S and T. From (2.2) and (2.3), we obtain

$$N(p, q) = \frac{\max\{d(p, q), d(p, Sp) + d(q, Tq), d(p, Tq) + d(q, Sp)\}}{d(p, Sp) + d(q, Tq) + 1} = 2d(p, q)$$

and

$$m(p, q) = \max\{d(p, q), d(p, Sp), d(q, Tq), \frac{d(p, Tq) + d(q, Sp)}{2}\} = d(p, q).$$

Thus, (2.1) becomes

$$\begin{aligned} \varphi(d(p, q)) &= \varphi(d(Sp, Tq)) \\ &\leq N(p, q) \cdot \varphi(m(p, q)) - \psi(m(p, q)) \leq 2d(p, q)\varphi(d(p, q)), \end{aligned}$$

which implies that $(1 - 2d(p, q))\varphi(d(p, q)) \leq 0$. Hence, $d(p, q) \geq \frac{1}{2}$. $\square$

If we put $\varphi(t) = t$ and $\psi(t) = 0$, then Theorem 2.2 can be modified as follows, which is the common fixed point theorem in [6].

Theorem 2.3. *Let (X, d) be a complete metric space, S, T selfmaps of X satisfying*

$$d(Sx, Ty) \leq N(x, y)m(x, y) \text{ for all } x, y \in X,$$

where

$$N(x, y) = \frac{\max\{d(x, y), d(x, Sx) + d(y, Ty), d(x, Ty) + d(y, Sx)\}}{d(x, Sx) + d(y, Ty) + 1}$$

and

$$m(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

Then
(a) S and T have at least one common fixed point $p \in X$.
(b) For even natural number n, $\{(ST)^{\frac{n}{2}}x\}$ and $\{T(ST)^{\frac{n}{2}}x\}$ converge to a common fixed point for $x \in X$.
(c) If p and q are distinct common fixed ponits of S and T, then $d(p, q) \geq \frac{1}{2}$.

Theorem 2.4. [3] *Let (X, d) be a complete metric space, T selfmaps of X satisfying*

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} d(x, y) \text{ for all } x, y \in X. \tag{2.14}$$

Then
(a) T has at least one fixed point $p \in X$.
(b) $\{T^n x\}$ converges to a fixed point for $x \in X$.
(c) If p and q are distinct fixed ponits of T, then $d(p, q) \geq \frac{1}{2}$.

Proof. If we put $\varphi = I$, $\psi(t) = 0$ and $S = T$ in (2.1), then the inequality (2.14) satisfies the hypotheses of Theorem 2.2 and so we obtain Theorem 2.4. □

3. Common Fixed Point Theorem for Multi-valued Mappings

We shall need the following notations for a common fixed point theorem on multi-valued mappings;

$$CB(X) = \{A | A \text{ is a nonempty closed and bounded subset of } X\},$$
$$D(a, B) = \inf\{d(a, b) | b \in B\} \text{ for } a \in X,$$
$$\delta(a, B) = \sup\{d(a, b) | b \in B\} \text{ for } a \in X,$$
$$H(A, B) = \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\}.$$

Lemma 3.1 ([5]). *Let $A, B \in CB(X)$, and let $x \in A$. Then, for each $\alpha > 0$, there exists $y \in B$ such that*

$$d(x, y) \leq H(A, B) + \alpha.$$

Lemma 3.2. *Let (X, d) be a complete metric space and let S and T be multi-valued mappings from X into $CB(X)$ satisfying, for all $x, y \in X$,*

$$\varphi(H(Sx, Ty)) \leq N(x, y) \cdot \varphi(m(x, y)) - \psi(m(x, y)), \tag{3.1}$$

where

$$N(x, y) = \frac{\max\{d(x, y), D(x, Sx) + D(y, Ty), D(x, Ty) + D(y, Sx)\}}{\delta(x, Sx) + \delta(y, Ty) + 1}, \tag{3.2}$$

$$m(x, y) = \max\{d(x, y), D(x, Sx), D(y, Ty), \frac{D(x, Ty) + D(y, Sx)}{2}\}, \tag{3.3}$$

$\varphi : [0, \infty) \to [0, \infty)$ a non-decreasing continuous function with $\varphi(ct) = c\varphi(t)$ and $\psi : [0, \infty) \to [0, \infty)$ a continuous function with $\psi(t) = 0$ if and only if $t = 0$. Then each fixed point of S is a fixed point of T, and vice versa.

Proof. Suppose that p is a fixed point of S. From (3.1) and the definition of H,

$$\varphi(D(p, Tp)) = \varphi(H(p, Tp)) = \varphi(H(Sp, Tp)) \quad \leq \quad N(p, p) \cdot \varphi(m(p, p)) - \psi(m(p, p)).$$

From (3.2) and (3.3), we have

$$\begin{aligned} N(p, p) &= \frac{\max\{d(p, p), D(p, Sp) + D(p, Tp), D(p, Tp) + D(p, Sp)\}}{\delta(p, Sp) + \delta(p, Tp) + 1} = \frac{D(p, Tp)}{\delta(p, Tp) + 1} \\ &\leq \frac{D(p, Tp)}{D(p, Tp) + 1} := \beta < 1 \end{aligned}$$

and

$$m(p, p) = \max\{d(p, p), D(p, Sp), D(p, Tp), \frac{D(p, Tp) + D(p, Sp)}{2}\} = D(p, Tp).$$

Therefore

$$\begin{aligned} \varphi(D(p, Tp)) &\leq \beta \cdot \varphi(D(p, Tp)) - \psi(D(p, Tp)) \\ &\leq \varphi(D(p, Tp)) - \psi(D(p, Tp)), \end{aligned}$$

which implies that $\psi(D(p, Tp)) \leq 0$. This contradicts the fact that the range of ψ is $[0, \infty)$. Therefore, p is a fixed point of T.

Similarly, it can be shown that, if q is a fixed point of T then it is a fixed point of S. □

Theorem 3.3. *Assume that S and T satisfy the hypotheses of Lemma 3.2. Then*
(a) S and T have at least one common fixed point $p \in X$.
(b) For even natural number n, $\{(ST)^{\frac{n}{2}}x\}$ and $\{T(ST)^{\frac{n}{2}}x\}$ converge to a common fixed point for $x \in X$.
(c) If p and q are distinct common fixed ponits of S and T, then $d(p,q) \geq \frac{1}{2}$.

Proof. (a) Let $x_0 \in X$ and $x_1 \in Sx_0$. Define h_n by

$$h_n = \sqrt{\beta_n} = \sqrt{\frac{d_{n-1}+d_n}{d_{n-1}+d_n+1}}. \tag{3.4}$$

From Lemma 3.1, for $0 < h_1 < 1$, i.e. $\frac{1}{h_1} - 1 > 0$, we can take $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \leq H(Sx_0,Tx_1) + (\frac{1}{h_1} - 1)H(Sx_0,Tx_1) = \frac{1}{h_1}H(Sx_0,Tx_1).$$

In a similar manner, for $0 < h_2 < 1$, we can take $x_3 \in Sx_2$ such that

$$d(x_2,x_3) = d(x_3,x_2) \leq H(Sx_2,Tx_1) + (\frac{1}{h_2} - 1)H(Sx_2,Tx_1) = \frac{1}{h_2}H(Sx_2,Tx_1).$$

Continuing this process, for $0 < h_{2n} < 1$, we can take $x_{2n+1} \in Sx_{2n}$ such that

$$d(x_{2n},x_{2n+1}) \leq \frac{1}{h_{2n}}H(Sx_{2n},Tx_{2n-1}) \tag{3.5}$$

and for $0 < h_{2n+1} < 1$, we can take $x_{2n+2} \in Tx_{2n+1}$ such that

$$d(x_{2n+1},x_{2n+2}) \leq \frac{1}{h_{2n+1}}H(Sx_{2n},Tx_{2n+1}). \tag{3.6}$$

If there exists $n \in \mathbb{N}$ such that $H(Sx_{2n},Tx_{2n-1}) = 0$, then $Sx_{2n} = Tx_{2n-1}$, which implies that $x_{2n} \in Sx_{2n}$, since $x_{2n} \in Tx_{2n-1}$, and x_{2n} is a fixed point of S. By Lemma 3.2, x_{2n} is a fixed point of T. Similarly, if there exists $n \in \mathbb{N}$ such that $H(Sx_{2n},Tx_{2n+1}) = 0$, then x_{2n+1} is a common fixed point of S and T. Therefore, we assume that $H(Sx_{2n},Tx_{2n-1}) \neq 0$ and $H(Sx_{2n},Tx_{2n+1}) \neq 0$.

On the other hand, if there exists $n \in \mathbb{N}$ such that $x_{2n} = x_{2n+1}$, then, since $x_{2n+1} \in Sx_{2n}$, x_{2n} is a fixed point of S. By Lemma 3.2, x_{2n} is a fixed point of T. Similarly, if there exists $n \in \mathbb{N}$ such that $x_{2n+1} = x_{2n+2}$, then x_{2n+1} is a common fixed point of S and T. Therefore, we also assume that $x_n \neq x_{n+1}$ for $n \in \mathbb{N}$.

From (3.2), (3.3) and (3.4), we get

$$\begin{aligned} N(x_{2n}, x_{2n-1}) &= \max\{d(x_{2n}, x_{2n-1}), D(x_{2n}, Sx_{2n}) + D(x_{2n-1}, Tx_{2n-1}),\\ &\quad D(x_{2n}, Tx_{2n-1}) + D(x_{2n-1}, Sx_{2n})\}/(\delta(x_{2n}, Sx_{2n})\\ &\quad +\delta(x_{2n-1}, Tx_{2n-1}) + 1)\\ &\le \max\{d_{2n-1}, d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n})\\ &\quad +d(x_{2n-1}, x_{2n+1})\}/(d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n}) + 1)\\ &\le \frac{\max\{d_{2n-1}, d_{2n} + d_{2n-1}, d_{2n} + d_{2n-1}\}}{d_{2n} + d_{2n-1} + 1} := \beta_{2n} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} m(x_{2n}, x_{2n-1}) &= \max\{d(x_{2n}, x_{2n-1}), D(x_{2n}, Sx_{2n}), D(x_{2n-1}, Tx_{2n-1}),\\ &\quad \frac{D(x_{2n}, Tx_{2n-1}) + D(x_{2n-1}, Sx_{2n})}{2}\}\\ &= \max\{d_{2n-1}, d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}),\\ &\quad \frac{d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})}{2}\}\\ &= \max\{d_{2n-1}, d_{2n}\}. \end{aligned} \tag{3.8}$$

Substituting (3.7), (3.8) into (3.5), we have

$$\begin{aligned} \varphi(d_{2n}) &= \varphi(d(x_{2n}, x_{2n+1})) \le \varphi(\frac{1}{h_{2n}} H(Sx_{2n}, Tx_{2n-1})) = \frac{1}{h_{2n}}\varphi(H(Sx_{2n}, Tx_{2n-1}))\\ &\le \frac{1}{h_{2n}}\beta_{2n} \cdot \varphi(max\{d_{2n-1}, d_{2n}\}) \le \sqrt{\beta_{2n}} \cdot \varphi(d_{2n-1}). \end{aligned}$$

A similar computation verifies that

$$\varphi(d_{2n+1}) \le \sqrt{\beta_{2n+1}} \cdot \varphi(d_{2n}).$$

From the above inequalities, we obtain

$$\varphi(d_{n+1}) \le \sqrt{\beta_{n+1}} \cdot \varphi(d_n) \text{ for } n \in \mathbb{N}. \tag{3.9}$$

Therefore, $\{d_n\}$ is a monotone decreasing positive real sequence. Taking the limit of both sides of (3.9) an $n \to \infty$, we have $\lim_{n\to\infty} d_n = 0$.

For any integers $m, n > 0$, using (3.9),

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} d_k \le \sum_{k=n}^{m-1} (\beta_{k-1} \cdots \beta_0) d_0 = d_0 \sum_{k=n}^{m-1} a_k,$$

where $a_k = \beta_{k-1} \cdots \beta_0$. Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \beta_k = 0$, $\sum a_n$ converges, which implies that $\{x_n\}$ is a Cauchy sequence, hence it converges to some point $p \in X$, since X is complete. Using (3.1)-(3.3), we have

$$\begin{aligned} N(x_{2n}, p) &\leq \max\{d(x_{2n}, p), d(x_{2n}, x_{2n+1}) + D(p, Tp), D(x_{2n}, Tp) \\ &\quad + d(p, x_{2n+1})\}/(\delta(x_{2n}, x_{2n+1}) + \delta(p, Tp) + 1), \end{aligned}$$

$$m(x_{2n}, p) \leq \max\{d(x_{2n}, p), d(x_{2n}, x_{2n+1}), D(p, Tp), \frac{D(x_{2n}, Tp) + d(p, x_{2n+1})}{2}\}$$

and

$$\begin{aligned} \varphi(d(x_{2n+1}, Tp)) &= \varphi(H(Sx_{2n}, Tp)) \\ &\leq N(x_{2n}, p) \cdot \varphi(m(x_{2n}, p)) - \psi(m(x_{2n}, p)). \end{aligned}$$

Taking the limit of both sides of the above inequalty as $n \to \infty$

$$\varphi(D(p, Tp)) \leq \frac{D(p, Tp)}{D(p, Tp) + 1} \cdot \varphi(D(p, Tp)),$$

which implies that p is a fixed point of T. From Lemma 3.2, p is a fixed point of S.

(b) For $x \in X$, let $x_1 \in Tx$. Define $x_{2n} \in Sx_{2n-1}$ and $x_{2n+1} \in Tx_{2n}$ for $n \in \mathbb{N}$. Then $\{(ST)^{\frac{n}{2}}x\}$ and $\{T(ST)^{\frac{n}{2}}x\}$ converge to common fixed point of S and T.

(c) Suppose that p and q are distinct common fixed points of S and T.

$$d(p, q) \leq D(p, Sp) + D(Sp, Tq) + D(Tq, q) \leq H(Sp, Tq). \tag{3.10}$$

From (3.2) and (3.3), we obtain

$$N(p, q) = \frac{\max\{d(p, q), D(p, Sp) + D(q, Tq), D(p, Tq) + D(q, Sp)\}}{\delta(p, Sp) + \delta(q, Tq) + 1} = 2d(p, q)$$

and

$$m(p, q) = \max\{d(p, q), D(p, Sp), D(q, Tq), \frac{D(p, Tq) + D(q, Sp)}{2}\} = d(p, q).$$

Thus, (2.1) becomes

$$\begin{aligned} \varphi(d(p, q)) &= \varphi(d(Sp, Tq)) \\ &\leq N(p, q) \cdot \varphi(m(p, q)) - \psi(m(p, q)) \leq 2d(p, q)\varphi(d(p, q)), \end{aligned}$$

which implies that $(1 - 2d(p, q))\varphi(d(p, q)) \leq 0$. Hence, $d(p, q) \geq \frac{1}{2}$. □

If we put $\varphi(t) = t$ and $\psi(t) = 0$, then Theorem 3.3 can be modified as follows, which is the common fixed point theorem in [6].

Theorem 3.4. *Let (X, d) be a complete metric space, S, T selfmaps of X satisfying*

$$H(Sx, Ty) \leq N(x, y)m(x, y) \text{ for all } x, y \in X,$$

where

$$N(x,y) = \frac{\max\{d(x,y), D(x,Sx)+D(y,Ty), D(x,Ty)+D(y,Sx)\}}{\delta(x,Sx)+\delta(y,Ty)+1}$$

and

$$m(x,y) = \max\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\}.$$

Then
(a) S and T have at least one common fixed point $p \in X$.
(b) For even natural number n, $\{(ST)^{\frac{n}{2}}x\}$ and $\{T(ST)^{\frac{n}{2}}x\}$ converge to a common fixed point for $x \in X$.
(c) If p and q are distinct common fixed ponits of S and T, then $d(p,q) \geq \frac{1}{2}$.

In Theorem 3.3, if φ is the identity function, ψ the zero function and $S = T$, then the main theorem of Khojasteh et. al. [3] for multi-valued mappings can be obtained as a corollary of Theorem 3.3.

Theorem 3.5. *Let (X,d) be a complete metric space, T selfmaps of X satisfying*

$$H(Tx,Ty) \leq \frac{D(x,Ty)+D(y,Tx)}{\delta(x,Tx)+\delta(y,Ty)+1} d(x,y) \text{ for all } x,y \in X.$$

Then T has a fixed point $p \in X$.

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