J. Korean Math. Soc. **61** (2024), No. 2, pp. 227–253
https://doi.org/10.4134/JKMS.j220380
pISSN: 0304-9914 / eISSN: 2234-3008

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO STOCHASTIC 3D GLOBALLY MODIFIED NAVIER-STOKES EQUATIONS WITH UNBOUNDED DELAYS

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Abstract. This paper studies the existence of weak solutions and the stability of stationary solutions to stochastic 3D globally modified Navier-Stokes equations with unbounded delays in the phase space $BCL_{-\infty}(H)$. We first prove the existence and uniqueness of weak solutions by using the classical technique of Galerkin approximations. Then we study stability properties of stationary solutions by using several approach methods. In the case of proportional delays, some sufficient conditions ensuring the polynomial stability in both mean square and almost sure senses will be provided.

1. Introduction

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{O}$. Let us define $F_N : [0,\infty) \to (0,1]$ by

$$F_N(r) = \min\left\{1, \frac{N}{r}\right\}, \quad r \in [0,\infty).$$

In this paper we consider the following stochastic 3D globally modified Navier-Stokes equations with infinite delays

$$(1.1) \qquad \begin{cases} du + [-\nu\Delta u + F_N(\|u\|)(u\cdot\nabla)u + \nabla p]dt & \\ = [f(t) + g_1(t,u_t)]dt + g_2(t,u_t)dW(t) & \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ u(t,x) = 0 & \text{in } \mathbb{R}^+ \times \partial\mathcal{O}, \\ u(\theta,x) = \phi(\theta,x), \quad \theta \in (-\infty,0],\ x \in \mathcal{O}, & \end{cases}$$

where $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2, u_3)$ is the velocity field of the fluid, p is the pressure, f is a nondelayed external force field, g_1 and g_2 are another external force terms and contain hereditary characteristics, ϕ is the initial datum and $W(t)$ is a Wiener process on a suitable probability

Received July 27, 2022; Revised November 12, 2023; Accepted December 28, 2023.
2020 *Mathematics Subject Classification.* Primary 60H15, 35Q35, 60H30, 35B35.
Key words and phrases. Stochastic globally modified Navier-Stokes equations, unbounded delays, stability, stationary solutions, polynomial stability.

space to be described below. Here, for a function $u : (-\infty, 0) \to H$, for each $t > 0$ we have denoted by u_t the function defined on $(-\infty, 0)$ by the relation $u_t(s) = u(t+s)$, $s \in (-\infty, 0)$.

We set

$$BCL_{-\infty}(H) = \big\{\varphi \in C((-\infty, 0]; H) : \lim_{\theta \to -\infty} \varphi(\theta) \text{ exists in } H\big\},$$

which is a Banach space with the norm

$$\|\varphi\|_{BCL_{-\infty}(H)} = \sup_{\theta \in (-\infty, 0]} \|\varphi(\theta)\|_H.$$

We assume that

(H1) For any $\xi \in BCL_{-\infty}(H)$, the mappings $[0, \infty) \ni t \mapsto g_i(t, \xi) \in H$, $i = 1, 2$, are measurable;

(H2) $f \in L^2_{\text{loc}}(0, \infty; V')$ with $\mathbb{E} \int_0^T \|f(t)\|_*^2 dt < \infty$ for any $T > 0$;

(H3) For the terms g_1, g_2, we assume that $g_1 : \mathbb{R} \times BCL_{-\infty}(H) \to L^2_{\text{loc}}(0, \infty; H)$ and $g_2 : \mathbb{R} \times BCL_{-\infty}(H) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(K_0, H))$. There exist $L_{g_1} > 0$ and $L_{g_2} > 0$ such that, for any $t \in [0, \infty)$ and all $\xi, \eta \in BCL_{-\infty}(H)$,

$$|g_1(t, \xi) - g_1(t, \eta)| \le L_{g_1} \|\xi - \eta\|_{BCL_{-\infty}(H)}; \tag{1.2}$$

$$\|g_2(t, \xi) - g_2(t, \eta)\|_{L^2(K_0, H)} \le L_{g_2} \|\xi - \eta\|_{BCL_{-\infty}(H)}; \tag{1.3}$$

and $g_1(\cdot, 0) = 0; g_2(\cdot, 0) = 0$.

Here, the spaces H, V, V' and other notations above are defined in Section 2 below.

The system (1.1) is indeed a globally modified version of the Navier-Stokes system since the modifying factor $F_N(\|u\|)$ depends on the norm $\|u\|$, which in turn depends on ∇u over the whole domain $\mathcal{O}$ and not just at or near the point $x \in \mathcal{O}$ under consideration. Essentially, it prevents large gradients dominating the dynamics and leading to explosions. The deterministic globally modified Navier-Stokes equations were first introduced by Caraballo et al. [5] and have been investigated in several papers since then, both for their own sake and as a means of obtaining results about the three-dimensional Navier-Stokes equations. The existence, uniqueness and numerical approximations of solutions were studied in [5, 6, 18]. The stability of solutions is studied in both the cases of finite and infinite delays, see e.g. [3, 4, 11, 14, 16, 17] and references therein. The existence of attractors in both autonomous and non-autonomous cases, and the existence of invariant measures have been investigated extensively in [8, 10, 15, 21, 24, 25]. In the stochastic case, in the work [7], the authors proved the existence, uniqueness and convergence of strong solutions as $N \to \infty$. And in a very recent paper [1], in the case without delays, the authors studied both mean square exponential stability and pathwise exponential stability of weak stationary solutions.

In [13], stability results for the 2D Navier-Stokes equations with unbounded variable delays in the phase space $BCL_{-\infty}(H)$ were studied. And then some extensions for other models in fluid mechanics were investigated in [19,20]. For the stochastic case, in [12], the authors studied the stability for weak solutions of 2D stochastic Navier-Stokes equations. A natural and important question arising is to study the stochastic 3D globally modified Navier-Stokes equations with unbounded variable delays. This is a main motivation of the present paper.

In this paper we will study the stability of a stationary solution to the deterministic globally modified Navier-Stokes equations, which is regarded as a solution of the stochastic equations (1.1). To do this, we follow the general lines of the approach carried out in [12] to investigate the mean square (local/asymptotic/polynomial) stability of stationary solutions. In the case of proportional delay, using the polynomial decay rate, we give a sufficient condition to obtain the pathwise polynomial stability of stationary solutions.

The rest of the paper is organized as follows. In the next section, for convenience of the reader, we recall some results on the function spaces and operators, cylindrical Wiener processes, which will be frequently used later. In Section 3, we prove the existence and uniqueness of a global weak solution to the problem. The stability results for the stationary solution are established in the last section.

2. Preliminaries

2.1. Function spaces and operators

Let $\mathcal{V} = \{u \in (C_0^\infty(\mathcal{O}))^3 : \nabla \cdot u = 0\}$. Denote by H the closure of $\mathcal{V}$ in $(L^2(\mathcal{O}))^3$, and by V the closure of $\mathcal{V}$ in $(H_0^1(\mathcal{O}))^3$. Then H and V are Hilbert spaces with inner products given by

$$(u,v) := \int_{\mathcal{O}} \sum_{j=1}^{3} u_j v_j dx, \text{ and } ((u,v)) := \int_{\mathcal{O}} \sum_{j=1}^{3} \nabla u_j \cdot \nabla v_j dx,$$

respectively, and the associated norms

$$|u|^2 := (u,u), \ \|u\|^2 := ((u,u)).$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V', and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V'. Denote by P the Helmholtz-Leray orthogonal projection in $(L^2(\mathcal{O}))^3$ onto the space H.

Set $A : V \to V'$ by $\langle Au, v\rangle = ((u,v))$. It is well-known that $Au = -P\Delta u$, with the domain

$$D(A) = (H^2(\mathcal{O}))^3 \cap V,$$

is a positive self-adjoint linear operator with a compact inverse. Thus, there exists a sequence $\{\phi_j : j = 1, 2, 3, \ldots\}$ of elements of H which forms an orthonormal basis in H, orthogonal in V corresponding the eigenvalue λ_j with $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty$ as $j \to \infty$.

We have the following Poincaré inequality

$$\|u\|^2 \geq \lambda_1 |u|^2 \quad \text{for all } u \in V. \tag{2.1}$$

We now define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} v_j dx,$$

and we denote

$$b_N(u,v,w) = F_N(\|v\|) b(u,v,w), \ \forall u,v,w \in V.$$

The form b_N is linear in u and w, but it is nonlinear in v. We have the following property

$$b(u,v,v) = 0, \ \ \forall u,v \in V. \tag{2.2}$$

We will also make use of the following inequality (see [14, p. 657])

$$b(u,v,w) \leq 2^{-1} |u|^{1/4} \|u\|^{3/4} \|v\| |w|^{1/4} \|w\|^{3/4}. \tag{2.3}$$

Using the Hölder inequality (with power exponents $6, 2, 3$), the Sobolev inequality and the Gagliardo-Nirenberg inequality we have the following inequality (see also in [18]),

$$|b(u,v,w)| \leq c_0 \|u\| \|v\| |w|^{1/2} \|w\|^{1/2}, \ \forall u,v,w \in V. \tag{2.4}$$

Moreover from the properties of b and the definition of F_N, we have

$$|b_N(u,v,w)| \leq c_0 \lambda_1^{-1/4} N \|u\| \|w\|, \ \forall u,v,w \in V. \tag{2.5}$$

If we denote

$$\langle B_N(u,v), w \rangle = b_N(u,v,w), \ \forall u,v,w \in V,$$

then from (2.5) we have

$$\|B_N(u,v)\|_* \leq c_0 \lambda_1^{-1/4} N \|u\|, \ \forall u,v \in V. \tag{2.6}$$

If $u = v$, we write $B_N(u) = B_N(u,u)$.

We recall the following important lemma.

Lemma 2.1 ([18]). *For every $u, v \in V$, and each $N > 0$,*

(1) $0 \leq \|u\| F_N(\|u\|) \leq N$,
(2) $|F_N(\|u\|) - F_N(\|v\|)| \leq \frac{1}{N} F_N(\|u\|) F_N(\|v\|) \|u - v\|$.

The following inequality is obtained from Lemma 2.1 and (2.4) (see [18, (2.4)])

$$|\langle B_N(u) - B_N(v), u - v \rangle| \leq \frac{\nu}{2} \|u - v\|^2 + C(\nu, c_0) N^4 |u - v|^2, \ \forall u,v \in V, \tag{2.7}$$

for some positive constant $C(\nu, c_0)$.

2.2. The cylindrical Wiener process

We first introduce stochastic integrals in Hilbert space (see [9]). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space on which an increasing and right continuous family $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ of complete sub-σ-algebra of $\mathcal{F}$ is defined. We assume that $\mathcal{F}_0$ contains all null set of $\mathcal{F}$, and further $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leq 0$. Let $\beta_n(t)$, $n = 1, 2, 3, \ldots$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, \mathbb{P}, \mathcal{F})$. Assume that $\{e_n\}$ $(n = 1, 2, 3, \ldots)$ is a complete orthonormal basis in the real and separable Hilbert space K. We denote by $\{W(t), t \geq 0\}$, the cylindrical Wiener process with value in K defined formally as

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda'_n}\beta_n(t)e_n,\ t \geq 0,$$

where λ'_n $(n = 1, 2, 3, \ldots)$ are nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda'_n < \infty$. Let $Q \in \mathcal{L}(K, K)$ be the operator defined by $Qe_n = \lambda'_n e_n$. Set $K_0 := Q^{\frac{1}{2}}K$, where $Q^{\frac{1}{2}}$ is the operator defined by $Q^{\frac{1}{2}}e_n = \sqrt{\lambda'_n}e_n$. Then K_0 is a Hilbert space with inner product

$$(u, v)_0 = (Q^{-1/2}u, Q^{-1/2}v),\ \forall u, v \in K_0.$$

Let $\|\cdot\|_0$ denote the norm in K_0. For another separable Hilbert space $\tilde{K}$, with scalar product $(\cdot,\cdot)_{\tilde{K}}$ and the associated norms $\|\cdot\|_{\tilde{K}}$, a linear operator Φ in $\mathcal{L}(K_0, \tilde{K})$ is called Hilbert-Schmidt from K_0 to $\tilde{K}$ if for every complete orthonormal basis $\{e_n^0\}$ of K_0,

$$\sum_{n=1}^{\infty} \|\Phi e_n^0\|_{\tilde{K}}^2 < \infty.$$

The value of the series is independent of the choice of $\{e_n^0\}$. Clearly, the imbedding of K_0 in K is Hilbert-Schmidt since Q is a trace class operator. The space of all Hilbert-Schmidt operators from K_0 into $\tilde{K}$, denoted by $L^2(K_0, \tilde{K})$, a separable Hilbert space with the scalar product

$$(\Phi, \Psi)_{L^2(K_0,\tilde{K})} = \sum_{n=1}^{\infty}(\Phi e_n^0, \Psi e_n^0)_{\tilde{K}},\ \forall \Phi, \Psi \in L^2(K_0, \tilde{K}).$$

Furthermore, by the definition of K_0, every complete orthonormal basis of K_0 can be represented by $\{\sqrt{\lambda'_n}e_n\}$ for some complete orthonormal basis $\{e_n\}$ of K. Then, the norm in $L^2(K_0, \tilde{K})$ is

$$\|\Phi\|_{L^2(K_0,\tilde{K})}^2 = \operatorname{Tr}\left[\left(\Phi Q^{\frac{1}{2}}\right)\left(\Phi Q^{\frac{1}{2}}\right)^*\right] = \sum_{n=1}^{\infty} \|\sqrt{\lambda'_n}\Phi e_n\|_{\tilde{K}}^2,\ \Phi \in L^2(K_0, \tilde{K}).$$

Now for an $L^2(K_0, \tilde{K})$-valued process $\Phi(t,\omega)$, $0 \le t \le T$, the stochastic integral $\int_0^T \Phi(s,\omega)dW(s)$ is well-defined if

$$\mathbb{E}\int_0^T \|\Phi(s)\|^2_{L^2(K_0,\tilde{K})}ds < \infty.$$

We will define some probabilistic evolution spaces necessary throughout the paper. For any separable Banach space X and $p \ge 1$, we consider the space $L^p(0,T;X)$ of X-valued measurable functions u defined on $[0,T]$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}} < \infty.$$

For $r, p \ge 1$, we will write $L^p(\Omega, \mathcal{F}, \mathbb{P}; L^r(0,T;X))$ to denote the space of all functions $u = u(t,x,\omega)$ with values in X defined on $[0,T]\times\mathcal{O}\times\Omega$ and such that u is measurable with respect to (t,ω) and for almost all t, u is $\mathcal{F}_t$ measurable. The space $L^p(\Omega,\mathcal{F},\mathbb{P};L^r(0,T;X))$ so defined is a Banach space with the following norm

$$\|u\|_{L^p(\Omega,\mathcal{F},\mathbb{P};L^r(0,T;X))} = \left[\mathbb{E}\left(\int_0^T \|u(t)\|_X^r dt\right)^{\frac{p}{r}}\right]^{\frac{1}{p}} < \infty,$$

where $\mathbb{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$.

We will write $L^2(\Omega,\mathcal{F},\mathbb{P};C((-\infty,T];X))$, to denote the space of all X-valued adapted process $u(t,\omega)$ defined on $(-\infty,T]\times\Omega$ which are continuous in $t\in(-\infty,T]$ for almost every $\omega\in\Omega$ and which satisfy $\mathbb{E}\sup_{t\in(-\infty,T]}\|u(t)\|_X^2 < \infty$. The norm in $L^2(\Omega,\mathcal{F},\mathbb{P};C((-\infty,T];X))$ is given by

$$\|u\|_{L^2(\Omega,\mathcal{F},\mathbb{P};C((-\infty,T];X))} = \left(\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|_X^2\right)^{\frac{1}{2}} < \infty.$$

3. Existence and uniqueness of weak solutions

We first give the definition of weak solutions.

Definition 3.1. Let $T > 0$ and $\phi \in L^2(\Omega,\mathcal{F},\mathbb{P};BCL_{-\infty}(H))$ be an initial process. A stochastic process $u(t)$, $t\in(-\infty,T]$, is said to be a weak solution of (1.1) on $[0,T]$, if

$(1a)$ $u(t)$ is $\mathcal{F}_t$-adapted, for $t \le T$,

$(1b)$ $u(\cdot)\in L^2(\Omega,\mathcal{F},\mathbb{P};C((-\infty,T];H))\cap L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V))$,

$(1c)$ The following equation holds as an identity in V', a.s.

$$u(t) = \phi(0) - \nu\int_0^t Au(s)ds - \int_0^t B_N(u(s))ds + \int_0^t (f(s)+g_1(s,u_s))ds + \int_0^t g_2(s,u_s)dW(s),\ t\in[0,T],$$

$(1d)$ $u(t) = \phi(t),\ t \in (-\infty, 0]$, a.s.

Theorem 3.1. *Let $T > 0$ and $\phi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; BCL_{-\infty}(H))$, be an initial condition. If* **(H1)-(H3)** *are fulfilled, then there exists a unique weak solution to the stochastic 3D system of globally modified equations (1.1) on $[0,T]$ that satisfies the following energy inequality*

$$\mathbb{E} \sup_{0\le t\le T} |u(s)|^2 + \nu \mathbb{E} \int_0^T \|u(s)\|^2 ds \le \mathcal{C}_0.$$

Here, $\mathcal{C}_0$ is a positive constant only depending on $\|f\|_*$, T and other parameters of system.

Proof. We will split the proof into five steps as follows.

Step 1. Construction of an approximating sequence.

For each integer $m \ge 1$, we denote by H_m the vector space spanned by $\{\phi_1, \ldots, \phi_m\}$. We define by $P_m : H \to H_m$ the orthogonal projection from H on H_m. Now we use the Galerkin approximation method to prove the existence of weak solution to (1.1). Set

$$u^m(t,x,\omega) := \sum_{j=1}^m \gamma_j(t,\omega)\phi_j(x),$$

where $\gamma_j(t,\omega)$ are determined by the following ordinary differential stochastic systems

$$\begin{aligned}(u^m(t), \phi_j) = (u_0^m, \phi_j) + \int_0^t \langle -\nu A u^m(s) - P_m B_N(u^m(s)) + P_m f(s), \phi_j\rangle\, ds \\ + \int_0^t (P_m g_1(s, u_s^m), \phi_j) ds + \int_0^t (P_m g_2(s, u_s^m), \phi_j) dW(s)\end{aligned} \tag{3.1}$$

for $j = 1, \ldots, m$ with an initial value $u^m(t) = P_m\phi(t),\ t \in (-\infty, 0]$, where $u_0^m = u^m(0) = P_m\phi(0)$. Here and from now on, for short, we only write $u^m(t)$ instead of $u^m(t,x,\omega)$ and $\gamma(t)$ instead of $\gamma(t,\omega)$.

The existence and uniqueness of $\gamma(t) = (\gamma_1(t), \ldots, \gamma_m(t))$ are proved similarly as in [22] (one can see in [23]). For convenience to the reader, we sketch the proof as follows.

We rewrite (3.1) as the following system

$$\begin{cases} d\gamma(t) = h_0(\gamma(t))dt + h_1(t,\gamma_t)dt + h_2(t,\gamma_t)dW(t),\ t > 0, \\ \gamma(\theta) = \xi(\theta),\ \theta \le 0, \end{cases} \tag{3.2}$$

where $\xi(\theta) := ((P_m\phi(\theta), \phi_1), \ldots, (P_m\phi(\theta), \phi_m))$, and the j-component of h_0, h_1 and h_2 are given in the following forms

$$\begin{aligned} h_0^j(\gamma(t)) &= -\nu(Au^m(t), \phi_j) - (P_m B_N(u^m(t)), \phi_j), \\ h_1^j(t,\gamma_t) &= (P_m g_1(t, u_t^m), \phi_j),\ j = 1, \ldots, m, \\ h_2^j(t,\gamma_t) &= (P_m g_2(t, u_t^m), \phi_j),\ j = 1, \ldots, m. \end{aligned}$$

We can check that h_0, h_1 and h_2 satisfy the following estimates:

$$|h_0(\gamma(t)) - h_0(\tilde{\gamma}(t))| \leq L_0 \|\gamma_t - \tilde{\gamma}_t\|_{BCL_{-\infty}(\mathbb{R}^m)}, \tag{3.3}$$

$$|h_1(t, \gamma_t) - h_1(t, \tilde{\gamma}_t)| \leq L_1 \|\gamma_t - \tilde{\gamma}_t\|_{BCL_{-\infty}(\mathbb{R}^m)}, \tag{3.4}$$

$$|h_2(t, \gamma_t) - h_2(t, \tilde{\gamma}_t)| \leq L_2 \|\gamma_t - \tilde{\gamma}_t\|_{BCL_{-\infty}(\mathbb{R}^m)}, \tag{3.5}$$

for a.e. $t \in [0, T]$, $\forall T > 0$, and three positive constants L_0, L_1, L_2 depending on m, ν, L_{g_1}, L_{g_2} and N.

Estimates (3.4) and (3.5) are obtained from the global Lipschitz conditions on g_1 and g_2. To prove (3.3) we first have

$$\begin{aligned} |h_0(\gamma(t)) - h_0(\tilde{\gamma}(t))| \leq \nu \sum_{j=1}^{m} &|(A(u^m(t) - \tilde{u}^m(t)), \phi_j)| \\ &+ \sum_{j=1}^{m} |(P_m B_N(u^m(t)), \phi_j) - (P_m B_N(\tilde{u}^m(t)), \phi_j)|. \end{aligned}$$

Here, $u^m(t) = \sum\limits_{j=1}^{m} \gamma_j(t)\phi_j$ and $\tilde{u}^m(t) = \sum\limits_{j=1}^{m} \tilde{\gamma}_j(t)\phi_j$.

We have

$$\begin{aligned} \sum_{j=1}^{m} |(A(u^m(t) - \tilde{u}^m(t)), \phi_j)| &\leq \sum_{j=1}^{m} \lambda_j |\gamma_j(t) - \tilde{\gamma}_j(t)| \\ &\leq \lambda_m \sum_{j=1}^{m} |\gamma_j(t) - \tilde{\gamma}_j(t)| \\ &\leq \lambda_m |\gamma(t) - \tilde{\gamma}(t)| \\ &\leq \lambda_m \|\gamma_t - \tilde{\gamma}_t\|_{BCL_{-\infty}(\mathbb{R}^m)}. \end{aligned} \tag{3.6}$$

Using (2.3), the Poincaré inequality and Lemma 2.1, we have

$$\begin{aligned} &\sum_{j=1}^{m} |(P_m B_N(u^m(t)), \phi_j) - (P_m B_N(\tilde{u}^m(t)), \phi_j)| \\ = &\sum_{j=1}^{m} \left| \int_{\mathcal{O}} [F_N(\|u^m(t)\|)(u^m(t) \cdot \nabla)u^m(t) - F_N(\|\tilde{u}^m(t)\|)(\tilde{u}^m(t) \cdot \nabla)\tilde{u}^m(t)] \cdot \phi_j dx \right| \\ \leq &\sum_{j=1}^{m} F_N(\|\tilde{u}^m(t)\|) |b(u^m(t) - \tilde{u}^m(t)), \tilde{u}^m(t), \phi_j)| \\ &+ \sum_{j=1}^{m} |F_N(\|u^m(t)\|) - F_N(\|\tilde{u}^m(t)\|)| \, |b(u^m(t), \tilde{u}^m(t), \phi_j)| \\ &+ \sum_{j=1}^{m} F_N(\|u^m(t)\|) |b(u^m(t), u^m(t) - \tilde{u}^m(t), \phi_j)| \end{aligned}$$

$$\leq 2^{-1}\lambda_1^{-1/4}\sum_{j=1}^m F_N(\|\tilde{u}^m(t)\|)\|\tilde{u}^m(t)\|\|u^m(t)-\tilde{u}^m(t)\|\|\phi_j\|$$
$$+\,2^{-1}\lambda_1^{-1/4}\sum_{j=1}^m F_N(\|u^m(t)\|)F_N(\|\tilde{u}^m(t)\|)\|u^m(t)-\tilde{u}^m(t)\|\|u^m(t)\|\|\tilde{u}^m(t)\|\|\phi_j\|$$
$$+\,2^{-1}\lambda_1^{-1/4}\sum_{j=1}^m F_N(\|u^m(t)\|)\|u^m(t)\|\|u^m(t)-\tilde{u}^m(t)\|\|\phi_j\|$$
$$\leq \frac{3}{2}N\lambda_1^{-1/4}\sum_{j=1}^m\|\phi_j\|\|u^m(t)-\tilde{u}^m(t)\|.$$

Since

$$u^m(t)-\tilde{u}^m(t)=\sum_{j=1}^m(\gamma_j(t)-\tilde{\gamma}_j(t))\phi_j,$$

then

$$\|u^m(t)-\tilde{u}^m(t)\|\leq\sum_{j=1}^m\|\phi_j\|\;|\gamma(t)-\tilde{\gamma}(t)|\leq\sum_{j=1}^m\|\phi_j\|\|\gamma_t-\tilde{\gamma}_t\|_{BCL_{-\infty}(\mathbb{R}^m)}.$$

Thus,

$$\sum_{j=1}^m|(P_mB_N(u^m(t)),\phi_j)-(P_mB_N(\tilde{u}^m(t)),\phi_j)|$$
$$\leq\frac{3}{2}N\lambda_1^{-1/4}\left(\sum_{j=1}^m\|\phi_j\|\right)^2\|\gamma_t-\tilde{\gamma}_t\|_{BCL_{-\infty}(\mathbb{R}^m)}. \tag{3.7}$$

Combining (3.6) and (3.7), we deduce that (this proves estimate (3.5)).

$$|h_0(\gamma(t))-h_0(\tilde{\gamma}(t))|\leq L_0\|\gamma_t-\tilde{\gamma}_t\|_{BCL_{-\infty}(\mathbb{R}^m)},$$

with

$$L_0=\lambda_m+\frac{3}{2}N\lambda_1^{-1/4}\left(\sum_{j=1}^m\|\phi_j\|\right)^2.$$

Now, using (3.3), (3.4) and (3.5), we can prove as same as [22, Lemma 3.1] that

$$\mathbb{E}(\sup_{-\infty<t\leq T}|\gamma(t)|^2)\leq C(T,\mathbb{E}\|\xi\|^2_{BCL_{-\infty}(\mathbb{R}^m)}).$$

Therefore, using the arguments as in [22, Theorem 3.1] we get the existence and uniqueness of a solution to (3.2).

We consider the following approximation of stochastic 3D globally modified Navier-Stokes equations

$$\begin{cases} u^m(t)=u_0^m+\int_0^t(-\nu Au^m(s)-P_mB_N(u^m(s))+P_mf(s)+P_mg_1(s,u_s^m))\,ds\\ \qquad\qquad+\int_0^tP_mg_2(s,u_s^m)dW(s),\\ u^m(t)=P_m\phi(t),\;\;t\in(-\infty,0].\end{cases}\tag{3.8}$$

Step 2. Estimates for the approximating sequence.
By the Itô formula for $|u^m(t)|^2$ and notice that $b_N(u^m(s), u^m(s), u^m(s)) = 0$, we obtain

$$|u^m(t)|^2 = |u_0^m|^2 - 2\nu \int_0^t \|u^m(s)\|^2 ds + 2\int_0^t \langle P_m f(s) + P_m g_1(s, u_s^m), u^m(s)\rangle\, ds$$
$$+ \int_0^t \|g_2(s, u_s^m)\|^2_{L^2(K_0,H)} ds + 2\int_0^t (P_m g_2(s, u_s^m), u^m(s)) dW(s). \tag{3.9}$$

Taking supremum with respect to t in (3.9) and expectation, we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{0\le s\le t} |u^m(s)|^2 + 2\nu \mathbb{E}\int_0^t \|u^m(s)\|^2 ds \\
&\le 2\mathbb{E}|\phi(0)|^2 + 4\mathbb{E} \sup_{0\le r\le t} \int_0^r \|f(s)\|_* \|u^m(s)\| ds \\
&\quad + 4\mathbb{E} \sup_{0\le r\le t} \int_0^r |u^m(s)|\, |g_1(s, u_s^m)| ds \\
&\quad + 4\mathbb{E} \sup_{0\le r\le t} \left| \int_0^r (u^m(s), P_m g_2(s, u_s^m)) dW(s) \right| \\
&\quad + 2\mathbb{E} \sup_{0\le r\le t} \int_0^r \|g_2(s, u_s^m)\|^2_{L^2(K_0,H)} ds \\
&= 2\mathbb{E}|\phi(0)|^2 + I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using the Cauchy inequality, we have

$$\begin{aligned}
I_1 &= 4\mathbb{E} \sup_{0\le r\le t} \int_0^r \|f(s)\|_* \|u^m(s)\| ds \\
&\le 2\nu \mathbb{E} \int_0^t \sup_{0\le r\le s} \|u^m(r)\|^2 ds + \frac{2}{\nu} \mathbb{E}\int_0^t \|f(s)\|_*^2 ds.
\end{aligned}$$

By the condition **(H3)** on g_1 and the Cauchy inequality, we have

$$\begin{aligned}
I_2 &= 4\mathbb{E} \sup_{0\le r\le t} \int_0^r |u^m(s)| |g_1(s, u_s^m)| ds \\
&\le \frac{1}{4}\mathbb{E} \sup_{0\le s\le t} |u^m(s)|^2 + 16 L_{g_1}^2 \mathbb{E}\int_0^t \|u_s^m\|^2_{BCL_{-\infty}(H)} ds \\
&\le \frac{1}{4}\mathbb{E} \sup_{0\le s\le t} |u^m(s)|^2 + 16 L_{g_1}^2 \mathbb{E}\int_0^t \sup_{0\le r\le s} |u^m(r)|^2 ds + 16 T L_{g_1}^2 \mathbb{E}\|\phi\|^2_{BCL_{-\infty}(H)}.
\end{aligned}$$

Now we use the Burkholder-Davis-Gundy inequality, condition **(H3)** on g_2 and the Cauchy inequality to obtain

$$I_3 = 4\mathbb{E} \sup_{0\le r\le t} \left| \int_0^r (u^m(s), P_m g_2(s, u_s^m)) dW(s) \right|$$

$$\leq 16\mathbb{E}\left(\int_0^t |u^m(s)|^2 \|g_2(s,u_s^m)\|^2_{L^2(K_0,H)} ds\right)^{1/2}$$
$$\leq \frac{1}{4}\mathbb{E}\sup_{0\leq s\leq t}|u^m(s)|^2 + 256L^2_{g_2}\mathbb{E}\int_0^t \|u_s^m\|^2_{BCL_{-\infty}(H)} ds$$
$$\leq \frac{1}{4}\mathbb{E}\sup_{0\leq s\leq t}|u^m(s)|^2 + 256L^2_{g_2}\mathbb{E}\int_0^t \sup_{0\leq r\leq s}|u^m(r)|^2 ds + 256TL^2_{g_2}\mathbb{E}\|\phi\|^2_{BCL_{-\infty}(H)}.$$

Finally, using condition **(H3)** on g_2, we have

$$I_4 = 2\mathbb{E}\sup_{0\leq r\leq t}\int_0^r \|g_2(s,u_s^m)\|^2_{L^2(K_0,H)} ds$$
$$\leq 2L^2_{g_2}\int_0^t \mathbb{E}\sup_{0\leq r\leq s}|u^m(r)|^2 ds + 2TL^2_{g_2}\mathbb{E}\|\phi\|^2_{BCL_{-\infty}(H)}.$$

Hence,

$$\mathbb{E}\sup_{0\leq s\leq t}|u^m(s)|^2 + \nu\mathbb{E}\int_0^t \|u^m(s)\|^2 ds$$
$$\leq 4\mathbb{E}|\phi(0)|^2 + \frac{4}{\nu}\mathbb{E}\int_0^T \|f(s)\|^2_* ds + 2T\left(16L^2_{g_1} + 258L^2_{g_2}\right)\mathbb{E}\|\phi\|^2_{BCL_{-\infty}(H)}$$
$$+ 2(2 + 16L^2_{g_1} + 256L^2_{g_2})\int_0^t \mathbb{E}\sup_{0\leq r\leq s}|u^m(r)|^2 ds. \tag{3.10}$$

By the Gronwall inequality, there exists a constant $\mathcal{C}_0 > 0$ such that

$$\mathbb{E}\sup_{0\leq s\leq t}|u^m(s)|^2 + \nu\mathbb{E}\int_0^t \|u^m(s)\|^2 ds \leq \mathcal{C}_0, \text{ uniformly in } m \geq 1. \tag{3.11}$$

Step 3. Taking limits in the finite-dimensional equations.

From (3.11), we have $\{u^m\}$ is bound in $L^2(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;H)\cap L^2(0,T;V))$. And therefore, $\{Au^m\}$ is bounded in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V'))$. From (2.6) and the properties of B_N we conclude that $\{B_N(u^m)\}$ is bounded in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V'))$. On the other hand, from (**H3**) and (3.11), we see that $\{g_1(t,u_t^m)\}$ is bounded in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;H))$ and

$$\{g_2(t,u_t^m)\} \text{ is bounded in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;L^2(K_0,H))).$$

From these uniform bounds, there exists a subsequence of $\{u^m\}$ (relabeled the same) such that

$$u^m \rightharpoonup u \text{ weakly in } L^2(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;H)\cap L^2(0,T;V)). \tag{3.12}$$

Moreover,

$$Au^m \rightharpoonup Au \text{ weakly in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V')), \tag{3.13}$$
$$B_N(u^m) \rightharpoonup \mathcal{B}_N \text{ weakly in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V')), \tag{3.14}$$
$$g_1(t,u_t^m) \rightharpoonup \xi \text{ weakly in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;H)), \tag{3.15}$$

$$g_2(t,u_t^m) \rightharpoonup \zeta \text{ weakly in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;L^2(K_0,H))). \tag{3.16}$$

Taking limits in (3.8) when $m \to \infty$, we obtain

$$u(t) = u_0 + \int_0^t (-\nu Au(s) - \mathcal{B}_N + f(s) + \xi)\, ds + \int_0^t \zeta dW(s). \tag{3.17}$$

Step 4. Proving that $-\nu Au - \mathcal{B}_N + \xi = -\nu Au - B_N(u) + g_1(t,u_t)$ *and* $\zeta = g_2(t,u_t)$.

Using the Itô formula to $e^{-\lambda t}|u(t)|^2$ and $e^{-\lambda t}|u^m(t)|^2$, respectively,

$$\begin{aligned}\mathbb{E}e^{-\lambda t}|u(t)|^2 = \mathbb{E}|u(0)|^2 &- \mathbb{E}\int_0^t \lambda e^{-\lambda s}|u(s)|^2 ds\\ &+ 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au(s) - \mathcal{B}_N + f(s), u(s)\right\rangle ds\\ &+ 2\mathbb{E}\int_0^t e^{-\lambda s}(\xi, u(s)) ds + \mathbb{E}\int_0^t e^{-\lambda s}\|\zeta\|^2_{L^2(K_0,H)} ds,\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\mathbb{E}e^{-\lambda t}|u^m(t)|^2 - \mathbb{E}|u_0^m|^2 - 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle P_m f(s), u^m(s)\right\rangle ds&\\ = -\mathbb{E}\int_0^t \lambda e^{-\lambda s}|u^m(s)|^2 ds&\\ + 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au^m(s) - B_N(u^m(s)), u^m(s)\right\rangle ds&\\ + 2\mathbb{E}\int_0^t e^{-\lambda s}(g_1(s,u_s^m), u^m(s)) ds&\\ + \mathbb{E}\int_0^t e^{-\lambda s}\|g_2(s,u_s^m)\|^2_{L^2(K_0,H)} ds := \beta_m.&\end{aligned} \tag{3.19}$$

Let $z \in L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;H))$ and $z(s) = \phi(s)$, $s \le 0$. We notice that $|u^m - z|^2 = |u^m|^2 + |z|^2 - 2(u^m, z)$, then from (3.19) we have

$$\beta_m + \gamma_m = \alpha_m,$$

where

$$\begin{aligned}\alpha_m = &- \mathbb{E}\int_0^t \lambda e^{-\lambda s}|u^m(s) - z(s)|^2 ds\\ &+ 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au^m(s) - B_N(u^m(s)), u^m(s) - z(s)\right\rangle ds\\ &- 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Az(s) - B_N(z(s)), u^m(s) - z(s)\right\rangle ds\\ &+ 2\mathbb{E}\int_0^t e^{-\lambda s}(g_1(s,u_s^m) - g_1(s,z_s), u^m(s) - z(s)) ds\end{aligned}$$

$$+\,\mathbb{E}\int_0^t e^{-\lambda s}\|g_2(s,u_s^m)-g_2(s,z_s)\|^2_{L^2(K_0,H)}ds,$$

and

$$\begin{aligned}\gamma_m = &-\mathbb{E}\int_0^t \lambda e^{-\lambda s}\left(|z(s)|^2-2(u^m(s),z(s))\right)ds\\ &+2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au^m(s)-B_N(u^m(s))+g_1(s,u_s^m),-z(s)\right\rangle ds\\ &-2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Az-B_N(z(s))+g_1(s,z_s),u^m(s)-z(s)\right\rangle ds\\ &+\mathbb{E}\int_0^t e^{-\lambda s}\left(g_2(s,z_s)-2g_2(s,u_s^m),g_2(s,z_s)\right)ds.\end{aligned}$$

By (2.7), there is a positive constant λ such that $\alpha_m \le 0$. Using (3.12)-(3.16), then

$$\begin{aligned}0\ge \liminf_{m\to\infty}\alpha_m \ge &-\mathbb{E}\int_0^t \lambda e^{-\lambda s}|u(s)-z(s)|^2ds\\ &+2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au(s)-\mathcal{B}_N,u(s)-z(s)\right\rangle ds\\ &-2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Az(s)-B_N(z(s)),u(s)-z(s)\right\rangle ds\\ &+2\mathbb{E}\int_0^t e^{-\lambda s}(\xi-g_1(s,z_s),u(s)-z(s))ds\\ &+\mathbb{E}\int_0^t e^{-\lambda s}\|\zeta-g_2(s,z_s)\|^2_{L^2(K_0,H)}ds.\end{aligned}\tag{3.20}$$

Taking $z(t)=u(t)$ in (3.20), it follows that $\zeta = g_2(t,u_t)$, $t\in[0,T]$, where we use the fact that $e^{-\lambda t}$ is bounded for $t\in[0,T]$.

From (3.19), we have

$$\beta_m = \mathbb{E}e^{-\lambda t}|u^m(t)|^2-\mathbb{E}|u^m(0)|^2-2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle f(s),u^m(s)\right\rangle ds.$$

Since (3.12), $|u^m(0)|^2\le |u(0)|^2$ and using (3.18) with notice that $\zeta = g_2(t,u_t)$ then

$$\begin{aligned}\liminf_{m\to\infty}\beta_m &\ge \mathbb{E}e^{-\lambda t}|u(t)|^2-\mathbb{E}|u(0)|^2-2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle f(s),u(s)\right\rangle ds\\ &=2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au(s)-\mathcal{B}_N+\xi,u(s)\right\rangle ds\\ &\quad-\mathbb{E}\int_0^t \lambda e^{-\lambda s}|u(s)|^2ds+\mathbb{E}\int_0^t\|g_2(s,u_s)\|^2_{L^2(K_0,H)}ds.\end{aligned}\tag{3.21}$$

Using (3.12)-(3.16) once again and notice that $\zeta = g_2(t, u_t)$, we deduce that

$$\begin{aligned}\liminf_{m\to\infty} \gamma_m \geq & -\mathbb{E}\int_0^t e^{-\lambda s}(|z(s)|^2 - 2(u(s), z(s)))ds \\ & + 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au(s) - \mathcal{B}_N + \xi, -z(s)\right\rangle ds \\ & - 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Az - B_N(z(s)) + g_1(s, z_s), u(s) - z(s)\right\rangle ds \\ & + \mathbb{E}\int_0^t e^{-\lambda s}(g_2(s, z_s) - 2g_2(s, u_s), g_2(s, z_s))ds. \end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22) with notice that $\alpha_m \leq 0$, one has

$$\begin{aligned} 0 \geq & \liminf_{m\to\infty} \alpha_m \geq \liminf_{m\to\infty} \beta_m + \liminf_{m\to\infty} \gamma_m \\ \geq & -\lambda\mathbb{E}\int_0^t e^{-\lambda s}|u(s) - z(s)|^2 ds \\ & + 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au(s) - \mathcal{B}_N + \xi, u(s) - z(s)\right\rangle ds \\ & - 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Az - B_N(z) + g_1(s, z_s), u(s) - z(s)\right\rangle ds \\ & + \mathbb{E}\int_0^t e^{-\lambda s}\|g_2(s, u_s) - g_2(s, z_s)\|^2_{L^2(K_0,H)} ds. \end{aligned}$$

Hence,

$$\begin{aligned} 0 \leq & \ \mathbb{E}\int_0^t e^{-\lambda s}\|g_2(s, u_s) - g_2(s, z_s)\|^2_{L^2(K_0,H)} ds \\ \leq & \ \lambda\mathbb{E}\int_0^t e^{-\lambda s}|u(s) - z(s)|^2 ds \\ & - 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Au(s) - \mathcal{B}_N + \xi, u(s) - z(s)\right\rangle ds \\ & + 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\nu Az - B_N(z(s)) + g_1(s, z_s), u(s) - z(s)\right\rangle ds. \end{aligned}$$

For any fixed $w \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; V))$, set $z(t) = u(t) - \eta w(t)$, then

$$\begin{aligned} 0 \leq & \ \eta\lambda\mathbb{E}\int_0^t e^{-\lambda s}|w|^2 ds - 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle -\mathcal{B}_N + \xi, w(s)\right\rangle ds \\ & + 2\mathbb{E}\int_0^t e^{-\lambda s}\left\langle \eta Aw(s) - B_N(u(s) - \eta w(s)) + g_1(s, u_s - \eta w(s)), w(s)\right\rangle ds. \end{aligned}$$

Let $\eta \to 0$, and since $L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; V))$ is dense in $L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; H))$,

$$e^{-\lambda t}\left(\mathcal{B}_N - \xi - B_N(u(t)) + g_1(t, u_t)\right) = 0, \ \text{a.e. } t \in [0, T], \ \omega \in \Omega.$$

Thus, we get from (3.17) that

$$u(t)=\phi(0)+\int_0^t (-\nu Au(s)-B_N(u(s))+f(s)+g_1(s,u_s))\,ds+\int_0^t g_2(s,u_s)dW(s),$$

a.e. $\omega \in \Omega$.

Step 5. Uniqueness of solutions.

Let $u(t)$ and $v(t)$ be two solutions to (1.1) with the same initial value. Applying the Itô formula to $|u(t)-v(t)|^2$, we have that

$$\begin{aligned}|u(t)-v(t)|^2 = {} & 2\int_0^t \langle -\nu A(u(s)-v(s)) - B_N(u(s)) + B_N(v(s),u(s)-v(s)\rangle\,ds \\ & + 2\int_0^t (g_1(s,u_s)-g_1(s,v_s),u(s)-v(s))ds \\ & + 2\int_0^t (g_2(s,u_s)-g_2(s,v_s),u(s)-v(s))\,dW(s) \\ & + \int_0^t \|g_2(s,u_s)-g_2(s,v_s)\|^2_{L^2(K_0,H)}ds. \end{aligned} \tag{3.23}$$

Thank to (2.7), we have

$$\begin{aligned} & \int_0^t \langle -\nu A(u(s)-v(s)) - B_N(u(s)) + B_N(v(s),u(s)-v(s)\rangle\,ds \\ & \le -\frac{\nu}{2}\int_0^t \|u(s)-v(s)\|^2ds + C(\nu,c_0)N^4\int_0^t |u(s)-v(s)|^2ds. \end{aligned} \tag{3.24}$$

We can now deduce from (1.2) and (1.3) that

$$\int_0^t \|g_2(s,u_s)-g_2(s,v_s)\|^2_{L^2(K_0,H)}ds \le L^2_{g_2}\int_0^t \|u_s-v_s\|^2_{BCL_{-\infty}(H)}ds, \tag{3.25}$$

and

$$\begin{aligned} & \int_0^t (g_1(s,u_s)-g_1(s,v_s),u(s)-v(s))ds \\ \le {} & \int_0^t |g_1(s,u_s)-g_1(s,v_s)||u(s)-v(s)|ds \\ \le {} & L_{g_1}\int_0^t \|u_s-v_s\|_{BCL_{-\infty}(H)}|u(s)-v(s)|ds. \end{aligned} \tag{3.26}$$

Combining (3.23), (3.24), (3.25), (3.26) and using the fact that $|u(s)-v(s)| \le \|u_s-v_s\|_{BCL_{-\infty}(H)}$, we obtain

$$\begin{aligned} & |u(t)-v(t)|^2 + \nu\mathbb{E}\int_0^t \|u(s)-v(s)\|^2ds \\ \le {} & (2L_{g_1}+L^2_{g_2}+2C(\nu,c_0)N^4)\int_0^t \|u_s-v_s\|^2_{BCL_{-\infty}(H)}ds \end{aligned}$$

$$+ 2\int_0^t (g_2(s,u_s) - g_2(s,v_s), u(s) - v(s))\, dW(s).$$

Then for any $t > 0$, we have

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t} |u(s) - v(s)|^2 \\
&\le \left(2L_{g_1} + L_{g_2}^2 + 2C(\nu, c_0)N^4\right) \mathbb{E}\int_0^t \|u_s - v_s\|^2_{BCL_{-\infty}(H)} ds \\
&\quad + 2\mathbb{E}\sup_{0\le r\le t}\left|\int_0^r (g_2(s,u_s) - g_2(s,v_s), u(s) - v(s))\, dW(s)\right|. \qquad (3.27)
\end{aligned}$$

Using the Bulkhoder-Davis-Gundy inequality and using (1.3), we have

$$\begin{aligned}
&2\mathbb{E}\sup_{0\le r\le t}\left|\int_0^r (g_2(s,u_s) - g_2(s,v_s), u(s) - v(s)) dW(s)\right| \\
&\le 8\mathbb{E}\left[\sup_{0\le s\le t} |u(s) - v(s)| \left(\int_0^t \|g_2(s,u_s) - g_2(s,v_s)\|^2_{L^2(K_0,H)} ds\right)^{1/2}\right] \\
&\le \frac{1}{2}\mathbb{E}\sup_{0\le s\le t} |u(s) - v(s)|^2 + 32L_{g_2}^2 \int_0^t \|u_s - v_s\|^2_{BCL_{-\infty}(H)} ds.
\end{aligned}$$

Substituting this inequality into (3.27) to obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t} |u(s) - v(s)|^2 \\
&\le 2(2L_{g_1} + 2C(\nu, c_0)N^4 + 33L_{g_2}^2)\mathbb{E}\int_0^t \|u_s - v_s\|^2_{BCL_{-\infty}(H)} ds. \qquad (3.28)
\end{aligned}$$

Since $u(s) = v(s) = \phi(s)$, $s \le 0$, we see that

$$\begin{aligned}
\mathbb{E}\int_0^t \|u_s - v_s\|^2_{BCL_{-\infty}(H)} ds &\le \mathbb{E}\int_0^t \sup_{-t<\theta\le 0} |u(s+\theta) - v(s+\theta)|^2 ds \\
&\le \mathbb{E}\int_0^t \sup_{0\le r\le s} |u(r) - v(r)|^2 ds.
\end{aligned}$$

Hence (3.28) becomes

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t} |u(s) - v(s)|^2 \\
&\le 2(2L_{g_1} + L_{g_2}^2 + 2C(\nu, c_0)N^4 + 32L_{g_2}^2)\mathbb{E}\int_0^t \sup_{0\le r\le s} |u(r) - v(r)|^2 ds.
\end{aligned}$$

By the Gronwall inequality we have

$$\mathbb{E}\sup_{0\le s\le t} |u(s) - v(s)|^2 = 0.$$

Consequently, $u(t) = v(t)$, a.e. $\omega \in \Omega$ for all $t \le T$. The proof is complete. $\square$

4. Stability of stationary solutions

This section is devoted to investigating the stability of stationary solutions to problem (1.1) with some extra conditions. More precisely, from now on, we assume f is independent of time, i.e., $f(t) \equiv f \in V'$, and g_i, $i = 1, 2$ are given by $g_i(t, u_t) = G_i(u(t - \rho(t))$ with $\rho \in C^1([0, \infty))$, $\rho(t) \geq 0$ for all $t \geq 0$ and $\rho_* = \sup_{t \geq 0} \rho'(t) < 1$. Here, $G_i : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $G_i(0) = 0$ and

$$\|G_i(u) - G_i(v)\|_{\mathbb{R}^3} \leq L_{G_i} \|u - v\|_{\mathbb{R}^3}$$

for some $L_{G_i} > 0$, $i = 1, 2$. One can check that all assumptions (**H1**)-(**H3**) are satisfied with L_{g_i} are replaced by L_{G_i}.

Using the above notations, we can rewrite the 3D globally modified Navier-Stokes equations with unbounded delays (1.1) in the following functional form

$$\begin{cases} du + [\nu Au + B_N(u, u)]dt & \\ = [PG_1(u(t - \rho(t))) + Pf]dt + PG_2(u(t - \rho(t)))dW(t) & \text{in } (0, \infty) \times \mathcal{O}, \\ u(\theta) = \phi(\theta), & \theta \in (-\infty, 0]. \end{cases} \tag{4.1}$$

By Theorem 3.1, for any $\phi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; BCL_{-\infty}(H))$ given, problem (4.1) has a unique globally solution $u(t)$, which is defined on the whole interval $[0, \infty)$.

The deterministic problem corresponding to (4.1) is the following problem

$$\begin{cases} du + [\nu Au + B_N(u, u)]dt = [PG_1(u(t - \rho(t))) + Pf]dt & \text{in } (0, \infty) \times \mathcal{O}, \\ u(\theta) = \phi(\theta), & \theta \in (-\infty, 0]. \end{cases} \tag{4.2}$$

Let us give the definition of stationary solutions to problem (4.2).

Definition 4.1. A weak stationary solution to problem (4.2) is an element $u_\infty \in V$ such that

$$\nu((u_\infty, v)) + b_N(u_\infty, u_\infty, v) = \langle f, v \rangle + (G_1(u_\infty), v), \ \forall v \in V.$$

The following theorem can be proved similarly as in [14].

Theorem 4.1. *If $\nu > L_{G_1} \lambda_1^{-1}$, then problem (4.2) admits at least one stationary solution u_∞ satisfying the following estimate*

$$\|u_\infty\| \leq \frac{\|f\|_*}{\nu - L_{G_1} \lambda_1^{-1}}. \tag{4.3}$$

Moreover, if the following condition holds

$$\nu > L_{G_1} \lambda_1^{-1} + \mu,$$

where

$$\mu := \lambda_1^{-1/4} \min \left\{ N, \frac{\|f\|_*}{\nu - L_{G_1} \lambda_1^{-1}} \right\}, \tag{4.4}$$

then the stationary solution u_∞ of (4.2) is unique.

Remark 4.1. Let u_∞ be a stationary solution of the deterministic problem (4.2). If $G_2(u_\infty) = 0$, then u_∞ is also a weak solution to the stochastic problem (4.1). From now on, we always impose this condition when studying the stability of the stationary solution u_∞.

4.1. Local stability via a direct approach

In this subsection, we prove the local stability of the stationary solution by a direct approach.

Theorem 4.2. *If $\nu > L_{G_1}\lambda_1^{-1}$, then there exists at least one stationary solution u_∞ to* (4.2)*. Moreover, if $G_2(u_\infty) = 0$ and*

$$\nu > \mu + \frac{L_{G_1}(2-\rho_*) + L_{G_2}^2}{2\lambda_1(1-\rho_*)}, \tag{4.5}$$

where μ is defined in (4.4)*, then the stationary solution u_∞ is unique and there exists $C = C(\rho_*, L_{G_1}, L_{G_2}) > 0$ such that any solution $u(t)$ to* (4.1) *satisfies*

$$\mathbb{E}|u(t) - u_\infty|^2 \le C\left(\mathbb{E}|\phi(0) - u_\infty|^2 + \mathbb{E}\int_{-\rho(0)}^0 |\phi(s) - u_\infty|^2 ds\right).$$

Proof. Applying the Itô formula to $|u(t) - u_\infty|^2$, we have

$$\begin{aligned}
|u(t)-u_\infty|^2 = |\phi(0)-u_\infty|^2 &+ 2\int_0^t \langle -\nu A(u-u_\infty) - B_N(u) + B_N(u_\infty), u-u_\infty\rangle\, ds \\
&+ 2\int_0^t (G_1(u(s-\rho(s))) - G_1(u_\infty), u-u_\infty) ds \\
&+ 2\int_0^t (G_2(u(s-\rho(s))) - G_2(u_\infty), u-u_\infty)\, dW(s) \\
&+ \int_0^t \|(G_2(u(s-\rho(s))) - G_2(u_\infty)\|^2_{L^2(K_0,H)} ds,
\end{aligned}$$

and taking the expectation

$$\begin{aligned}
\mathbb{E}|u(t) - u_\infty|^2 = \mathbb{E}|\phi(0) - u_\infty|^2 &- 2\nu\mathbb{E}\int_0^t \|u(s) - u_\infty\|^2 ds \\
&- 2\mathbb{E}\int_0^t \langle B_N(u) - B_N(u_\infty), u(s) - u_\infty\rangle\, ds \\
&+ 2\mathbb{E}\int_0^t (G_1(u(s-\rho(s))) - G_1(u_\infty), u(s) - u_\infty) ds \\
&+ \mathbb{E}\int_0^t \|(G_2(u(s-\rho(s))) - G_2(u_\infty)\|^2_{L^2(K_0,H)} ds.
\end{aligned} \tag{4.6}$$

Using (2.2), (2.3) and Lemma 2.1, we have

$$\begin{aligned}
&\langle B_N(u(s)) - B_N(u_\infty), u(s) - u_\infty\rangle \\
&= F_N(\|u_\infty\|) b(u(s) - u_\infty, u_\infty, u(s) - u_\infty)
\end{aligned}$$

$$
\begin{aligned}
&+ (F_N(\|u(s)\|) - F_N(\|u_\infty\|))\, b(u(s), u_\infty, u(s) - u_\infty) \\
&+ F_N(\|u(s)\|) b(u(s), u(s) - u_\infty, u(s) - u_\infty) \\
\le\ & \min\left\{1, \frac{N}{\|u_\infty\|}\right\} 2^{-1}\lambda_1^{-1/4}\|u_\infty\|\|u(s) - u_\infty\|^2 \\
&+ \frac{1}{N} F_N(\|u(s)\|) F_N(\|u_\infty\|) 2^{-1}\lambda_1^{-1/4}\|u(s)\|\|u_\infty\|\|u(s) - u_\infty\|^2 \\
\le\ & \min\left\{1, \frac{N}{\|u_\infty\|}\right\} 2^{-1}\lambda_1^{-1/4}\|u_\infty\|\|u(s) - u_\infty\|^2 \\
&+ \frac{1}{N}\min\left\{1, \frac{N}{\|u(s)\|}\right\}\|u(s)\|\min\left\{1, \frac{N}{\|u_\infty\|}\right\}\|u_\infty\| 2^{-1}\lambda_1^{-1/4}\|u(s) - u_\infty\|^2 \\
\le\ & \min\left\{\|u_\infty\|\lambda_1^{-1/4}, N\lambda_1^{-1/4}\right\}\|u(s) - u_\infty\|^2.
\end{aligned}
$$

Hence, using estimate (4.3), we find that

$$
\langle B_N(u(s)) - B_N(u_\infty), u(s) - u_\infty\rangle \le \mu\|u(s) - u_\infty\|^2, \tag{4.7}
$$

where μ is defined in (4.4). On the other hand, using condition on G_1, the Cauchy inequality and Poincaré inequality (2.1), then

$$
\begin{aligned}
&2\mathbb{E}\int_0^t (G_1(u(s - \rho(s))) - G_1(u_\infty), u(s) - u_\infty) ds \\
\le\ & 2\mathbb{E}\int_0^t L_{G_1}|u(s - \rho(s)) - u_\infty||u(s) - u_\infty| ds \\
\le\ & \frac{L_{G_1}}{\lambda_1}\mathbb{E}\int_0^t \|u(s) - u_\infty\|^2 ds + L_{G_1}\mathbb{E}\int_0^t |u(s - \rho(s)) - u_\infty|^2 ds.
\end{aligned}
$$

The last term on the right-hand side of (4.6) is bounded by

$$
\mathbb{E}\int_0^t \|(G_2(u(s-\rho(s))) - G_2(u_\infty)\|^2_{L^2(K_0,H)} ds \le L^2_{G_2}\mathbb{E}\int_0^t |u(s-\rho(s)) - u_\infty|^2 ds.
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}|u(t) - u_\infty|^2 \le\ & \mathbb{E}|\phi(0) - u_\infty|^2 + \left(-2\nu + 2\mu + \frac{L_{G_1}}{\lambda_1}\right)\mathbb{E}\int_0^t \|u(s) - u_\infty\|^2 ds \\
&+ \left(L_{G_1} + L^2_{G_2}\right)\mathbb{E}\int_0^t |u(s - \rho(s)) - u_\infty|^2 ds.
\end{aligned} \tag{4.8}
$$

Taking $\eta = s - \rho(s)$ then

$$
\mathbb{E}\int_0^t |u(s - \rho(s)) - u_\infty|^2 ds \le \frac{1}{1 - \rho^*}\mathbb{E}\int_{-\rho(0)}^t |u(\eta) - u_\infty|^2 d\eta.
$$

Thus

$$\mathbb{E}|u(t)-u_\infty|^2 \le \mathbb{E}|\phi(0)-u_\infty|^2 + \frac{L_{G_1}+L_{G_2}^2}{(1-\rho_*)}\mathbb{E}\int_{-\rho(0)}^0 |\phi(s)-u_\infty|^2 ds + \left(-2\nu+2\mu+\frac{L_{G_1}(2-\rho_*)+L_{G_2}^2}{\lambda_1(1-\rho_*)}\right)\mathbb{E}\int_0^t \|u(s)-u_\infty\|^2 ds.$$

Therefore, by (4.5) we have

$$\mathbb{E}|u(t)-u_\infty|^2 \le \mathbb{E}|\phi(0)-u_\infty|^2 + \frac{L_{G_1}+L_{G_2}^2}{(1-\rho_*)}\mathbb{E}\int_{-\rho(0)}^0 |\phi(s)-u_\infty|^2 ds.$$

Therefore, we complete the proof. □

4.2. Asymptotic stability via the construction of Lyapunov functionals

Theorem 4.3. *If $\nu > L_{G_1}\lambda_1^{-1}$, then there exists at least one stationary solution u_∞ to (4.2). Moreover, if $G_2(u_\infty)=0$ and suppose that*

$$\nu \ge \mu + \frac{2L_{G_1}\sqrt{1-\rho_*}+L_{G_2}^2}{2\lambda_1(1-\rho_*)}, \tag{4.9}$$

where μ is defined in (4.4), then the stationary solution u_∞ is unique and locally stable, that is,

$$\mathbb{E}|u(t)-u_\infty|^2 \le \mathbb{E}|\phi(0)-u_\infty|^2 + \frac{L_{G_1}\sqrt{1-\rho_*}+L_{G_2}^2}{1-\rho_*}\mathbb{E}\|\phi-u_\infty\|^2_{L^2(-\rho(0),0;H)} \tag{4.10}$$

for any solution $u(t)$ to (4.1) with $\phi \in L^2(\Omega,\mathcal{F},\mathbb{P};BCL_{-\infty}(H))$. Furthermore, if

$$\nu > \mu + \frac{2L_{G_1}\sqrt{1-\rho_*}+L_{G_2}^2}{2\lambda_1(1-\rho_*)},$$

then u_∞ is asymptotically stable in mean square, i.e.,

$$\lim_{t\to\infty}\mathbb{E}|u(t)-u_\infty|^2 = 0.$$

Proof. Let $w(t)=u(t)-u_\infty$ and

$$U(t,\phi) = |\phi(0)|^2 + \frac{c}{1-\rho_*}\int_{-\rho(t)}^0 |\phi(s)|^2 ds$$

for a suitable constant c to be specified later on. Then we replace ϕ by u_t-u_∞, and obtain

$$U(t,w_t) = |u(t)-u_\infty|^2 + \frac{c}{1-\rho_*}\int_{t-\rho(t)}^t |u(s)-u_\infty|^2 ds. \tag{4.11}$$

Applying the Itô formula to $U(t, w_t)$ and taking the expectation, we have

$$\begin{aligned}\mathbb{E}U(t,w_t) \le \mathbb{E}U(0,w_0) &+ 2\mathbb{E}\int_0^t \langle -\nu Aw(s) - B_N(u(s)) + B_N(u_\infty), w(s)\rangle\, ds \\ &+ 2\mathbb{E}\int_0^t (G_1(u(s-\rho(s))) - G_1(u_\infty), w(s))ds \\ &+ \mathbb{E}\int_0^t \|G_2(u(s-\rho(s)))\|^2_{L^2(K_0,H)}ds \\ &+ \frac{c}{1-\rho_*}\mathbb{E}\int_0^t |w(s)|^2 ds - \mathbb{E}\int_0^t \frac{c(1-\rho'(s))}{1-\rho_*}|w(s-\rho(s))|^2 ds.\end{aligned}$$

By the Cauchy inequality and using (4.7), we obtain

$$\begin{aligned}&\mathbb{E}U(t,w_t) \\ &\le \mathbb{E}U(0,w_0) - 2(\nu-\mu)\mathbb{E}\int_0^t \|w(s)\|^2 ds + 2L_{G_1}\mathbb{E}\int_0^t |w(s-\rho(s))||w(s)|ds \\ &\quad + \frac{c}{1-\rho_*}\mathbb{E}\int_0^t |w(s)|^2 ds + L_{G_2}^2\mathbb{E}\int_0^t |w(s-\rho(s))|^2 ds - c\mathbb{E}\int_0^t |w(s-\rho(s))|^2 ds \\ &\le \mathbb{E}U(0,w_0) - 2(\nu-\mu)\mathbb{E}\int_0^t \|w(s)\|^2 ds + \frac{L_{G_1}^2}{c-L_{G_2}^2}\mathbb{E}\int_0^t |w(s)|^2 ds \\ &\quad + (c-L_{G_2}^2)\mathbb{E}\int_0^t |w(s-\rho(s))|^2 ds + \frac{c}{1-\rho_*}\mathbb{E}\int_0^t |w(s)|^2 ds \\ &\quad - c\mathbb{E}\int_0^t |w(s-\rho(s))|^2 ds + L_{G_2}^2\mathbb{E}\int_0^t |w(s-\rho(s))|^2 ds \\ &\le \mathbb{E}U(0,w_0) - 2(\nu-\mu)\mathbb{E}\int_0^t \|w(s)\|^2 ds + \left(\frac{L_{G_1}^2}{c-L_{G_2}^2} + \frac{c}{1-\rho_*}\right)\mathbb{E}\int_0^t |w(s)|^2 ds.\end{aligned}$$

Choose $c = L_{G_1}\sqrt{1-\rho_*} + L_{G_2}^2$, the coefficient in the last term of the right-hand side reaches minimum. Using Poincaré inequality (2.1), we conclude that (4.12)

$$\mathbb{E}U(t,w_t) + 2\left(\nu - \mu - \frac{2L_{G_1}\sqrt{1-\rho_*} + L_{G_2}^2}{2\lambda_1(1-\rho_*)}\right)\int_0^t \mathbb{E}\|w(s)\|^2 ds \le \mathbb{E}U(0,w_0).$$

From (4.11), we have

$$\mathbb{E}U(t,w_t) \ge \mathbb{E}|u(t) - u_\infty|^2,$$

and

$$\mathbb{E}U(0,w_0) = \mathbb{E}|\phi(0) - u_\infty|^2 + \frac{L_{G_1} + L_{G_2}^2}{1-\rho_*}\mathbb{E}\|\phi - u_\infty\|^2_{L^2(-\rho(0),0;H)}.$$

Using Poincaré inequality (2.1), then (4.12) becomes

$$2\lambda_1\left(\nu - \mu - \frac{2L_{G_1}\sqrt{1-\rho_*} + L_{G_2}^2}{2\lambda_1(1-\rho_*)}\right)\int_0^t \mathbb{E}|u(s) - u_\infty|^2 ds$$

$$\leq \mathbb{E}|\phi(0)-u_\infty|^2 + \frac{L_{G_1}\sqrt{1-\rho_*}+L_{G_2}^2}{1-\rho_*}\mathbb{E}\|\phi-u_\infty\|^2_{L^2(-\rho(0),0;H)}. \tag{4.13}$$

Therefore, if $\nu \geq \mu + \frac{2L_{G_1}\sqrt{1-\rho_*}+L_{G_2}^2}{2\lambda_1(1-\rho_*)}$, then the stationary solution u_∞ is stable and satisfies (4.10).

If $\nu > \mu + \frac{2L_{G_1}\sqrt{1-\rho_*}+L_{G_2}^2}{2\lambda_1(1-\rho_*)}$, from (4.13) we obtain

$$\mathbb{E}\int_0^\infty |u(s)-u_\infty|^2 ds \leq \mathbb{E}|\phi(0)-u_\infty|^2 + \frac{L_{G_1}\sqrt{1-\rho_*}+L_{G_2}^2}{1-\rho_*}\mathbb{E}\|\phi-u_\infty\|^2_{L^2(-\rho(0),0;H)}.$$

By the continuity in time of u in H, we deduce that $\lim_{t\to\infty}\mathbb{E}|u(t)-u_\infty|^2 = 0$, i.e. the stationary solution u_∞ is asymptotically stable in mean square. □

Remark 4.2. Since $\frac{L_{G_1}(2-\rho_*)+L_{G_2}^2}{2\lambda_1(1-\rho_*)} > \frac{2L_{G_1}\sqrt{1-\rho_*}+L_{G_2}^2}{2\lambda_1(1-\rho_*)}$ for $\rho_* \in (0,1)$, we can see that Theorem 4.3 is an improvement of Theorem 4.2.

4.3. Polynomial stability: the proportional delay case

We now consider problem (4.1) with proportional delay, a particular case of unbounded variable delay. More precisely, we assume $\rho(t) = (1-q)t$ with $q \in (0,1)$. We will show the polynomial stability of the stationary solution.

The following lemma is key tool in the proof of polynomial stability results.

Lemma 4.1 ([2, Lemma 3.6(i)]). *Let $a < 0$, $b > 0$ and $q \in (0,1)$. Suppose $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$D^+h(t) \leq ah(t) + bh(qt),\ t > 0,\ h(0) = h_0,\ q \in (0,1),$$

with $h_0 \geq 0$ and where D^+h denotes the Dini derivative of h at t in the following sense

$$D^+h = \limsup_{\delta\downarrow 0}\frac{h(t+\delta)-h(t)}{\delta}.$$

Then there exist $C = C(a,b,q) > 0$ such that

$$h(t) \leq Ch(0)(1+t)^\alpha,\ \forall t \geq 0,$$

where α obeys $a + bq^\alpha = 0$.

We first prove the following theorem concerning with the polynomial stability in mean square.

Theorem 4.4. *Assume that $f \in V'$ and we consider (4.1) with $\rho(t) = (1-q)t$ for $q \in (0,1)$. If $\nu > L_{G_1}\lambda_1^{-1}$, then there exist at least one weak stationary solution u_∞ to (4.2) satisfying (4.3). Furthermore, if $G_2(u_\infty) = 0$ and*

$$\nu > \mu + L_{G_1}\lambda_1^{-1} + \frac{1}{2}L_{G_2}^2\lambda_1^{-1}, \tag{4.14}$$

where μ is defined in (4.4), *then u_∞ is asymptotically stable in mean square with polynomial rate, that is, there exists $C = C(\nu, \lambda_1, \|f\|_*, L_{G_1}, L_{G_2}, q) > 0$ such that for any solution $u(t)$ to* (4.1),

$$\mathbb{E}|u(t) - u_\infty|^2 \leq C\mathbb{E}|\phi(0) - u_\infty|^2(1 + t)^\alpha, \tag{4.15}$$

where

$$\alpha = \log_q \left(\frac{2\nu - 2\mu - L_{G_1}\lambda_1^{-1}}{(L_{G_1} + L_{G_2}^2)\lambda_1^{-1}} \right) < 0. \tag{4.16}$$

Proof. The existence and uniqueness of a stationary solution u_∞ to (4.2) follows immediately from Theorem 4.1. Let $w(t) = u(t) - u_\infty$, then applying the Itô formula to $|w(t)|^2$, and as same as the estimate (4.8) by taking $\rho(t) = (1 - q)t$, we have

$$\begin{aligned}\mathbb{E}|w(t+\delta)|^2 = \mathbb{E}|w(t)|^2 + (-2\nu + 2\mu + L_{G_1}\lambda_1^{-1}) \int_t^{t+\delta} \mathbb{E}\|w(s)\|^2 ds \\ + \left(L_{G_1} + L_{G_2}^2\right) \int_t^{t+\delta} |w(qs)|^2 ds\end{aligned} \tag{4.17}$$

for any $\delta > 0$. Denoting $h(t) = \mathbb{E}|w(t)|^2$ and noting that $-2\nu+2\mu+L_{G_1}\lambda_1^{-1} < 0$ since condition (4.14), by using Poincaré inequality (2.1), we obtain from (4.17) by taking $\delta \downarrow 0$ that

$$D^+h(t) \leq \lambda_1 \left(-2\nu + 2\mu + L_{G_1}\lambda_1^{-1}\right) h(t) + \left(L_{G_1} + L_{G_2}^2\right) h(qt).$$

Applying Lemma 4.1, there exists $C = C(\nu, \lambda_1, \|f\|_*, L_{G_1}, L_{G_2}, q) > 0$ such that

$$h(t) \leq Ch(0)(1 + t)^\alpha, \ \forall t \geq 0,$$

where α satisfies

$$\lambda_1 \left(-2\nu + 2\mu + L_{G_1}\lambda_1^{-1}\right) + \left(L_{G_1} + L_{G_2}^2\right) q^\alpha = 0,$$

that is, α is given by (4.16). Since $\nu > \mu + L_{G_1}\lambda_1^{-1} + \frac{1}{2}L_{G_2}^2\lambda_1^{-1}$, it holds that $\alpha < 0$ and due to condition (4.9), we get (4.15). This ends the proof. ☐

The following theorem is the pathwise polynomial stability of stationary solution u_∞.

Theorem 4.5. *Assume that $f \in V'$ and we consider* (4.1) *with $\rho(t) = (1-q)t$, for $q \in (0, 1)$. Furthermore, if $G_2(u_\infty) = 0$ and assume that*

$$\nu > \mu + \frac{1}{2}\left(1 + \frac{1}{q}\right) L_{G_1}\lambda_1^{-1} + \frac{1}{2q}L_{G_2}^2\lambda_1^{-1}, \tag{4.18}$$

where μ is defined in (4.4), *then any solution $u(t)$ to* (4.1) *converges to the stationary solution u_∞ in H, almost surely at a polynomial rate.*

Proof. Let $\mathcal{K}$ be a positive integer and any $t \geq \mathcal{K}$. Using the Itô formula to the function $|w(t)|^2 = |u(t) - u_\infty|^2$, we have

$$
\begin{aligned}
|w(t)|^2 = |w(\mathcal{K})|^2 &- 2\nu \int_{\mathcal{K}}^t \|w(s)\|^2 ds + 2\int_{\mathcal{K}}^t \langle -B_N(u(s)) + B_N(u_\infty), w(s)\rangle \, ds \\
&+ 2\int_{\mathcal{K}}^t (G_1(u(qs)) - G_1(u_\infty), w(s))\, ds \\
&+ 2\int_{\mathcal{K}}^t (G_2(u(qs)) - G_2(u_\infty), w(s))\, dW(s) \\
&+ \int_{\mathcal{K}}^t \|G_2(u(qs)) - G_2(u_\infty))\|^2_{L^2(K_0,H)}\, ds.
\end{aligned}
$$

By using the Burkholder-Davis-Gundy inequality, the Cauchy inequality and condition on G_2 we obtain

$$
\begin{aligned}
&2\mathbb{E}\left[\sup_{\mathcal{K}\leq t\leq \mathcal{K}+1} \int_{\mathcal{K}}^t (G_2(u(qs)) - G_2(u_\infty), w(s))\, dW(s)\right] \\
&\leq 8\mathbb{E}\left[\sup_{\mathcal{K}\leq t\leq \mathcal{K}+1} |w(t)|^2 \int_{\mathcal{K}}^{\mathcal{K}+1} \|G_2(u(qs)) - G_2(u_\infty)\|^2_{L^2(K_0,H)}\, ds\right]^{\frac{1}{2}} \\
&\leq 32\mathbb{E}\int_{\mathcal{K}}^{\mathcal{K}+1} \|G_2(u(qs)) - G_2(u_\infty)\|^2_{L^2(K_0,H)}\, ds + \frac{1}{2}\mathbb{E}\sup_{\mathcal{K}\leq t\leq \mathcal{K}+1} |w(t)|^2 \\
&\leq 32L_{G_2}^2 \mathbb{E}\int_{\mathcal{K}}^{\mathcal{K}+1} |w(qs)|^2 ds + \frac{1}{2}\mathbb{E}\sup_{\mathcal{K}\leq t\leq \mathcal{K}+1} |w(t)|^2.
\end{aligned}
$$

Using condition on G_1, the Cauchy inequality and Poincaré inequality (2.1) then

$$
\begin{aligned}
&2\mathbb{E}\sup_{\mathcal{K}\leq t\leq \mathcal{K}+1} \left|\int_{\mathcal{K}}^t \langle G_1(u(qs)) - G_1(u_\infty), w(s)\rangle\, ds\right| \\
&\leq 2\mathbb{E}\int_{\mathcal{K}}^{\mathcal{K}+1} |\langle G_1(u(qs)) - G_1(u_\infty), w(s)\rangle|\, ds \\
&\leq 2\mathbb{E}\int_{\mathcal{K}}^{\mathcal{K}+1} L_{G_1}|w(qs)||w(s)| ds \\
&\leq L_{G_1}\mathbb{E}\int_{\mathcal{K}}^{\mathcal{K}+1} |w(qs)|^2 ds + L_{G_1}\lambda_1^{-1}\mathbb{E}\int_{\mathcal{K}}^{\mathcal{K}+1} \|w(s)\|^2 ds.
\end{aligned}
$$

As same as (4.7) we have

$$
\sup_{\mathcal{K}\leq t\leq \mathcal{K}+1} \left|-2\int_{\mathcal{K}}^t \mathbb{E}\left(\langle B_N(u(s)) - B_N(u_\infty), w(s)\rangle\right) ds\right| \leq 2\mu \int_{\mathcal{K}}^{\mathcal{K}+1} \mathbb{E}\|w(s)\|^2 ds.
$$

Combining all above inequalities, we obtain that

$$\frac{1}{2}\mathbb{E}\sup_{\mathcal{K}\le t\le \mathcal{K}+1}|w(t)|^2 \le \mathbb{E}|w(\mathcal{K})|^2 + \left(-2\nu+2\mu+L_{G_1}\lambda_1^{-1}\right)\mathbb{E}\int_{\mathcal{K}}^{\mathcal{K}+1}\|w(s)\|^2 ds$$

$$+\left(L_{G_1}+32L_{G_2}^2\right)\mathbb{E}\int_{\mathcal{K}}^{\mathcal{K}+1}|w(qs)|^2 ds. \tag{4.19}$$

Condition (4.18) implies that $-2\nu+2\mu+L_{G_1}\lambda_1^{-1}<0$ and condition (4.14) is satisfied. Hence, we deduce from (4.19) that

$$\mathbb{E}\sup_{\mathcal{K}\le t\le \mathcal{K}+1}|w(t)|^2 \le M(q\mathcal{K}+1)^{\alpha}, \tag{4.20}$$

where

$$M = C\mathbb{E}|\phi(0)-u_\infty|^2\left(1+L_{G_1}+32L_{G_2}^2\right).$$

Since condition (4.18) then $\alpha<-1$, and therefore we can choose $\varepsilon<0$ such that $\alpha-\varepsilon<-1$. Using the Markov inequality, we deduce from (4.20) that

$$\mathbb{P}\left\{\omega : \sup_{\mathcal{K}\le t\le \mathcal{K}+1}|w(t)|^2 > (1+q\mathcal{K})^{\varepsilon}\right\} \le M(1+q\mathcal{K})^{\alpha-\varepsilon}.$$

Hence, we can apply the Borel-Cantelli lemma to obtain an integer $\mathcal{K}_0=\mathcal{K}_0(\omega)>0$ such that

$$\sup_{\mathcal{K}\le t\le \mathcal{K}+1}|w(t)|^2 < (1+q\mathcal{K})^{\varepsilon}, \text{ a.s., for all } \mathcal{K}\ge\mathcal{K}_0.$$

The proof is complete. □

Acknowledgements. This work is financially supported by the Vietnam Ministry of Education and Training under grant number B2023-SPH-13.

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