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REAL HYPERSURFACES IN THE COMPLEX HYPERBOLIC QUADRIC WITH CYCLIC PARALLEL STRUCTURE JACOBI OPERATOR

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Abstract. Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. The Riemannian curvature tensor field R of M allows us to define a symmetric Jacobi operator with respect to the Reeb vector field ξ, which is called the structure Jacobi operator $R_\xi = R(\,\cdot\,,\xi)\xi \in \mathrm{End}(TM)$. On the other hand, in [20], Semmelmann showed that the cyclic parallelism is equivalent to the Killing property regarding any symmetric tensor. Motivated by his result above, in this paper we consider the cyclic parallelism of the structure Jacobi operator R_ξ for a real hypersurface M in the complex hyperbolic quadric Q^{m*}. Furthermore, we give a complete classification of Hopf real hypersurfaces in Q^{m*} with such a property.

Introduction

As a typical example of the Hermitian symmetric spaces of rank 2, we can give the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$. It can be also regarded as a kind of a real Grassmannian manifold of compact type with rank 2 (see [2], [6], [8], and [19]). The characterization problems of real hypersurfaces of such a real Grassmannian manifold have been studied from various geometrical perspectives (see [1], [9], [10], [12], [16], [17], [21], [22], [25], and so on).

In this paper we investigate a classification problem of real hypersurfaces in the Hermitian symmetric space of non-compact type, so-called the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$. It is a dual space of the complex quadric Q^m. By virtue of the study due to Klein and Suh given in [7], the complex hyperbolic quadric Q^{m*} admits both a complex structure J

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and an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$, consisting of real structures on the tangent spaces of Q^{m*}. That is, $\mathfrak{A}$ is given by $\mathfrak{A} = \{\lambda A \,|\, \lambda \in S^1\}$, where A stands for a real structure of Q^{m*}. Such two geometric structures of Q^{m*} satisfy the anti-commuting property $AJ = -JA$. The complex hyperbolic quadric Q^{1^*} is isometric to the real hyperbolic space $\mathbb{R}H^2 = SO^0_{1,2}/SO_2$, and Q^{2^*} is isometric to the Hermitian product $\mathbb{C}H^1 \times \mathbb{C}H^1$ of complex hyperbolic spaces. For this reason, in this paper we will assume $m \geq 3$. Then for $m \geq 3$ the triple (Q^{m*}, J, g) is a Hermitian symmetric space of non-compact type with rank 2 whose minimal sectional curvature is equal to -4 (see [2] and [7]).

It is well known that J and $\mathfrak{A}$ are parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of Q^{m*}, which means that J and $A \in \mathfrak{A}$ satisfy $(\bar{\nabla}_U J)V = 0$ and $(\bar{\nabla}_U A)V = q(U)JAV$, respectively, where q denotes a certain real-valued 1-form on TQ^{m*} and U, V are any tangent vector fields of Q^{m*} (see [7]). Since $A \in \mathfrak{A}$ is a self-adjoint involution, it holds $A^2U = U$ for any $U \in TQ^{m*}$. From this, the tangent vector space T_pQ^{m*} at any point $p \in Q^{m*}$ splits into two orthogonal, maximal totally real subspaces of T_pQ^{m*}. We denote by $T_pQ^{m*} = V(A) \oplus JV(A)$, where $V(A)$ and $JV(A)$ are the $(+1)$-eigenspace and (-1)-eigenspace of A, respectively. So, a unit tangent vector $W \in T_pQ^{m*}$ can be expressed as $W = \cos(t)Z_1 + \sin(t)JZ_2$ for orthonormal unit vectors Z_1, $Z_2 \in V(A)$. Here, $t \in [0, \frac{\pi}{4}]$ is uniquely determined by W (see [19]). We say that the vector W is singular if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. In particular, the vectors with $t = 0$ are called $\mathfrak{A}$-principal, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-isotropic.

Let M be a real hypersurface with unit normal vector field N in a Kähler manifold Q^{m*}. Let (ϕ, ξ, η, g) be the almost contact metric structure on M induced by the complex structure of Q^{m*}. We define the Reeb vector field $\xi = -JN$ and the structure tensor field $\phi = J_{|TM} - N \otimes \eta$, where η is a 1-form defined by $\eta(X) = g(X, \xi)$ for any tangent vector field X of M in Q^{m*}. We denote by S the shape operator of M given as $\bar{\nabla}_X N = -SX$. The real hypersurface M is said to be *Hopf* if the Reeb vector field ξ is principal, that is, $S\xi = \alpha\xi$, $\alpha = g(S\xi, \xi)$.

When the shape operator S of M in Q^{m*} commutes with the structure tensor ϕ, that is, $S\phi = \phi S$, we say that the Reeb flow on M is *isometric* (or M has isometric Reeb flow). With respect to this concept, a remarkable classification for real hypersurfaces in complex hyperbolic quadric Q^{m*} was introduced in [23], as follows.

Theorem A. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. The Reeb flow on M is isometric if and only if M is locally congruent to an open part of the following real hypersurfaces in the complex hyperbolic quadric Q^{m*}:*

($\mathcal{T}_A^*$) *A tube with radius $r \in \mathbb{R}_+$ around the complex totally geodesic embedding of the complex hyperbolic space $\mathbb{C}H^k$ into Q^{2k^*}, $m = 2k$.*

($\mathcal{H}_A^*$) *A horosphere whose center at infinity is singular and of type $\mathfrak{A}$-isotropic.*

Indeed, [23], it is known that the normal vector field N of a real hypersurface in Theorem A is always $\mathfrak{A}$-isotropic.

On the other hand, as a typical characterization for real hypersurfaces with the $\mathfrak{A}$-principal normal vector field in Q^{m*}, we introduce the following result due to Klein and Suh given in [7]. We say that M is a *contact hypersurface* of a Kähler manifold if there exists an everywhere nonzero smooth function ρ such that $d\eta(X, Y) = 2\rho g(\phi X, Y)$ holds on M. It can be easily verified that a real hypersurface M is contact if and only if there exists an everywhere nonzero constant function ρ on M such that $S\phi + \phi S = 2\rho\phi$.

Theorem B. *Let M be a real hypersurface with constant mean curvature in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of one of the following real hypersurfaces in Q^{m*}:*

($\mathcal{T}_{B_1}^*$) *A tube with radius $r \in \mathbb{R}_+$ around the complex totally geodesic embedding of the complex hyperbolic quadric Q^{m-1^*} into Q^{m*}.*

($\mathcal{T}_{B_2}^*$) *A tube with radius $r \in \mathbb{R}_+$ around the totally real totally geodesic embedding of the real hyperbolic quadric $\mathbb{R}H^m$ into Q^{m*}.*

($\mathcal{H}_B^*$) *A horosphere in Q^{m*} with $\mathfrak{A}$-principal center at infinity.*

In addition to the above-mentioned theorems, a number of results have been given from the classification or characterization studies on a real hypersurface of Q^{m*} regarding various symmetric operators (see [5], [18], [24], [26], [29], and [31] etc.). Motivated by these results, in this paper we want to give a classification result of real hypersurfaces in Q^{m*} with the symmetric operator named the structure Jacobi operator. To do so, we first introduce the structure Jacobi operator R_ξ of M which is defined as the Jacobi operator with respect to the Reeb vector field ξ, as we explain below.

In general, the Jacobi operator on a Riemannian manifold with respect to X is defined by $R_X = R(\,\cdot\,, X)X$, where R denotes the Riemannian curvature tenor of M in Q^{m*}. Specially, we will call the Jacobi operator on a real hypersurface M in Q^{m*} with respect to the Reeb vector field ξ the *structure Jacobi operator* of M. Then it satisfies $g(R_\xi X, Y) = g(X, R_\xi Y)$ for any X, $Y \in TM$, which means that the linear operator $R_\xi \in \mathrm{End}(TM)$ is symmetric. As a characterization of M in Q^{m*} under the condition of R_ξ, it is well known that there are no Hopf real hypersurfaces with parallel structure Jacobi operator $\nabla R_\xi = 0$ in Q^{m*} (see [28]). Moreover, as a weaker condition of parallel structure Jacobi operator, the notion of Reeb parallelism and Codazzi type regarding R_ξ were studied in [10] and [27], respectively.

Motivated by the results mentioned above, it is natural to investigate real hypersurfaces in Q^{m*} by using another condition (on the derivative of R_ξ), so-called cyclic parallel structure Jacobi operator. Here, the structure Jacobi operator R_ξ of a real hypersurface M in Q^{m*} is said to be *cyclic parallel* if it satisfies

$$(\dagger) \qquad g\big((\nabla_X R_\xi)Y, Z\big) + g\big((\nabla_Y R_\xi)Z, X\big) + g\big((\nabla_Z R_\xi)X, Y\big) = 0$$

for any X, Y, $Z \in TM$. By using the linearization, the equation ($\dagger$) of cyclic parallel structure Jacobi operator is equivalent to $g\big((\nabla_X R_\xi)X, X\big) = 0$ for any $X \in TM$. According to the definition of the Killing tensor introduced in [3] and [20], it means that the structure Jacobi operator being cyclic parallel should be Killing. From such viewpoints, we can give its geometric meaning, as follows: *Let γ be any geodesic curve on M such that $\gamma(0) = p$ and $\dot\gamma(0) = X$ as the initial conditions. Then the structure Jacobi curvature $\mathbb{R}_\xi(\dot\gamma, \dot\gamma) := g(R_\xi\dot\gamma, \dot\gamma)$ is constant along the geodesic γ of the vector field X.* Here we denote by $\mathbb{R}_\xi$ the structure Jacobi tensor of type (0,2) defined by $\mathbb{R}_\xi(X, Y) = g(R_\xi X, Y)$ for any tangent vectors X, $Y \in T_pM$ at any point p of M (see Lemma 2.8 in [20]). On real hypersurfaces in complex Grassmannian manifolds with rank 1 or 2, the classification problems in terms of cyclic parallel structure Jacobi operator have been studied in [4], [13] and [14], respectively. Recently, in [11], Lee and Suh classified Hopf real hypersurfaces with cyclic parallel structure Jacobi operator in the complex quadric Q^m, as follows.

Theorem C. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then, the structure Jacobi operator R_ξ on M is cyclic parallel if and only if M is locally congruent to an open part of the following hypersurfaces in the complex quadric Q^m:*

(i) *A tube of radius $r = \frac{\pi}{4}$ around a totally geodesic $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$.*
(ii) *A tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the m-dimensional sphere S^m satisfying $\tan^2(\sqrt{2}r) = 2$.*

Based on what has been mentioned above, in this paper we want to give a classification of Hopf real hypersurfaces with cyclic parallel structure Jacobi operator in the complex hyperbolic quadric Q^{m*}. In order to do this, we first prove that the unit normal vector field N of M in Q^{m*} is singular, where M is a Hopf real hypersurface with constant mean curvature $\varepsilon = \frac{1}{2m-1}\mathrm{Tr}S$, as follows:

Theorem 1. *Let M be a Hopf real hypersurface with constant mean curvature in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. If the structure Jacobi operator R_ξ of M is cyclic parallel, then the unit normal vector field N of M in Q^{m*} is singular.*

As a consequence of Theorem 1 together with Propositions 3.3 and 4.6, we can prove the following:

Theorem 2. *There does not exist a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with constant mean curvature and cyclic parallel structure Jacobi operator.*

Each section of this paper covers the following topics: In Section 1, we recall the Riemannian geometry of a real hypersurface in the complex hyperbolic quadric Q^{m*} and introduce some fundamental equations which play an important role in proving our theorems. The formula for the structure Jacobi operator R_ξ and its covariant derivative ∇R_ξ will be shown explicitly in Section 1. In Sections 2, we will give some general information about Hopf real hypersurfaces in Q^{m*} with cyclic parallel structure Jacobi operator (see Lemma 2.3). Moreover, in the same section by using this fact we will give a complete proof of Theorem 1. According to two kinds of singular normal vector fields of M, so-called $\mathfrak{A}$-isotropic and $\mathfrak{A}$-principal, in Sections 3 and 4 we will consider a classification problem of Hopf real hypersurfaces in Q^{m*} with cyclic parallel structure Jacobi operator. Consequently, combining Sections 2, 3, and 4, we give a complete proof of Theorem 2.

1. Preliminaries

Throughout this paper, all manifolds, vector fields, and so on are considered of class C^∞. Let M be a connected real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. As mentioned before, the ambient space Q^{m*} is equipped with a Kähler structure J and a parallel rank two vector bundle $\mathfrak{A}$. By the Kähler structure J, we shall write

$$JX = \phi X + \eta(X)N \quad \text{and} \quad JN = -\xi, \tag{1.1}$$

where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathcal{C}^\perp$, where $\mathcal{C} = \mathrm{Ker}(\eta)$ is the maximal holomorphic subbundle of TM. Let us denote by ∇ and S the Levi-Civita connection and the shape operator of M, respectively. Then, by using $\bar\nabla_X Y = \nabla_X Y + g(SX, Y)N$ and $\bar\nabla_X N = -SX$, the properties $J^2 = -I$ and $\bar\nabla J = 0$ gives us

$$\phi^2 Y = -Y + \eta(Y)\xi, \quad (\nabla_X \phi)Y = \eta(Y)SX - g(SX, Y)\xi \tag{1.2}$$

and

$$\nabla_X \xi = \phi SX, \tag{1.3}$$

where $\bar\nabla$ is the Levi-Civita connection of Q^{m*} and I stands for the identity mapping of TQ^{m*}.

For any real structure $A \in \mathfrak{A}$ of the complex hyperbolic quadric Q^{m*}, we can decompose AX for any $X \in TM$ into the tangential and the normal parts as follows:

$$AX = BX + g(AX, N)N, \tag{1.4}$$

where BX is the tangential component of AX. Since A is symmetric, that is, $g(AX, Y) = g(X, AY)$, we see that B is also symmetric.

At each point $p \in Q^{m*}$, the real structure $A \in \mathfrak{A}_p$ induces a splitting $T_pQ^{m*} = V(A) \oplus JV(A)$ into two orthogonal, maximal totally real subspaces of the tangent space T_pQ^{m*}. Here $V(A)$ (resp. $JV(A)$) is the $(+1)$-eigenspace (resp. the (-1)-eigenspace) of A (see [7]). It then follows that we can choose a real structure $A \in \mathfrak{A}_p$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2 \tag{1.5}$$

for some orthonormal vectors Z_1, $Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. By using $JN = -\xi$ and $JA = -AJ$, it implies

$$\begin{cases} JN = \cos(t)JZ_1 - \sin(t)Z_2 \ \ (\text{i.e., } \xi = \sin(t)Z_2 - \cos(t)JZ_1), \\ AN = \cos(t)Z_1 - \sin(t)JZ_2, \\ A\xi = \cos(t)JZ_1 + \sin(t)Z_2, \end{cases} \tag{1.6}$$

and therefore $g(A\xi, N) = g(AN, \xi) = 0$ and $g(A\xi, \xi) = -g(AN, N) = -\cos(2t)$ on M. So, we see that the unit vector $A\xi$ is always tangent to M. Furthermore, the anti-commuting property $JA = -AJ$ regarding the real structure A and the Kähler structure J provides

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N, \tag{1.7}$$

and

$$\phi BX + g(X, \phi A\xi)\xi = JAX = -AJX = -B\phi X + \eta(X)\phi A\xi \tag{1.8}$$

for any tangent vector field X of M. In addition, from the property of $A^2 = I$, we get

$$B^2X = X - g(\phi A\xi, X)\phi A\xi, \quad B\phi A\xi = g(A\xi, \xi)\phi A\xi, \tag{1.9}$$

together (1.4) and (1.7). In [7], the Riemannian curvature tensor $\bar{R}$ of Q^{m*} is introduced, as follows.

$$\begin{aligned} \bar{R}(U, V)W = & -g(V, W)U + g(U, W)V - g(JV, W)JU \\ & + g(JU, W)JV + 2g(JU, V)JW - g(AV, W)AU \\ & + g(AU, W)AV - g(JAV, W)JAU + g(JAU, W)JAV \end{aligned} \tag{1.10}$$

for arbitrary $A \in \mathfrak{A}$ and U, V, $W \in TQ^{m*}$. By using the Gauss and Weingarten formulas, it is easy to see that the left side of (1.10) becomes

$$\begin{aligned} \bar{R}(X, Y)Z &= \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \\ &= R(X, Y)Z - g(SY, Z)SX + g(SX, Z)SY \\ &\quad + g((\nabla_X S)Y, Z)N - g((\nabla_Y S)X, Z)N \end{aligned}$$

for any X, Y, $Z \in TM$. Using this fact and $JAX = \phi BX + g(\phi A\xi, X)\xi + g(A\xi, X)N$, together with (1.1), (1.4) and (1.7), we can obtain the following two equations as the tangential and normal parts of (1.10), which are called the equations of Gauss and Codazzi, respectively, for a real hypersurface M in Q^m.

$$R(X, Y)Z = -g(Y, Z)X + g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y$$

$$
\begin{aligned}
&+ 2g(\phi X, Y)\phi Z - g(BY, Z)BX + g(BX, Z)BY \\
&- g(\phi BY, Z)\phi BX - g(\phi BY, Z)g(X, \phi A\xi)\xi \\
&+ g(\phi BX, Z)\phi BY + g(\phi BX, Z)g(Y, \phi A\xi)\xi \\
&- g(Y, \phi A\xi)\eta(Z)\phi BX + g(X, \phi A\xi)\eta(Z)\phi BY \\
&+ g(SY, Z)SX - g(SX, Z)SY \qquad (1.11)
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla_X S)Y - (\nabla_Y S)X = &-\eta(X)\phi Y + \eta(Y)\phi X + 2g(\phi X, Y)\xi \\
&+ g(\phi A\xi, X)BY - g(\phi A\xi, Y)BX \\
&- g(A\xi, X)\phi BY - g(\phi A\xi, Y)g(A\xi, X)\xi \\
&+ g(A\xi, Y)\phi BX + g(\phi A\xi, X)g(A\xi, Y)\xi, \qquad (1.12)
\end{aligned}
$$

where R and S denote the Riemannian curvature tensor and the shape operator of M, respectively. In this paper, the right-hand side of (1.12) is denoted by $\Xi(X, Y)$ for the sake of convenience. By means of this notation, (1.12) is written as $\Xi(X, Y) = (\nabla_X S)Y - (\nabla_Y S)X$, which will be used in Section 4.

Now let us focus our attention on a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}. The Reeb vector field ξ of a real hypersurface M in Q^{m*} is said to be *Hopf* if it is invariant under the shape operator S of M. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be the *Hopf foliation* of M. We say that M is *Hopf real hypersurface* in Q^{m*} if and only if the Hopf foliation of M is totally geodesic. By using the fact of $\phi\xi = 0$, together with (1.2) and (1.3), it can be easily seen that a real hypersurface M in Q^{m*} is Hopf if and only if the Reeb vector field ξ is Hopf. From this point of view, the fact of M being Hopf means that the Reeb vector field ξ satisfies $S\xi = \alpha\xi$, where $\alpha = g(S\xi, \xi)$. Hereafter, we call the smooth function $\alpha = g(S\xi, \xi)$ the Reeb curvature function of M. Then, by differentiating $S\xi = \alpha\xi$ and using the equation of Codazzi (1.12), we get the following:

Lemma 1.1 ([23]). *Let M be a Hopf real hypersurface in Q^{m*}, $m \geq 3$. Then we obtain*

$$
Y\alpha = (\xi\alpha)\eta(Y) + 2g(A\xi, \xi)g(\phi A\xi, Y) \qquad (1.13)
$$

and

$$
\begin{aligned}
2S\phi SY = &\alpha(S\phi + \phi S)Y - 2\phi Y + 2g(A\xi, \xi)g(\phi A\xi, Y)\xi \\
&- 2\eta(Y)g(A\xi, \xi)\phi A\xi - 2g(\phi A\xi, Y)A\xi + 2g(A\xi, Y)\phi A\xi \qquad (1.14)
\end{aligned}
$$

for any tangent vector field Y on M.

On the other hand, as mentioned above, from (1.6) we obtain $g(A\xi, N) = 0$. It means that the vector field $A\xi$ is tangent to M in Q^m, that is, $A\xi \in T_pM$, $p \in M$. Taking the covariant derivatives of this formula with respect to the

Levi-Civita connection ∇ and using the formula $(\bar{\nabla}_X A)Y = q(X)JAY$ for any tangent vector fields X and Y on M, we get

$$\begin{aligned}\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - g(SX, A\xi)N \\ &= (\bar{\nabla}_X A)\xi + A(\bar{\nabla}_X \xi) - g(SX, A\xi)N \\ &= q(X)JA\xi + A(\nabla_X \xi) + g(SX, \xi)AN - g(SX, A\xi)N \\ &= q(X)\big\{\phi A\xi + g(A\xi, \xi)N\big\} + B\phi SX + g(A\phi SX, N)N \\ &\quad - g(SX, \xi)\big\{\phi A\xi + g(A\xi, \xi)N\big\} - g(SX, A\xi)N.\end{aligned}$$

Then, by comparing the tangential and the normal components of the above equation, we get

$$\nabla_X(A\xi) = q(X)\phi A\xi + B\phi SX - g(SX, \xi)\phi A\xi \tag{1.15}$$

and

$$\begin{aligned}q(X)g(A\xi, \xi) &= -g(\phi SX, AN) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi) \\ &= g(\phi SX, \phi A\xi) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi) \\ &= g(SX, A\xi) - g(A\xi, \xi)g(SX, \xi) \\ &\quad + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi) \\ &= 2g(SX, A\xi). \qquad (1.16)\end{aligned}$$

In particular, if M is Hopf, then (1.15) and (1.16), respectively, become

$$\nabla_X(A\xi) = \big(q(X) - \alpha\eta(X)\big)\phi A\xi + B\phi SX = \kappa(X)\phi A\xi + B\phi SX \tag{1.17}$$

and

$$q(\xi)g(A\xi, \xi) = 2\alpha g(A\xi, \xi), \tag{1.18}$$

where $\kappa(X) = q(X) - \alpha\eta(X)$ for any $X \in TM$.

Let us consider the Hessian tensor of the Reeb curvature function $\alpha = g(S\xi, \xi)$ which is defined by

$$\mathrm{Hess}(\alpha)(X, Y) = g(\nabla_X \mathrm{grad}\, \alpha, Y)$$

for any X and Y tangent to M. Then, it satisfies

$$\mathrm{Hess}(\alpha)(X, Y) = \mathrm{Hess}(\alpha)(Y, X),$$

that is, $g(\nabla_X \mathrm{grad}\alpha, Y) = g(\nabla_Y \mathrm{grad}\alpha, X)$. Related to this property, in [30] Woo-Lee-Suh gave:

Lemma 1.2 (see Lemma 3.5 in [30])**.** *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then we obtain*

$$Y\beta = \nabla_Y\big(g(A\xi, \xi)\big) = -2g(S\phi A\xi, Y)$$

and

$$Y(\xi\alpha) = 2\beta g(SA\xi, Y) + \xi(\xi\alpha)\eta(Y) - 2\alpha\beta g(A\xi, Y), \tag{1.19}$$

where $\alpha = g(S\xi, \xi)$ and $\beta = g(A\xi, \xi)$. Moreover, by using these formulas, the Hessian property which is given by $g(\nabla_X grad\alpha, Y) = g(\nabla_Y grad\alpha, X)$ can be rearranged as

$$\begin{aligned}
&2\beta g(SA\xi, X)\eta(Y) - 2\alpha\beta g(A\xi, X)\eta(Y) + (\xi\alpha)g(\phi SX, Y) \\
&\quad - 4g(S\phi A\xi, X)g(\phi A\xi, Y) - 4g(SA\xi, X)g(A\xi, Y) + 2\beta g(BSX, Y) \\
&= 2\beta g(SA\xi, Y)\eta(X) - 2\alpha\beta g(A\xi, Y)\eta(X) + (\xi\alpha)g(\phi SY, X) \\
&\quad - 4g(S\phi A\xi, Y)g(\phi A\xi, X) - 4g(SA\xi, Y)g(A\xi, X) + 2\beta g(BSY, X),
\end{aligned}$$

where $BX = (AX)^T$ denotes the tangential part of AX.

Now, let us define the structure Jacobi operator R_ξ of a real hypersurface M in Q^{m*}. Indeed, the structure Jacobi operator R_ξ is a Jacobi operator with respect to the structure vector field ξ given by $R_\xi Y = R(Y, \xi)\xi$ for any $Y \in TM$. Bearing in mind (1.11), it follows

$$\begin{aligned}
R_\xi Y = &-Y + \eta(Y)\xi - g(A\xi, \xi)BY + g(A\xi, Y)A\xi \\
&+ g(\phi A\xi, Y)\phi A\xi + \alpha SY - g(S\xi, Y)S\xi,
\end{aligned} \tag{1.20}$$

where we have used $A\xi = B\xi \in TM$ and $\alpha = g(S\xi, \xi)$. In particular, if M is Hopf, then (1.20) becomes

$$\begin{aligned}
R_\xi Y = &-Y + \eta(Y)\xi - g(A\xi, \xi)BY + g(A\xi, Y)A\xi \\
&+ g(\phi A\xi, Y)\phi A\xi + \alpha SY - \alpha^2\eta(Y)\xi
\end{aligned} \tag{1.21}$$

for any tangent vector field Y of M. Moreover, by taking the covariant derivative of (1.21) along the direction of $Z \in TM$, we can obtain

$$\begin{aligned}
(\nabla_Z R_\xi)Y &= \nabla_Z(R_\xi Y) - R_\xi(\nabla_Z Y) \\
&= (1-\alpha^2)g(Y, \phi SZ)\xi + (1-\alpha^2)\eta(Y)\phi SZ \\
&\quad + 2g(S\phi A\xi, Z)BY - \beta(\nabla_Z B)Y + g(B\phi SZ, Y)A\xi \\
&\quad + g(A\xi, Y)B\phi SZ + \beta g(SY, Z)\phi A\xi - \eta(Y)g(SA\xi, Z)\phi A\xi \\
&\quad + \beta\kappa(Z)\eta(Y)\phi A\xi - g(B\phi SZ, \phi Y)\phi A\xi \\
&\quad + \beta g(\phi A\xi, Y)SZ - g(\phi A\xi, Y)g(SA\xi, Z)\xi \\
&\quad + \beta\kappa(Z)g(\phi A\xi, Y)\xi + g(\phi A\xi, Y)\phi B\phi SZ \\
&\quad + (Z\alpha)SY + \alpha(\nabla_Z S)Y - 2\alpha(Z\alpha)\eta(Y)\xi,
\end{aligned} \tag{1.22}$$

where we have used (1.2), (1.3) and (1.17).

On the other hand, by using (1.1), (1.4) and (1.7), together with $(\bar\nabla_Z A)Y = q(Z)JAY$ and $JAY = \phi BY + g(\phi A\xi, Y)\xi + g(A\xi, Y)N$, we get

$$\begin{aligned}
(\nabla_Z B)Y &= \nabla_Z(BY) - B(\nabla_Z Y) \\
&= \bar\nabla_Z(BY) - g(SZ, BY)N - B(\nabla_Z Y) \\
&= \bar\nabla_Z\big(AY - g(AY, N)N\big) - g(SZ, BY)N - B(\nabla_Z Y) \\
&= (\bar\nabla_Z A)Y + A(\bar\nabla_Z Y) - g\big((\bar\nabla_Z A)Y + A(\bar\nabla_Z Y), N\big)N
\end{aligned}$$

$$\begin{aligned}
&- g(AY, \bar{\nabla}_Z N)N - g(AY, N)\bar{\nabla}_Z N - g(SZ, BY)N - B(\nabla_Z Y) \\
&= q(Z)JAY + A(\nabla_Z Y) + g(SZ, Y)AN - q(Z)g(JAY, N)N \\
&\quad - g(\nabla_Z Y, AN)N - g(SZ, Y)g(N, AN)N + g(AY, SZ)N \\
&\quad + g(AY, N)SZ - g(SZ, BY)N - B(\nabla_Z Y) \\
&= q(Z)\phi BY + q(Z)g(\phi A\xi, Y)\xi - g(SZ, Y)\phi A\xi - g(Y, \phi A\xi)SZ,
\end{aligned} \tag{1.23}$$

where we have used $\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N$ and $\bar{\nabla}_X N = -SX$ for any tangent vector fields X and Y of M (we call them the Gauss and Weingarten formulas, respectively). In addition, the equation of Codazzi (1.12) gives

$$\begin{aligned}
(\nabla_Z S)Y = (\nabla_Y S)Z &- \eta(Z)\phi Y + \eta(Y)\phi Z + 2g(\phi Z, Y)\xi \\
&+ g(\phi A\xi, Z)BY - g(\phi A\xi, Y)BZ \\
&- g(A\xi, Z)\phi BY - g(\phi A\xi, Y)g(A\xi, Z)\xi \\
&+ g(A\xi, Y)\phi BZ + g(\phi A\xi, Z)g(A\xi, Y)\xi.
\end{aligned} \tag{1.24}$$

Substituting (1.23) and (1.24) into (1.22) and using $\kappa(Z) = q(Z) - \alpha\eta(Z)$, together with (1.13) and (1.16), it can be rearranged as

$$\begin{aligned}
&(\nabla_Z R_\xi)Y \\
&= (1-\alpha^2)g(Y, \phi SZ)\xi + (1-\alpha^2)\eta(Y)\phi SZ + 2g(S\phi A\xi, Z)BY \\
&\quad - 2g(SA\xi, Z)\phi BY + \beta g(SZ, Y)\phi A\xi + 2\beta g(Y, \phi A\xi)SZ \\
&\quad + g(B\phi SZ, Y)A\xi + g(A\xi, Y)B\phi SZ + \beta g(SY, Z)\phi A\xi \\
&\quad + \eta(Y)g(SA\xi, Z)\phi A\xi - \alpha\beta\eta(Z)\eta(Y)\phi A\xi - g(B\phi SZ, \phi Y)\phi A\xi \\
&\quad - g(\phi A\xi, Y)g(SA\xi, Z)\xi - \alpha\beta\eta(Z)g(\phi A\xi, Y)\xi + g(\phi A\xi, Y)\phi B\phi SZ \\
&\quad + (\xi\alpha)\eta(Z)SY + 2\beta g(\phi A\xi, Z)SY + \alpha(\nabla_Y S)Z - \alpha\eta(Z)\phi Y \\
&\quad + \alpha\eta(Y)\phi Z + 2\alpha g(\phi Z, Y)\xi + \alpha g(\phi A\xi, Z)BY - \alpha g(\phi A\xi, Y)BZ \\
&\quad - \alpha g(A\xi, Z)\phi BY - \alpha g(\phi A\xi, Y)g(A\xi, Z)\xi + \alpha g(A\xi, Y)\phi BZ \\
&\quad + \alpha g(\phi A\xi, Z)g(A\xi, Y)\xi - 2\alpha(\xi\alpha)\eta(Z)\eta(Y)\xi - 4\alpha\beta g(\phi A\xi, Z)\eta(Y)\xi
\end{aligned} \tag{1.25}$$

for any tangent vector fields Y and Z of M.

With the help of these facts and formulas, in the remaining part of this paper we will give a complete proof of our results given in the Introduction.

2. Proof of Theorem 1
- The singularity of unit normal vector field -

Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. In this section, we will prove that if M has a constant mean curvature and its structure Jacobi operator R_ξ is cyclic parallel, then the unit normal vector field N of M is singular. In order to do this, let us denote $\alpha := g(S\xi, \xi)$ and $\beta := g(A\xi, \xi)$. The following facts are known for such smooth functions α and β. Hereafter, let $\mathcal{V}$ be any open set of M.

Remark 2.1. By (1.6), the fact that $\beta = g(A\xi, \xi)$ identically vanishes on an open set $\mathcal{V}$ gives the unit normal vector field N is $\mathfrak{A}$-isotropic on $\mathcal{V}$. In fact, bearing in mind (1.6), our assumption of $\beta = 0$ follows

$$0 = g(A\xi, \xi) = g\big(\cot(t)JZ_1 + \sin(t)Z_2, \sin(t)Z_2 - \cos(t)JZ_1\big) = -\cos(2t),$$

where $0 \leq t \leq \frac{\pi}{4}$. That is, it implies $t = \frac{\pi}{4}$. So, by (1.5), the unit normal vector field N becomes

$$N = \cos\Big(\frac{\pi}{4}\Big)Z_1 + \sin\Big(\frac{\pi}{4}\Big)JZ_2 = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for some orthonormal vectors Z_1 and $Z_2 \in V(A) = \{Z \in TQ^{m*} \,|\, AZ = Z\}$, which means that N is $\mathfrak{A}$-isotropic.

Remark 2.2. On Hopf real hypersurface M in Q^{m*}, the fact that the geodesic Reeb flow $\alpha = g(S\xi, \xi)$ is either constant or vanishing on $\mathcal{V}$ implies that N is singular on $\mathcal{V}$. In fact, by means of (1.13), we get $\beta\phi A\xi = 0$ for these two cases regarding α. If $\beta = 0$, by virtue of Remark 2.1, then N is $\mathfrak{A}$-isotropic. On the other hand, if $\beta \neq 0$, then we obtain $\phi A\xi = 0$. Applying the structure tensor field ϕ to this formula and using (1.2), it follows that

$$A\xi = \beta\xi. \tag{2.1}$$

As mentioned in the Introduction, it is known that the real structure $A \in \mathfrak{A}$ is an anti-linear involution on TQ^{m*}, that is, $A^2X = X$ for any $X \in TQ^{m*}$. So, using this fact and (2.1) again, we get

$$\xi = A^2\xi = \beta A\xi = \beta^2\xi,$$

that is, $\beta^2 = 1$. Now, by our assumption $\beta \neq 0$, the smooth function β satisfies $\beta = g(A\xi, \xi) = -\cos(2t)$, $0 \leq t < \frac{\pi}{4}$. Thus, we consequently have $t = 0$ for the case of $\beta \neq 0$. From this, the unit normal vector field N is rewritten as

$$N = \cos(0)Z_1 + \sin(0)JZ_2 = Z_1 \in V(A),$$

which means that N is $\mathfrak{A}$-principal. Combining the discussions mentioned in above two cases, it gives a complete proof of Remark 2.2.

Now, we want to derive some basic equations regarding the cyclic parallelism of the structure Jacobi operator R_ξ of M. As it is well known, our assumption that the structure Jacobi operator R_ξ of M in Q^{m*} is cyclic parallel means that R_ξ satisfies

$$0 = g((\nabla_X R_\xi)Y, Z) + g((\nabla_Y R_\xi)Z, X) + g((\nabla_Z R_\xi)X, Y) \tag{$\dagger$}$$

for any tangent vector fields X, Y and Z of M. Taking $X = Y = Z$ in the above (†) becomes $g((\nabla_Z R_\xi)Z, Z) = 0$. From this, we get

$$g((\nabla_{X+Y} R_\xi)(X + Y), (X + Y)) = 0,$$

which gives

$$0 = g((\nabla_Y R_\xi)X, X) + g((\nabla_X R_\xi)Y, X) + g((\nabla_Y R_\xi)Y, X)$$

$$
\begin{aligned}
&+ g((\nabla_X R_\xi)X, Y) + g((\nabla_Y R_\xi)X, Y) + g((\nabla_X R_\xi)Y, Y) \\
&= g((\nabla_Y R_\xi)X, X) + 2g((\nabla_X R_\xi)X, Y) + 2g((\nabla_Y R_\xi)Y, X) \\
&\quad + g((\nabla_X R_\xi)Y, Y). \qquad (2.2)
\end{aligned}
$$

Actually, the structure Jacobi operator R_ξ of M is symmetric. From this, we get $g((\nabla_X R_\xi)Y, Z) = g(Y, (\nabla_X R_\xi)Z)$. By virtue of this property, we get

$$g((\nabla_X R_\xi)X, Y) = g((\nabla_X R_\xi)Y, X)$$

and

$$g((\nabla_Y R_\xi)X, Y) = g((\nabla_Y R_\xi)Y, X).$$

So, the second equality in (2.2) holds. Using this formula, we prove:

Lemma 2.3. *Let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then, we get*

$$(\xi\alpha)(h - \alpha) = 0,$$

where h is the trace of the shape operator S of M, that is, $h := TrS$.

Proof. By our assumption that the structure Jacobi operator R_ξ of M is cyclic parallel, (2.2) gives

$$
\begin{aligned}
&g((\nabla_\xi R_\xi)X, X) + 2g((\nabla_X R_\xi)X, \xi) \\
&+ 2g((\nabla_\xi R_\xi)\xi, X) + g((\nabla_X R_\xi)\xi, \xi) = 0 \qquad (2.3)
\end{aligned}
$$

for $Y = \xi$ and $X \in TM$. Using (1.24), we get

$$
\begin{aligned}
(\nabla_\xi R_\xi)X = &-3\alpha\beta\phi BX + 2\alpha\beta\eta(X)\phi A\xi + (\xi\alpha)SX - \alpha(\xi\alpha)\eta(X)\xi \\
&+ \alpha\beta g(\phi A\xi, X)\xi + \alpha^2\phi SX - \alpha S\phi SX - \alpha\phi X \\
&- \alpha g(\phi A\xi, X)A\xi + \alpha g(A\xi, X)\phi A\xi \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla_X R_\xi)\xi = &(1 - \alpha^2)\phi SX + g(S\phi A\xi, X)A\xi - 2g(SA\xi, X)\phi A\xi \\
&+ \alpha\beta\eta(X)\phi A\xi + \beta B\phi SX + g(SA\xi, X)\phi A\xi + \alpha(\xi\alpha)\eta(X)\xi \\
&- \alpha\beta g(\phi A\xi, X)\xi + \alpha(\nabla_\xi S)X + \alpha\phi X + \alpha g(\phi A\xi, X)A\xi \\
&- \alpha g(A\xi, X)\phi A\xi + \alpha\beta\phi BX - 2\alpha(\xi\alpha)\eta(X)\xi.
\end{aligned}
$$

From these two formulas and (1.12), we obtain the following four equations.

$$
\begin{aligned}
g((\nabla_\xi R_\xi)X, X) = &-3\alpha\beta g(\phi BX, X) + 2\alpha\beta\eta(X)g(\phi A\xi, X) \\
&+ (\xi\alpha)g(SX, X) - \alpha(\xi\alpha)\eta(X)\eta(X) \\
&+ \alpha\beta g(\phi A\xi, X)\eta(X) + \alpha^2 g(\phi SX, X) \\
&- \alpha g(S\phi SX, X) - \alpha g(\phi X, X) \\
&- \alpha g(\phi A\xi, X)g(A\xi, X) + \alpha g(A\xi, X)g(\phi A\xi, X), \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
2g((\nabla_X R_\xi)X,\xi) &= 2g((\nabla_X R_\xi)\xi, X) \\
&= 2(1-\alpha^2)g(\phi SX, X) + 2g(S\phi A\xi, X)g(A\xi, X) \\
&\quad - 2g(SA\xi, X)g(\phi A\xi, X) + 2\alpha\beta\eta(X)g(\phi A\xi, X) \\
&\quad + 2\beta g(B\phi SX, X) + 2\alpha^2 g(\phi SX, X) \\
&\quad - 2\alpha g(S\phi SX, X) - g(\phi A\xi, X)g(A\xi, X) \\
&\quad - 2\alpha\beta g(\phi BX, X) + 2\alpha g(A\xi, X)g(\phi A\xi, X) \\
&\quad + 2\alpha g(\phi A\xi, X)g(A\xi, X) - 2\alpha g(A\xi, X)g(\phi A\xi, X) \\
&\quad + 2\alpha\beta g(\phi BX, X), \qquad (2.6)
\end{aligned}
$$

$$2g((\nabla_\xi R_\xi)\xi, X) = g((\nabla_\xi R_\xi)X, \xi) = 0, \qquad (2.7)$$

and

$$
\begin{aligned}
g((\nabla_X R_\xi)\xi,\xi) &= \alpha g((\nabla_\xi S)X,\xi) - \alpha(\xi\alpha)\eta(X) \\
&= \alpha g((\nabla_\xi S)\xi, X) - \alpha(\xi\alpha)\eta(X) = 0. \qquad (2.8)
\end{aligned}
$$

By means of (2.7) and (2.8), the equation (2.3) is rewritten as

$$g((\nabla_\xi R_\xi)X, X) + 2g((\nabla_X R_\xi)X, \xi) = 0. \qquad (2.9)$$

Let $\{e_1, e_2, \ldots, e_{2m-1} := \xi\}$ be a basis of the tangent vector space T_pM of M at $p \in M$. From (2.5) and (2.6), contracting X on M, (2.9) gives $(\xi\alpha)(h-\alpha) = 0$, where we have used that $h := \mathrm{Tr}(S)$ and $\mathrm{Tr}(C) = \sum_{i=1}^{2m-1} g(Ce_i, e_i) = 0$ for any skew-symmetric operator C of M. Here we say that a tensor C is skew-symmetric if C satisfies $g(CX,Y) = -(CY,X)$ for any tangent vector fields X and Y of M. It completes a proof of our lemma. □

By virtue of Remarks 2.1 and 2.2, we know that when the smooth function α (or β, respectively) identically vanishes on M, the unit normal vector field N of M in Q^{m*} is singular. So, in the following lemmas, let us consider for a Hopf real hypersurface satisfying $\alpha \neq 0$ and $\beta \neq 0$. With this understood, we first prove:

Lemma 2.4. *Let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. If the smooth functions $\beta = g(A\xi,\xi)$ and $\alpha = g(S\xi,\xi)$ satisfying $\xi\alpha = 0$ are non-vanishing on $\mathcal{V}$, then the unit normal vector field N of $\mathcal{V}$ in Q^{m*} is singular.*

Proof. By our assumptions that $(\xi\alpha) = 0$ and $\beta \neq 0$, (1.19) gives

$$SA\xi = \alpha A\xi. \qquad (2.10)$$

Putting $Y = A\xi$ in (1.14) and using (2.10), we get $\alpha S\phi A\xi = (\alpha^2 - 2\beta^2)\phi A\xi$. Because the Reeb function α satisfies $\alpha \neq 0$ on $\mathcal{V}$, it follows that

$$S\phi A\xi = \frac{\alpha^2 - 2\beta^2}{\alpha}\phi A\xi =: \sigma\phi A\xi. \qquad (2.11)$$

On the other hand, putting ξ instead of Y in (1.21) gives $R_\xi\xi = 0$. From this and (1.3), we get

$$(\nabla_X R_\xi)\xi = \nabla_X(R_\xi\xi) - R_\xi(\nabla_X\xi) = -R_\xi\phi SX$$

for any tangent vector field X of M. Using this formula and the symmetric property of R_ξ, the cyclic parallelism of R_ξ for $Y = \xi$ becomes

$$\begin{aligned}
0 &= g((\nabla_X R_\xi)\xi, Z) + g((\nabla_\xi R_\xi)Z, X) + g((\nabla_Z R_\xi)X, \xi) \\
&= g((\nabla_X R_\xi)\xi, Z) + g((\nabla_\xi R_\xi)Z, X) + g((\nabla_Z R_\xi)\xi, X) \\
&= -g(R_\xi\phi SX, Z) + g((\nabla_\xi R_\xi)Z, X) - g(R_\xi\phi SZ, X) \\
&= g(-R_\xi\phi SX + (\nabla_\xi R_\xi)X + S\phi R_\xi X, Z),
\end{aligned}$$

which yields

$$-R_\xi\phi SX + (\nabla_\xi R_\xi)X + S\phi R_\xi X = 0. \tag{2.12}$$

By using (2.4), the previous equation (2.12) becomes

$$\begin{aligned}
&S\phi R_\xi X - R_\xi\phi SX - 2\alpha\beta\phi BX + 2\alpha\beta\eta(X)\phi A\xi + (\xi\alpha)SX \\
&- \alpha(\xi\alpha)\eta(X)\xi + \alpha\beta g(\phi A\xi, X)\xi + \alpha^2\phi SX - \alpha S\phi SX \\
&- \alpha\phi X - \alpha g(\phi A\xi, X)A\xi - \alpha\beta\phi BX + \alpha g(A\xi, X)\phi A\xi = 0.
\end{aligned} \tag{2.13}$$

Moreover, by using (1.21) and $\phi^2 A\xi = -A\xi + \beta\xi$, we get

$$\begin{aligned}
S\phi R_\xi X = &-S\phi X - \beta S\phi BX + g(A\xi, X)S\phi A\xi \\
&- g(\phi A\xi, X)SA\xi + \alpha\beta g(\phi A\xi, X)\xi + \alpha S\phi SX
\end{aligned}$$

and

$$\begin{aligned}
R_\xi\phi SX = &-\phi SX - \beta B\phi SX + g(A\xi, \phi SX)A\xi \\
&+ g(A\xi, SX)\phi A\xi - \alpha\beta\eta(X)\phi A\xi + \alpha S\phi SX.
\end{aligned}$$

So, (2.13) can be rearranged as

$$\begin{aligned}
&- S\phi X - \beta S\phi BX + g(A\xi, X)S\phi A\xi - g(\phi A\xi, X)SA\xi \\
&+ \alpha\beta g(\phi A\xi, X)\xi + \phi SX + \beta B\phi SX - g(A\xi, \phi SX)A\xi \\
&- g(A\xi, SX)\phi A\xi + \alpha\beta\eta(X)\phi A\xi - 2\alpha\beta\phi BX \\
&+ 2\alpha\beta\eta(X)\phi A\xi + (\xi\alpha)SX - \alpha(\xi\alpha)\eta(X)\xi + \alpha\beta g(\phi A\xi, X)\xi \\
&+ \alpha^2\phi SX - \alpha S\phi SX - \alpha\phi X - \alpha g(\phi A\xi, X)A\xi \\
&- \alpha\beta\phi BX + \alpha g(A\xi, X)\phi A\xi = 0.
\end{aligned} \tag{2.14}$$

Bearing in mind (2.10) and (2.11), putting $X = A\xi$ in (2.14) gives

$$(4\alpha\beta^2 + \alpha^3 - \alpha^2\sigma)\phi A\xi = 0,$$

where we have used $BA\xi = \xi$, $g(A\xi, A\xi) = 1$, $g(\phi A\xi, A\xi) = 0$, $B\phi A\xi = \beta\phi A\xi$, and $\xi\alpha = 0$. From our assumptions $\alpha \neq 0$ and $\beta \neq 0$ on $\mathcal{V}$, together with $\alpha\sigma = \alpha^2 - 2\beta^2$ given in (2.11), it follows $\phi A\xi = 0$. By using the proof given in Remark 2.2, we see that the unit normal vector field N is $\mathfrak{A}$-principal. □

In fact, combining the above discussions and remarks mentioned in this section, we are ready to give a complete proof of our Theorem 1 as follows.

Proof of Theorem 1. For the proof, we first assume that the structure Jacobi operator R_ξ of a Hopf real hypersurface M with constant mean curvature in Q^{m*} is cyclic parallel. And, as an open subset of M, we take

$$\mathcal{U} = \{p \in M \mid \alpha(p) \neq 0\}.$$

Then, we have $M = \mathcal{U} \cup \mathrm{Int}(M \backslash \mathcal{U}) \cup \partial(M \backslash \mathcal{U})$. Here, $\mathrm{Int}(M\backslash\mathcal{U})$ and $\partial(M\backslash\mathcal{U})$ stand for the interior and boundary sets of $M\backslash\mathcal{U}$, respectively, where $M\backslash\mathcal{U}$ denotes the orthogonal complement of the set $\mathcal{U}$ in M.

Case 1. On $\mathcal{U} \subset M$

Let $\mathcal{W} = \{p \in \mathcal{U} \mid \beta(p) \neq 0\}$. Then, $\mathcal{W}$ is an open subset of $\mathcal{U}$. So, we can consider the following three cases.

- Subcase 1-1. $p \in \mathcal{W} \subset \mathcal{U}$

  At any point $p \in \mathcal{W}$, it holds that $\alpha(p) \neq 0$ and $\beta(p) \neq 0$. It follows from Lemma 2.3 that the cyclic parallelism of the structure Jacobi operator R_ξ gives

$$(\xi\alpha)(p) \cdot (h - \alpha)(p) = 0. \tag{2.15}$$

  So, we may put an open set $\Gamma := \{p \in \mathcal{W} \mid (\xi\alpha)(p) \neq 0\}$, which means that $\mathcal{W} = \Gamma \cup \mathrm{Int}(\mathcal{W}\backslash\Gamma) \cup \partial(\mathcal{W}\backslash\Gamma)$. With this set-up, we now consider the following three cases:

  – Subcase 1-1-(i). $p \in \Gamma \subset \mathcal{W}$

    At any point $p \in \Gamma$, it holds that $\alpha(p) \neq 0$, $\beta(p) \neq 0$ and $(\xi\alpha)(p) \neq 0$. Thus, (2.15) says that the Reeb curvature function $\alpha = g(S\xi, \xi)$ satisfies $\alpha = h$ on Γ. By our assumption that M has constant mean curvature h, it implies that α is constant on Γ. From this and Remark 2.2, we can see that the unit normal vector N_p is singular. Hence, the normal vector field N is singular on Γ.

  – Subcase 1-1-(ii). $p \in \mathrm{Int}(\mathcal{W}\backslash\Gamma) \subset \mathcal{W}$

    At $p \in \mathrm{Int}(\mathcal{W}\backslash\Gamma)$, it holds that $\alpha(p) \neq 0$, $\beta(p) \neq 0$, and $(\xi\alpha)(p) = 0$. Then, by virtue of Lemma 2.4, we see that the unit normal vector N_p is $\mathfrak{A}$-principal. Consequently, the unit normal vector field N is singular on $\mathrm{Int}(\mathcal{W}\backslash\Gamma)$.

  – Subcase 1-1-(iii). $p \in \partial(\mathcal{W}\backslash\Gamma) \subset \mathcal{W}$

    Given a point $p \in \partial(\mathcal{W}\backslash\Gamma)$, there is a sequence (s_n) such that $s_n \to p$ for each point $s_n \in (\mathcal{W}\backslash\Gamma)$. Then, for each s_n we obtain $\alpha(s_n) \neq 0$, $\beta(s_n) \neq 0$, and $(\xi\alpha)(s_n) = 0$. By virtue of the proof of Lemma 2.4, we get $\phi A\xi(s_n) = 0$ for any s_n. Thus, the continuity of vector field $\phi A\xi$ gives

$$0 = \lim_{n\to\infty} \phi A\xi(s_n) = \phi A\xi(\lim_{n\to\infty} s_n) = \phi A\xi(p).$$

Applying the proof used in Remark 2.2, it means that $\beta(p) = -1$. So, we see that the unit normal vector N_p is $\mathfrak{A}$-principal, which means that the unit normal vector field N is singular on $\partial(\mathcal{W}\backslash\Gamma)$.

By combining Subcase 1-1-(i)–(iii) altogether, we can see that the unit normal vector field N is singular on $\mathcal{W} = \{p \in \mathcal{U} \mid \beta(p) \neq 0\}$.

- Subcase 1-2. $p \in \mathrm{Int}(\mathcal{U}\backslash\mathcal{W}) \subset \mathcal{U}$

  We get $\beta(p) = 0$ for all $p \in \mathrm{Int}(\mathcal{U}\backslash\mathcal{W})$. By using Remark 2.1, we obtain that the unit normal vector field N is $\mathfrak{A}$-isotropic. So, we see that N is singular on $\mathrm{Int}(\mathcal{U}\backslash\mathcal{W})$.
- Subcase 1-3. $p \in \partial(\mathcal{U}\backslash\mathcal{W}) \subset \mathcal{U}$

  Since p is a boundary point of $\mathcal{U}\backslash\mathcal{W}$, there is a sequence (p_n) such that $p_n \to p$, where each point p_n belongs to $\mathcal{U}\backslash\mathcal{W}$. It means that $\beta(p_n) = 0$ for each p_n. From this fact and the continuity of $\beta = g(A\xi, \xi)$, we get

  $$0 = \lim_{n\to\infty} \beta(p_n) = \beta(\lim_{n\to\infty} p_n) = \beta(p).$$

  By using again the proof of Remark 2.1, we see that the unit normal vector N_p at $p \in \partial(\mathcal{U}\backslash\mathcal{W})$ is $\mathfrak{A}$-isotropic. So, we can assert that the unit normal vector field N on $\partial(\mathcal{U}\backslash\mathcal{W})$ is singular.

Summing up above three cases, we conclude that the unit normal vector field N is singular on $\mathcal{U} = \{p \in M \mid \alpha(p) \neq 0\}$.

Case 2. On $\mathrm{Int}(M\backslash\mathcal{U})$

Let p be a point of $\mathrm{Int}(M\backslash\mathcal{U})$. Then, it holds $\alpha(p) = 0$ for all p. It means that the Reeb curvature function $\alpha = g(S\xi, \xi)$ vanishes on $\mathrm{Int}(M\backslash\mathcal{U})$. So, by virtue of Remark 2.2 we conclude that the unit normal vector field N is singular on $\mathrm{Int}(M\backslash\mathcal{U})$.

Case 3. On $\partial(M\backslash\mathcal{U})$

Given a point $p \in \partial(M\backslash\mathcal{U})$, there is a sequence (q_n) such that $q_n \to p$ for each point $q_n \in M\backslash\mathcal{U}$. From this, we obtain $\alpha(q_n) = 0$ for all q_n. So, by (1.13) we see that $(\beta\phi A\xi)(q_n) = 0$ for all q_n. By the continuity of β and $\phi A\xi$ on M, it follows that

$$\begin{aligned} 0 = \lim_{n\to\infty} (\beta\phi A\xi)(q_n) &= \lim_{n\to\infty} \big(\beta(q_n)\phi A\xi(q_n)\big) \\ &= \beta(\lim_{n\to\infty} q_n)\phi A\xi(\lim_{n\to\infty} q_n) \\ &= \beta(p)\phi A\xi(p) \\ &= (\beta\phi A\xi)(p). \end{aligned}$$

By using the same proof of Remark 2.2, we obtain that the normal vector N_p at p is singular, which means that the unit normal vector field N on $\partial(M\backslash\mathcal{U})$ is also singular.

Summarizing the above Cases 1, 2 and 3, we give a complete proof of our Theorem 1 mentioned in the Introduction. □

3. Proof of Theorem 2
- with the unit $\mathfrak{A}$-isotropic normal vector field -

Let M be a Hopf real hypersurface with constant mean curvature in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, whose the structure Jacobi operator R_ξ is cyclic parallel. By virtue of Theorem 1, we have known that the unit normal vector field N of M is singular. That is, N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.

So, in this section, we consider the case that the normal vector field N of a Hopf real hypersurface M with cyclic parallel structure Jacobi operator in Q^{m*} is $\mathfrak{A}$-isotropic. It means that N is expressed as $N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$ for some orthonormal unit vector fields Z_1, Z_2 which belong to the distribution $V(A) = \{Z \in TQ^{m*} \mid AZ = Z\}$. Bearing in mind the property of real structure $A \in \mathfrak{A}$, it gives $t = \frac{\pi}{4}$ in (1.5). Applying this fact to (1.6) and using them, we get

$$\begin{aligned} g(A\xi, N) &= \frac{1}{2} g\big(JZ_1 + Z_2, Z_1 + JZ_2\big) = 0, \\ g(A\xi, \xi) &= \frac{1}{2} g\big(JZ_1 + Z_2, Z_2 - JZ_1\big) = 0, \\ g(AN, N) &= \frac{1}{2} g\big(Z_1 - JZ_2, Z_1 + JZ_2\big) = 0. \end{aligned}$$

By such equations, (1.4) and (1.7) assure that the vector fields $A\xi$ and AN are tangent to M, that is, $A\xi = B\xi$ and $AN = -\phi A\xi$. Furthermore, taking the covariant derivative to $g(A\xi, N) = 0$ and $g(AN, N) = 0$ with respect to the Levi-Civita connection $\bar{\nabla}$ and using the Weingarten and Gauss formulas, we obtain

$$SA\xi = 0 \quad \text{and} \quad SAN = -S\phi A\xi = 0. \tag{3.1}$$

So, the tangent vector space T_pM at any point $p \in M$ is composed of three distributions $\mathrm{Span}\{\xi\}$, $\mathrm{Span}\{A\xi, \phi A\xi\}$ and $\mathcal{Q}$, that is, $T_pM = \mathrm{Span}\{\xi\} \oplus \mathrm{Span}\{A\xi, \phi A\xi\} \oplus \mathcal{Q}$. Here, $\mathcal{Q}$ is a $(2m-4)$-dimensional distribution given by $\mathcal{Q} = \{X \in T_pM \mid X \perp \xi, A\xi, \phi A\xi\}$.

On the other hand, by (1.21), the structure Jacobi operator R_ξ of a Hopf real hypersurface M with $\mathfrak{A}$-isotropic unit normal vector field N in Q^{m*} and its derivative are given as

$$R_\xi Y = -Y + \eta(Y)\xi + g(A\xi, Y)A\xi + g(\phi A\xi, Y)\phi A\xi + \alpha SY - \alpha^2\eta(Y)\xi \tag{3.2}$$

and

$$\begin{aligned} (\nabla_X R_\xi)Y &= \nabla_X(R_\xi Y) - R_\xi(\nabla_X Y) \\ &= g(Y, \phi SX)\xi + \eta(Y)\phi SX + g(\nabla_X(A\xi), Y)A\xi \\ &\quad + g(A\xi, Y)\nabla_X(A\xi) + g((\nabla_X\phi)A\xi, Y)\phi A\xi \\ &\quad - g(\nabla_X(A\xi), \phi Y)\phi A\xi + g(\phi A\xi, Y)(\nabla_X\phi)A\xi \\ &\quad + g(\phi A\xi, Y)\phi(\nabla_X A\xi) + (X\alpha)SY + \alpha(\nabla_X S)Y \end{aligned}$$

$$-2\alpha(X\alpha)\eta(Y)\xi - \alpha^2 g(Y, \phi SX)\xi - \alpha^2\eta(Y)\phi SX \tag{3.3}$$

for any X and $Y \in TM$, respectively.

In Section 2, we obtained that regardless of the singularity of the unit normal vector field N of M the cyclic parallelism of R_ξ gives (2.14) for $Y = \xi$. Accordingly, applying our assumption that N is $\mathfrak{A}$-isotropic, that is, (3.1) and $\beta := g(A\xi, \xi) = 0$, (2.14) becomes

$$\begin{aligned} &- S\phi X + \phi SX + (\xi\alpha)SX - \alpha(\xi\alpha)\eta(X)\xi + \alpha^2\phi SX \\ &- \alpha S\phi SX - \alpha\phi X - \alpha g(\phi A\xi, X)A\xi + \alpha g(A\xi, X)\phi A\xi = 0. \end{aligned} \tag{3.4}$$

On the other hand, as N is $\mathfrak{A}$-isotropic, (1.14) gives

$$2S\phi SX = \alpha S\phi X + \alpha\phi SX - 2\phi X - 2g(\phi A\xi, X)A\xi + 2g(A\xi, X)\phi A\xi.$$

From this, (3.4) can be arranged as

$$(\alpha^2 + 2)\phi SX + 2(\xi\alpha)SX - (\alpha^2 + 2)S\phi X - 2\alpha(\xi\alpha)\eta(X)\xi = 0 \tag{3.5}$$

for any tangent vector field X of M.

Now, let us take some unit tangent vector field $X_0 \in \mathcal{Q}$ such that $SX_0 = \lambda X_0$. Here, the distribution $\mathcal{Q}$ is given by $\mathcal{Q} = \{X \in TM \mid X \perp \xi, A\xi, \phi A\xi\}$. Then, for such an $X_0 \in \mathcal{Q}$ the equation (3.5) becomes

$$(\alpha^2 + 2)S\phi X_0 = \lambda(\alpha^2 + 2)\phi X_0 + 2\lambda(\xi\alpha)X_0,$$

which is the same as

$$S\phi X_0 = \lambda\phi X_0 + 2(\xi\alpha)\frac{\lambda}{\alpha^2 + 2}X_0. \tag{3.6}$$

Moreover, putting $Y = X_0$ in (1.14) and using $SX_0 = \lambda X_0$, we get

$$(2\lambda - \alpha)S\phi X_0 = (\alpha\lambda - 2)\phi X_0. \tag{3.7}$$

Substituting (3.6) into (3.7) provides

$$(2\lambda - \alpha)\Big\{\lambda\phi X_0 + 2(\xi\alpha)\frac{\lambda}{\alpha^2 + 2}X_0\Big\} = (\alpha\lambda - 2)\phi X_0. \tag{3.8}$$

Taking the inner product of (3.8) with X_0 yields

$$2\lambda(\xi\alpha)\frac{(2\lambda - \alpha)}{\alpha^2 + 2} = 0,$$

which implies

$$(\xi\alpha)\lambda(2\lambda - \alpha) = 0. \tag{3.9}$$

Let us denote $\mathcal{U} = \{p \in M \mid (\xi\alpha)(p) \neq 0\}$, which is an open subset of M. For this open set $\mathcal{U}$, let us consider three cases as follows.

Case I. On $\mathcal{U} \subset M$

Since $(\xi\alpha)(p) \neq 0$ at every point $p \in \mathcal{U}$, (3.9) gives us

$$\alpha\lambda = 2\lambda^2. \tag{3.10}$$

On the other hand, taking the inner product of (3.8) with ϕX_0 and using the fact of $g(\phi X_0, \phi X_0) = 1$, we get

$$\lambda^2 - \alpha\lambda + 1 = 0.$$

Substituting (3.10) into this formula becomes $\lambda^2 = 1$, that is, $\lambda = \pm 1$. From this and (3.10), we obtain $\alpha = \pm 2$. It means that $\alpha := g(S\xi, \xi)$ is constant on $\mathcal{U}$.

Case II. On $\mathrm{Int}(M \backslash \mathcal{U})$

Now, M is a Hopf real hypersurface with $\mathfrak{A}$-isotropic unit normal vector field N in Q^{m*}. By (1.13), we obtain

$$Y\alpha = (\xi\alpha)\eta(Y) \tag{3.11}$$

for any tangent vector field Y of M.

On the other hand, it holds that $(\xi\alpha)(p) = 0$ at any point $p \in \mathrm{Int}(M \backslash \mathcal{U})$. By virtue of this fact, (3.11) assure that the smooth function $\alpha := g(S\xi, \xi)$ is constant on $\mathrm{Int}(M \backslash \mathcal{U})$.

Case III. On $\partial(M \backslash \mathcal{U})$

Let p be a point of $\partial(M \backslash \mathcal{U})$, where $\partial(M \backslash \mathcal{U})$ denotes the boundary set of $M \backslash \mathcal{U}$ in M. Then, there is a sequence (p_n) such that $p_n \to p$, that is, $(\xi\alpha)(p_n) = 0$ for each point $p_n \in M \backslash \mathcal{U}$. So, (3.11) yields $(Y\alpha)(p_n) = 0$. From this and the continuity of $Y\alpha$ on M, we get

$$0 = \lim_{n \to \infty} (Y\alpha)(p_n) = (Y\alpha)(\lim_{n \to \infty} p_n) = (Y\alpha)(p),$$

which means that α is constant on $\partial(M \backslash \mathcal{U})$.

Summing up above three cases, we assert:

Lemma 3.1. *Let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. If the unit normal vector field N of M is $\mathfrak{A}$-isotropic, then a smooth function $\alpha = g(S\xi, \xi)$ is constant on M.*

By virtue of Lemma 3.1, the equation (3.5) is rearranged as

$$(\alpha^2 + 2)(\phi SX - S\phi X) = 0 \tag{3.12}$$

for any $X \in TM$. From this and Theorem A, we obtain:

Lemma 3.2. *Let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. If the unit normal vector field N of M is $\mathfrak{A}$-isotropic, then M has an isometric Reeb flow. Moreover, M is locally congruent to an open part of the following Hopf real hypersurfaces in Q^{m*}:*

($\mathcal{T}_A^*$) *(only if $m = 2k$ is even) A tube with radius $r \in \mathbb{R}^+$ around the complex totally geodesic embedding of the complex hyperbolic space $\mathbb{C}H^k$ into Q^{2k^*}.*

($\mathcal{H}_A^*$) *A horosphere in Q^{m*} whose center at infinity is singular and of type $\mathfrak{A}$-isotropic.*

We call such model spaces given in Lemma 3.2 the real hypersurfaces of Type (A) in Q^{m*}, which is denoted by M_A. We introduce some characterizations of M_A, as follows.

Proposition A ([23]). *Let M_A be the real hypersurfaces of Type* (A) *in Q^{m*}, $m \geq 3$. Then the following holds*:

(i) *A real hypersurface M_A is Hopf.*

(ii) *The unit normal vector field N of M_A is singular and $\mathfrak{A}$-isotropic.*

(iii) *The shape operator S of M_A commutes with the structure tensor field ϕ, that is, $S\phi = \phi S$. It means that M_A has isometric Reeb flow.*

(iv) *M_A has constant principal curvatures, and in particular constant mean curvature. Then the principal curvatures of M_A with respect to the unit normal vector field N and the corresponding principal curvature spaces are given in Table 1. Here, $\mathcal{C}$ and $\mathcal{Q}$ are the maximal holomorphic subbundle and the maximal $\mathfrak{A}$-invariant subbundle of TM_A, respectively. In addition, $T\mathbb{C}H^k$ and $\nu\mathbb{C}H^k$ stand for the tangent and normal bundle of $\mathbb{C}H^k$, respectively.*

Table 1. Principal curvatures of model spaces of M_A

Type	*Eigenvalues*	*Eigenspace*	*Multiplicity*
($\mathcal{T}_A^*$)	$\alpha=2\coth(2r)$	$T_\alpha=\mathbb{R}JN$	$m_\alpha=1$
	$\beta=0$	$T_\beta=\mathcal{C}\ominus\mathcal{Q}=Span\{A\xi,\phi A\xi\}$	$m_\beta=2$
	$\lambda=\tanh(r)$	$T_\lambda=T\mathbb{C}H^k\ominus(\mathcal{C}\ominus\mathcal{Q})$	$m_\lambda=2k-2$
	$\mu=\coth(r)$	$T_\mu=\nu\mathbb{C}H^k\ominus\mathbb{C}\nu M_A$	$m_\mu=2k-2$
($\mathcal{H}_A^*$)	$\alpha=2$	$T_\alpha=\mathbb{R}JN$	$m_\alpha=1$
	$\beta=0$	$T_\beta=\mathcal{C}\ominus\mathcal{Q}=Span\{A\xi,\phi A\xi\}$	$m_\beta=2$
	$\sigma=1$	$T_\sigma=\mathcal{Q}$	$m_\sigma=2m-4$

In particular, on a model space ($\mathcal{T}_A^*$) it holds

(v) *$B(T_\lambda) = T_\mu$ and $B(T_\mu) = T_\lambda$, that is, $SBX = \mu BX$ for $X \in T_\lambda$ and $SBX = \lambda BX$ for $X \in T_\mu$.*

(vi) *$\phi(T_\lambda) = T_\lambda$ and $\phi(T_\mu) = T_\mu$, that is, $S\phi X = \lambda\phi X$ for $X \in T_\lambda$ and $S\phi X = \mu\phi X$ for $X \in T_\mu$.*

By using the information of M_A given in Proposition A, in the remaining part of this section, we consider the converse statement of Lemma 3.2, that is, whether a real hypersurface of Type (A) in Q^{m*} satisfies all conditions given in Lemma 3.2 or not? In fact, by (i) and (ii) in Proposition A, we see that a real hypersurface M_A is Hopf with $\mathfrak{A}$-isotropic unit normal vector field N in Q^{m*}, $m \geq 3$. So, from now on, we want to show whether or not the model space M_A has the cyclic parallel structure Jacobi operator.

In order to do this, let us assume that the structure Jacobi operator R_ξ of M_A is cyclic parallel. Then, it holds that

$$(\dagger)\qquad g((\nabla_X R_\xi)Y, Z) + g((\nabla_Y R_\xi)Z, X) + g((\nabla_Z R_\xi)X, Y) = 0$$

for any X, Y and $Z \in TM_A = \mathrm{Span}\{\xi\} \oplus \mathrm{Span}\{A\xi, \phi A\xi\} \oplus \mathcal{Q}$.

Putting $Y = A\xi$ in (3.3) and using $SA\xi = S\phi A\xi = 0$ gives

$$\begin{aligned}(\nabla_X R_\xi)A\xi &= g(\nabla_X(A\xi), A\xi)A\xi + \nabla_X(A\xi)\\ &\quad - g(\nabla_X(A\xi), \phi A\xi)\phi A\xi + \alpha(\nabla_X S)A\xi\\ &= B\phi SX - \alpha SB\phi SX,\end{aligned}\tag{3.13}$$

where we have used $\nabla_X(A\xi) = (q(X) - \alpha\eta(X))\phi A\xi + B\phi SX$, $(\nabla_X\phi)A\xi = 0$, $BA\xi = B^2\xi = \xi$, $B\phi A\xi = 0$ and

$$\begin{aligned}(\nabla_X S)A\xi &= \nabla_X(SA\xi) - S(\nabla_X A\xi)\\ &= -(q(X) - \alpha\eta(X))S\phi A\xi - SB\phi SX = -SB\phi SX.\end{aligned}\tag{3.14}$$

And, substituting $X = A\xi$ and $Y = Z$ into (3.3) we get

$$(\nabla_{A\xi} R_\xi)Z = \alpha(\nabla_{A\xi} S)Z,\tag{3.15}$$

where we have used $\nabla_{A\xi}(A\xi) = q(A\xi)\phi A\xi$ and $(\nabla_{A\xi}\phi)A\xi = 0$. From (3.13) and (3.15), the cyclic parallelism of R_ξ for $Y = A\xi$ and $X, Z \in TM_A$ gives

$$\begin{aligned}0 &= g((\nabla_X R_\xi)A\xi, Z) + g((\nabla_{A\xi} R_\xi)Z, X) + g(X, (\nabla_Z R_\xi)A\xi)\\ &= g(B\phi SX - \alpha SB\phi SX, Z) + g(\alpha(\nabla_{A\xi} S)Z, X) + g(X, B\phi SZ - \alpha SB\phi SZ)\\ &= g(B\phi SX - \alpha SB\phi SX + \alpha(\nabla_{A\xi} S)X - S\phi BX + \alpha S\phi BSX, Z),\end{aligned}$$

which is the same as

$$B\phi SX - \alpha SB\phi SX + \alpha(\nabla_{A\xi} S)X - S\phi BX + \alpha S\phi BSX = 0.\tag{3.16}$$

In addition, by using (1.12) and (3.14), together with $BA\xi = \xi$ and $g(A\xi, \xi) = 0$, we get

$$\begin{aligned}(\nabla_{A\xi} S)X &= (\nabla_X S)(A\xi) + \eta(X)\phi A\xi - \phi BX\\ &= -SB\phi SX + \eta(X)\phi A\xi - \phi BX.\end{aligned}$$

From this, (3.16) can be rearranged as

$$B\phi SX - 2\alpha SB\phi SX + \alpha\eta(X)\phi A\xi - \alpha\phi BX - S\phi BX + \alpha S\phi BSX = 0\tag{3.17}$$

for any tangent vector field X of M_A.

Now, let us take X_0 instead of X in (3.17), where X_0 is a unit vector field X_0 belonging to $\mathcal{Q} = \{X \in TM_A \mid X \perp \xi, A\xi, \phi A\xi\}$. Then, by virtue of Proposition A, we can put $SX_0 = \tau X_0$, where

$$\tau = \begin{cases} \lambda & \text{for } X_0 \in T_\lambda \subset \mathcal{Q} \subset T(\mathcal{T}_A^*),\\ \mu & \text{for } X_0 \in T_\mu \subset \mathcal{Q} \subset T(\mathcal{T}_A^*),\\ \sigma & \text{for } X_0 \in \mathcal{Q} \subset T(\mathcal{H}_A^*).\end{cases}$$

By using (1.8), we get $\phi BX_0 = -B\phi X_0$. So, (3.17) gives

$$-(\alpha+\tau)\phi BX_0 + (3\alpha\tau - 1)S\phi BX_0 = 0. \tag{3.18}$$

As $X_0 \in \mathcal{Q}$, together with (1.9) and $BA\xi = B^2\xi$, the tangent vector field BX_0 of M_A satisfies $g(BX_0, BX_0) = g(B^2X_0, X_0) = 1$ and $g(BX_0, \xi) = g(BX_0, \phi A\xi) = g(BX_0, A\xi) = 0$. It implies that BX_0 is a unit tangent vector field belongs to $\mathcal{Q}$. From this and Proposition A, we see that BX_0 also becomes a principal vector field. Therefore, we may write $SBX_0 = \delta BX_0$. Moreover, from this fact and (iii) in Proposition A, we obtain

$$S\phi BX_0 = \phi SBX_0 = \delta\phi BX_0.$$

Hence, (3.18) which is the cyclic parallelism of R_ξ with respect to $X_0 \in \mathcal{Q}$ and $Y = A\xi$ is rearranged as

$$(-\alpha - \tau + 3\alpha\tau\delta - \delta)\phi BX_0 = 0, \tag{3.19}$$

where $SX_0 = \tau X_0$ and $SBX_0 = \delta BX_0$.

• On $(\mathcal{H}_A^*)$

By using the information of principal curvatures of $(\mathcal{H}_A^*)$, we get $SX = X$ and $SBX = BX$ for any $X \in \mathcal{Q}$. Since X_0 and BX_0 belong to $\mathcal{Q}$, it follows that $SX_0 = \tau X_0 = X$ and $SBX_0 = \delta BX_0 = BX_0$, that is, $\tau = \delta = 1$. Using these facts, (3.19) provides

$$2(\alpha - 1)\phi BX_0 = 0. \tag{3.20}$$

On the other hand, by (1.2) and (1.9), together with $A\xi = B\xi$, we know that ϕBX_0 is a unit tangent vector field belongs to $\mathcal{Q} \subset T(\mathcal{H}_A^*)$. So, (3.20) tells us $\alpha = 1$, which makes a contradiction for $\alpha = 2$. Hence, we assert that the structure Jacobi operator R_ξ of $(\mathcal{H}_A^*)$ is not cyclic parallel.

• On $(\mathcal{T}_A^*)$

Since the distribution $\mathcal{Q}$ of $(\mathcal{T}_A^*)$ can be decomposed as $\mathcal{Q} = T_\lambda \oplus T_\mu$ and $B(T_\lambda) = T_\mu$ (see (v) in Proposition A), we get $\tau = \lambda$ and $\delta = \mu$ for $X_0 \in T_\lambda \subset \mathcal{Q} \subset T(\mathcal{T}_A^*)$. Bearing in mind $\phi BX_0 \neq 0$, the equation (3.19) gives

$$\alpha = 0,$$

where we have used $\alpha = \lambda + \mu$ and $\lambda\mu = 1$. It makes a contradiction for $\alpha = 2\coth(2r)$, $r \in \mathbb{R}^+$. Therefore, we conclude that the structure Jacobi operator R_ξ of $(\mathcal{T}_A^*)$ does not satisfy the property of cyclic parallel.

Summing up the discussions mentioned in Section 3, we conclude:

Proposition 3.3. *There does not exist any Hopf real hypersurface M with cyclic parallel structure Jacobi operator and $\mathfrak{A}$-isotropic unit normal vector field in Q^{m*}, $m \geq 3$.*

Remark. Theorem 1 and Proposition 3.3 assure that the unit normal vector field N of M in Q^{m*} is $\mathfrak{A}$-principal if M is a Hopf real hypersurface with constant mean curvature in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, whose structure Jacobi operator is cyclic parallel.

4. Proof of Theorem 2 - with unit $\mathfrak{A}$-principal normal vector field -

Let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. In this section, we consider the case that a unit normal vector field N of M in Q^{m*} is $\mathfrak{A}$-principal. By virtue of the definition of an $\mathfrak{A}$-principal tangent vector field of Q^{m*}, the unit normal vector field N of M is expressed as $N = Z_1 \in V(A)$, that is, $t = 0$ in (1.5). Moreover, by (1.6), it gives

$$A\xi = JZ_1 = -\xi \quad \text{and} \quad AN = Z_1 = N. \tag{4.1}$$

From these facts, we obtain some useful equations regarding $\mathfrak{A}$-principal normal vector field, as follows.

Lemma 4.1 (see Lemma 5.1 in [30]). *Let M be a real hypersurface with $\mathfrak{A}$-principal unit normal vector field N in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then, the following facts hold on M.*

(i) $AX = BX$ *where BX is a tangential part of AX,*
(ii) $A\phi X = -\phi AX$,
(iii) $A\phi SX = -\phi SX$ *and* $q(X) = 2g(SX, \xi)$,
(iv) $ASX = SX - 2g(SX, \xi)\xi$ *and* $SAX = SX - 2\eta(X)S\xi$

for any tangent vector field X of M.

Furthermore, it is well known that a Hopf real hypersurface with $\mathfrak{A}$-principal unit normal vector field in Q^{m*} becomes a contact real hypersurface with constant mean curvature (see Proposition 5.3 in [30]). Therefore, by this fact and Theorem B mentioned in Section 1, we obtain the following:

Proposition 4.2. *Let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. If the unit normal vector field N of M in Q^{m*} is $\mathfrak{A}$-principal, then M is locally congruent to an open part of the following contact real hypersurfaces in Q^{m*}:*

($\mathcal{T}^*_{B_1}$) *A tube of radius $r > 0$ around the $(m-1)$-dimensional complex hyperbolic quadric Q^{m-1*} which is embedded in Q^{m*} as a totally geodesic complex hypersurface.*
($\mathcal{T}^*_{B_2}$) *A tube of radius $r > 0$ around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form of Q^{m*}.*
($\mathcal{H}^*_B$) *A horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*}.*

We call such contact hypersurfaces the real hypersurfaces of Type (B) in Q^{m*}, which is denoted by M_B. For the model spaces M_B, we give their geometric structures in detail, as follows.

Proposition B ([7]). *Let M_B be a tubes ($\mathcal{T}^*_{B_1}$), ($\mathcal{T}^*_{B_2}$) and a horosphere ($\mathcal{H}^*_B$) in Q^{m*}, $m \geq 3$. For M_B the following statements hold*:

(i) *Every unit normal vector N of M_B is $\mathfrak{A}$-principal.*
(ii) *M_B is a Hopf hypersurface, that is, $S\xi = \alpha\xi$.*
(iii) *The shape operator S and the structure tensor field ϕ satisfy $S\phi+\phi S = \frac{2}{\alpha}\phi$ (it means that M_B is contact).*
(iv) *A contact hypersurface M_B has constant principal curvatures, and in particular constant mean curvature. Then the principal curvatures of M_B with respect to the unit normal vector field N and the corresponding principal curvature spaces are given in Table 2.*

TABLE 2. Principal curvatures of model spaces of M_B

Type	*Eigenvalues*	*Eigenspace*	*Multiplicity*
$(\mathcal{T}^*_{B_1})$	$\alpha=-\sqrt{2}\coth(\sqrt{2}r)$	$T_\alpha=\mathbb{R}JN$	$m_\alpha=1$
	$\lambda=0$	$T_\lambda=JV(A)\cap\mathcal{C}=\{X\in\mathcal{C} \mid AX=-X\}$	$m_\lambda=m-1$
	$\mu=-\sqrt{2}\tanh(\sqrt{2}r)$	$T_\mu=V(A)\cap\mathcal{C}=\{X\in\mathcal{C} \mid AX=X\}$	$m_\mu=m-1$
$(\mathcal{T}^*_{B_2})$	$\alpha=-\sqrt{2}\tanh(\sqrt{2}r)$	$T_\alpha=\mathbb{R}JN$	$m_\alpha=1$
	$\lambda=0$	$T_\lambda=JV(A)\cap\mathcal{C}=\{X\in\mathcal{C} \mid AX=-X\}$	$m_\lambda=m-1$
	$\mu=-\sqrt{2}\coth(\sqrt{2}r)$	$T_\mu=V(A)\cap\mathcal{C}=\{X\in\mathcal{C} \mid AX=X\}$	$m_\mu=m-1$
$(\mathcal{H}^*_B)$	$\alpha(=\mu)=-\sqrt{2}$	$T_\alpha(=T_\mu)=(V(A)\cap\mathcal{C})\oplus\mathbb{R}JN$	$m_\alpha(=m_\mu)=m$
	$\lambda=0$	$T_\lambda=JV(A)\cap\mathcal{C}$	$m_\lambda=m-1$

Remark 4.3. The fact of M_B being contact assures that the structure tensor ϕ maps T_λ onto T_μ, and vice versa. That is, $\phi(T_\lambda) = T_\mu$ and $\phi(T_\mu) = T_\lambda$. On the other hand, the fact of (iv) in Lemma 4.1 tells us that the eigenspaces T_λ and T_μ are invariant under the real structure A, i.e., $A(T_\lambda) = T_\lambda$ and $A(T_\mu) = T_\mu$.

Now, by using the information of M_B given in Proposition B, in the remaining part of this section, let us check whether or not the structure Jacobi operator R_ξ of M_B is cyclic parallel.

In fact, by (i) in Proposition B, a contact real hypersurface M_B has an $\mathfrak{A}$-principal unit normal vector field N in Q^{m*}. So, bearing in mind (1.21) and (4.1), the structure Jacobi operator R_ξ of M_B is

$$R_\xi Y = -Y + 2\eta(Y)\xi + BY + \alpha SY - \alpha^2\eta(Y)\xi. \tag{4.2}$$

Taking the covariant derivative of (4.2) in the direction of Z and using Lemma 4.1, together with (1.23) and $Z\alpha = 0$, we get

$$\begin{aligned}(\nabla_Z R_\xi)Y &= 2g(Y,\phi SZ)\xi + 2\eta(Y)\phi SZ + (\nabla_Z B)Y \\ &\quad + \alpha(\nabla_Z S)Y - \alpha^2 g(Y,\phi SZ)\xi - \alpha^2\eta(Y)\phi SZ \\ &= (2-\alpha^2)g(\phi SZ, Y)\xi + (2-\alpha^2)\eta(Y)\phi SZ\end{aligned}$$

$$+\,2\alpha\eta(Z)\phi BY + \alpha(\nabla_Z S)Y. \tag{4.3}$$

By the symmetric property of R_ξ and (4.3), the left-side of cyclic parallelism of structure Jacobi operator satisfies

$$\begin{aligned}
&g\big((\nabla_X R_\xi)Y, Z\big) + g\big((\nabla_Y R_\xi)Z, X\big) + g\big((\nabla_Z R_\xi)X, Y\big)\\
&= g\big((\nabla_X R_\xi)Y, Z\big) + g\big((\nabla_Y R_\xi)X, Z\big) + g\big((\nabla_Z R_\xi)Y, X\big)\\
&= g\big((\nabla_X R_\xi)Y, Z\big) + g\big((\nabla_Y R_\xi)X, Z\big)\\
&\quad - (2-\alpha^2)g(S\phi Y, Z)\eta(X) - (2-\alpha^2)\eta(Y)g(S\phi X, Z)\\
&\quad + 2\alpha\eta(Z)g(\phi BY, X) - \alpha g((\nabla_Z S)Y, X)\\
&= g\big((\nabla_X R_\xi)Y, Z\big) + g\big((\nabla_Y R_\xi)X, Z\big)\\
&\quad - (2-\alpha^2)g(S\phi Y, Z)\eta(X) - (2-\alpha^2)\eta(Y)g(S\phi X, Z)\\
&\quad + 2\alpha\eta(Z)g(\phi BY, X) + \alpha g((\nabla_Y S)X, Z)\\
&\quad - \alpha\eta(Z)g(\phi Y, X) + \alpha\eta(Y)g(\phi Z, X) + 2\alpha g(\phi Z, Y)\eta(X)\\
&\quad + \alpha g(\phi BY, X)\eta(Z) + \alpha\eta(Y)g(B\phi X, Z),
\end{aligned} \tag{4.4}$$

where we have used

$$\begin{aligned}
g((\nabla_Z S)Y, X) &= g((\nabla_Y S)Z, X) + g(\Xi(Z, Y), X)\\
&= g((\nabla_Y S)X, Z) - \eta(Z)g(\phi Y, X) + \eta(Y)g(\phi Z, X)\\
&\quad + 2g(\phi Z, Y)\eta(X) + g(\phi BY, X)\eta(Z) + \eta(Y)g(B\phi X, Z)
\end{aligned}$$

for any tangent vector fields X, Y, and Z on M. Deleting Z from (4.4) and using $(\nabla_Y S)X = (\nabla_X S)Y + \Xi(Y, X)$, we get

$$\begin{aligned}
\Theta_X Y &:= (\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X\\
&\quad - (2-\alpha^2)\eta(X)S\phi Y - (2-\alpha^2)\eta(Y)S\phi X - 2\alpha g(B\phi X, Y)\xi\\
&\quad + \alpha(\nabla_Y S)X + \alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X - 2\alpha\eta(X)\phi Y\\
&\quad - \alpha g(B\phi X, Y)\xi + \alpha\eta(Y)B\phi X\\
&= (2-\alpha^2)g(\phi SX, Y)\xi + (2-\alpha^2)\eta(Y)\phi SX + 2\alpha\eta(X)\phi BY\\
&\quad + 3\alpha(\nabla_X S)Y - (2-\alpha^2)g(S\phi X, Y)\xi + (2-\alpha^2)\eta(X)\phi SY\\
&\quad + 2\alpha\eta(Y)\phi BX + 2\alpha\Xi(Y, X) - (2-\alpha^2)\eta(X)S\phi Y\\
&\quad - (2-\alpha^2)\eta(Y)S\phi X - 2\alpha g(B\phi X, Y)\xi + \alpha g(\phi X, Y)\xi\\
&\quad - \alpha\eta(Y)\phi X - 2\alpha\eta(X)\phi Y - \alpha g(B\phi X, Y)\xi + \alpha\eta(Y)B\phi X.
\end{aligned} \tag{4.5}$$

We denote this formula by $\Theta_X Y$ for any tangent vector fields X and Y of M_B.

In order to give a complete classification of cyclic parallel structure Jacobi operator, we want to consider each step in detail, as follows. By virtue of Proposition B, we take

$$\mathfrak{B} = \{\underbrace{e_1, e_2, \dots, e_{m-1}}_{\in V(A)\cap\mathcal{C}}, \underbrace{e_m, \dots, e_{2m-2}}_{\in JV(A)\cap\mathcal{C}}, e_{2m-1} = \xi\}$$

as a basis of the tangent vector space T_pM_B of M_B at any point $p \in M_B$. We put

$$E_{+1} := V(A) \cap \mathcal{C} = \{X \in \mathcal{C} \,|\, AX = X\} = \mathrm{Span}\{e_i \,|\, i = 1, 2, \dots, m-1\}$$

and

$$E_{-1} := JV(A) \cap \mathcal{C} = \{X \in \mathcal{C} \,|\, AX = -X\} = \mathrm{Span}\{e_i \,|\, i = m, \dots, 2m-2\},$$

which means $T_pM_B = \mathrm{Span}\{\xi\} \cup E_{+1} \cup E_{-1}$. By using such construction of $\mathfrak{B}$, let us calculate $\Theta_X Y$ regarding the subspace containing X and Y.

First, taking $X = \xi$ in (4.5) and using (1.12), we get

$$\begin{aligned} \Theta_\xi Y &= 2\alpha\phi BY + 3\alpha(\nabla_\xi S)Y + (2-\alpha^2)\phi SY \\ &\quad + 2\alpha\Xi(Y,\xi) - (2-\alpha^2)S\phi Y - 2\alpha\phi Y \\ &= 3\alpha\phi BY - 3\alpha S\phi SY - 3\alpha\phi Y + 2(\alpha^2+1)\phi SY - (2-\alpha^2)S\phi Y, \end{aligned} \tag{4.6}$$

where we have used $A\xi = -\xi$, $\Xi(Y,\xi) = \phi Y - \phi BY$ and

$$\begin{aligned} (\nabla_\xi S)Y &= (\nabla_Y S)\xi + \Xi(\xi, Y) \\ &= (Y\alpha)\xi + \alpha\phi SY - S\phi SY - \phi Y + \phi BY \\ &= \alpha\phi SY - S\phi SY - \phi Y + \phi BY \quad (\because\ \alpha : \text{constant on } M_B). \end{aligned}$$

By using this equation, we get:

Lemma 4.4. *Let M_B be a real hypersurface of Type* (B) *in Q^{m*}, $m \geq 3$. Then we have*

$$\Theta_\xi Y = \begin{cases} 0 & \text{for } Y = \xi, \\ (-6\alpha - 2\mu + \alpha^2\mu)\phi Y & \text{for } Y \in E_{-1}, \\ (2\mu\alpha^2 + 2\mu)\phi Y & \text{for } Y \in E_{+1}. \end{cases}$$

Proof. Putting $Y = \xi$ in (4.6) and using M being Hopf with $\mathfrak{A}$-principal unit normal vector field, it follows

$$\Theta_\xi \xi = 0.$$

Let us take $Y \in E_{-1} = JV(A) \cap \mathcal{C}$. By virtue of Proposition B, we obtain $E_{-1} = T_\lambda$. From this and Remark 4.3, the following facts hold that

$$\eta(Y) = 0, \quad AY = BY = -Y, \quad SY = \lambda Y \ (\lambda = 0), \quad S\phi Y = \mu\phi Y$$

for any $Y \in E_{-1}$. Applying these facts to (4.6) becomes

$$\Theta_\xi Y (\in E_{-1}) = (\alpha^2\mu - 2\mu - 6\alpha)\phi Y.$$

Now, let us take $Y \in E_{+1} = V(A) \cap \mathcal{C}$. From Proposition B and Remark 4.3, we get $\eta(Y) = 0$, $AY = BY = Y$, $SY = \mu Y$, and $S\phi Y = \lambda Y$ $(\lambda = 0)$. By using these facts, (4.6) can be arranged as

$$\Theta_\xi Y (\in E_{+1}) = (2\mu\alpha^2 + 2\mu)\phi Y.$$

It completes the proof of our lemma. □

Now, let us consider the case of $X \in \mathcal{C} = E_{-1} \cup E_{+1}$. Then:

Lemma 4.5. *On M_B, we get*

$$\Theta_{X\in E_{-1}}Y = \begin{cases} (-6\alpha - 2\mu + \alpha^2\mu)\phi X & \text{for } Y = \xi, \\ (-6\alpha - 2\mu + \alpha^2\mu)g(\phi X, Y)\xi & \text{for } Y \in \mathcal{C} \end{cases}$$

and

$$\Theta_{X\in E_{+1}}Y = \begin{cases} (2\alpha^2\mu + 2\mu)\phi X & \text{for } Y = \xi, \\ (2\alpha^2\mu + 2\mu)g(\phi X, Y)\xi & \text{for } Y \in \mathcal{C}. \end{cases}$$

Proof. Let us consider the tensor field $\Theta_X Y$ for any $X \in E_{-1} \subset TM_B$ and $Y \in TM_B$. By means of Proposition B, we get $E_{-1} = JV(A) \cap \mathcal{C} = T_\lambda$. From this and Remark 4.3, it follows that $\eta(X) = 0$, $AX = BX = -X$, $SX = \lambda X = 0$ and $S\phi X = \mu\phi X$. By using these facts, (4.5) can be rearranged as

$$\begin{aligned} \Theta_{X(\in E_{-1})}Y &= 3\alpha(\nabla_X S)Y - \mu(2-\alpha^2)g(\phi X, Y)\xi + 2\alpha\eta(Y)\phi BX \\ &\quad + 2\alpha\Xi(Y,X) - \mu(2-\alpha^2)\eta(Y)\phi X - 3\alpha g(B\phi X, Y)\xi \\ &\quad + \alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X + \alpha\eta(Y)B\phi X \\ &= 3\alpha(\nabla_X S)Y + (-6\alpha - 2\mu + \alpha^2\mu)g(\phi X, Y)\xi \\ &\quad + (-6\alpha - 2\mu + \alpha^2\mu)\eta(Y)\phi X, \end{aligned} \tag{4.7}$$

where we have used $B\phi X = -\phi BX = \phi X$ and $\Xi(Y,X) = -2\eta(Y)\phi X - 2g(\phi X, Y)\xi$.

Similarly, for the case of $X \in E_{+1} = V(A) \cap \mathcal{C}$ and $Y \in TM_B$, together with Proposition B and Remark 4.3, we get $E_{+1} = T_\mu$. This fact means that $\eta(X) = 0$, $AX = BX = X$, $SX = \mu X$ and $S\phi X = \lambda\phi X = 0$. So, (4.5) becomes

$$\begin{aligned} \Theta_{X(\in E_{+1})}Y &= \mu(2-\alpha^2)g(\phi X, Y)\xi + \mu(2-\alpha^2)\eta(Y)\phi X \\ &\quad + 3\alpha(\nabla_X S)Y + 2\alpha\eta(Y)\phi BX + 2\alpha\Xi(Y,X) \\ &\quad - 3\alpha g(B\phi X, Y)\xi + \alpha g(\phi X, Y)\xi \\ &\quad - \alpha\eta(Y)\phi X + \alpha\eta(Y)B\phi X \\ &= 3\alpha(\nabla_X S)Y + (2\mu - \alpha^2\mu)g(\phi X, Y)\xi \\ &\quad + (2\mu - \alpha^2\mu)\eta(Y)\phi X, \end{aligned} \tag{4.8}$$

where we have used $B\phi X = -\phi BX = -\phi X$ and $\Xi(Y,X) = -2g(\phi X, Y)\xi$.

On the other hand, Theorem 1.4 in [15] assures that the shape operator S of a real hypersurface M_B in Q^{m*}, $m \geq 3$, is *η-parallel*, which means that the shape operator S of M_B satisfies

$$g((\nabla_X S)Y, Z) = 0 \quad \text{for any} \quad X, Y, Z \in \mathcal{C}.$$

For any $X, Y \in \mathcal{C} = E_{+1} \cup E_{-1}$, by virtue of η-parallelism regarding the shape operator of a Hopf real hypersurface M_B, a vector field $(\nabla_X S)Y \in TM_B$ is expressed as

$$\begin{aligned}(\nabla_X S)Y &= \sum_{i=1}^{2m-2} g((\nabla_X S)Y, e_i)e_i + g((\nabla_X S)Y, \xi)\xi \\ &= g((\nabla_X S)\xi, Y)\xi = \alpha g(\phi SX, Y)\xi - g(S\phi SX, Y)\xi\end{aligned}$$

with respect to a basis $\mathfrak{B} = \{\underbrace{e_1, e_2, \dots, e_{m-1}}_{\in E_{+1}}, \underbrace{e_m, \dots, e_{2m-2}}_{\in E_{-1}}, e_{2m-1} = \xi\}$. According to $X \in E_{+1} \cup E_{-1}$, it yields

$$(4.9) \qquad (\nabla_X S)Y = \begin{cases} 0 & \text{for } X \in E_{-1} = T_\lambda,\ Y \in \mathcal{C}, \\ \alpha\mu g(\phi X, Y)\xi & \text{for } X \in E_{+1} = T_\mu,\ Y \in \mathcal{C}. \end{cases}$$

By (4.9), the equations (4.7) and (4.8) become

$$(4.10) \qquad \Theta_{X(\in E_{-1})}Y = (-6\alpha - 2\mu + \alpha^2\mu)g(\phi X, Y)\xi$$

and

$$(4.11) \qquad \Theta_{X(\in E_{+1})}Y = (2\mu + 2\alpha^2\mu)g(\phi X, Y)\xi$$

for any $Y \in \mathcal{C}$.

On the other hand, if $Y = \xi$, then we get $(\nabla_X S)\xi = \alpha\phi SX - S\phi SX$ together with our assumption of M_B being Hopf with constant principal curvatures. This implies

$$(\nabla_X S)\xi = \begin{cases} 0 & \text{for } X \in E_{-1} = T_\lambda, \\ \alpha\mu\phi X & \text{for } X \in E_{+1} = T_\mu. \end{cases}$$

So, (4.7) and (4.8) give

$$(4.12) \quad \Theta_{X(\in E_{-1})}\xi = (-6\alpha - 2\mu + \alpha^2\mu)\phi X \quad \text{and} \quad \Theta_{X(\in E_{+1})}\xi = (2\mu + 2\alpha^2\mu)\phi X.$$

By combining (4.10), (4.11) and (4.12), we complete a proof of Lemma 4.5. □

Since the structure Jacobi operator R_ξ of M_B is cyclic parallel, it holds that $\Theta_X Y = 0$ for all $X, Y \in TM_B$. So, we obtain $\alpha^2\mu = 2\mu + 6\alpha$ and $2\alpha^2\mu = -2\mu$ from Lemmas 4.4 and 4.5. By the direct calculations, it gives

$$(4.13) \qquad 2\alpha = -\mu.$$

That is, if the principal curvatures α and μ of a contact real hypersurface M_B satisfies (4.13), then R_ξ of M_B becomes cyclic parallel.

On the other hand, the cases of $(\mathcal{T}^*_{B_1})$, $(\mathcal{T}^*_{B_2})$, and $(\mathcal{H}^*_B)$ do not occur. In fact, the principal curvatures α and μ of $(\mathcal{T}^*_{B_1})$ are $\alpha = -\sqrt{2}\coth(\sqrt{2}r)$ and $\mu = -\sqrt{2}\tanh(\sqrt{2}r)$. So, (4.13) gives $\tanh^2(\sqrt{2}r) = -2$, which makes a contradiction.

On $(\mathcal{T}^*_{B_2})$, by virtue of Proposition B, the principal curvatures α and μ are $\alpha = -\sqrt{2}\tanh(\sqrt{2}r)$ and $\mu = -\sqrt{2}\coth(\sqrt{2}r)$. So, (4.13) becomes $\tanh^2(\sqrt{2}r) = -\frac{1}{2}$. It makes a contradiction. On the other hand, for $(\mathcal{H}^*_B)$, bearing in mind Proposition B, we get $\alpha = \mu = -\sqrt{2}$. It arises a contradiction with (4.13).

Summing up above discussions, we can assert that the structure Jacobi operator R_ξ of M_B is not cyclic parallel. From this and Proposition 4.2, we obtain:

Proposition 4.6. *There does not exist any Hopf real hypersurface M with cyclic parallel structure Jacobi operator and $\mathfrak{A}$-principal unit normal vector field in Q^{m*}, $m \geq 3$.*

Finally, combining Theorem 1 and Propositions 3.3 and 4.6 gives a complete proof of our Theorem 2 in the introduction.

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