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INEQUALITIES FOR INTEGRAL MEAN ESTIMATE OF POLYNOMIALS

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Abstract. In this paper, we obtain integral analogues of inequalities concerning polynomials proved by Soraisam et al. [33]. The results improve other known inequalities as well.

1. Introduction

Let $p(z)$ be a polynomial of degree n over the set of complex numbers and for each real number $r > 0$, we define the integral mean of $p(z)$ on the unit circle $|z| = 1$ by

$$\|p\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}.$$

If we take limit as $r \to \infty$ in the above equality and make use of the well-known fact from analysis [29, 36] that

$$\lim_{r \to \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} = \max_{|z|=1} |p(z)|,$$

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we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

Similarly, we can define

$$\|p\|_0 = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| d\theta \right\},$$

and it follows easily that $\lim_{r\to 0^+} \|p\|_r = \|p\|_0$. It would be of further interest that by taking limit as $r \to 0^+$ the stated results on integral mean inequalities holding for $r > 0$, hold for $r = 0$ as well.

A classical inequality that relates an estimate to the size of the derivative of a polynomial to that of the polynomial itself in the uniform-norm on the unit circle in the plane was shown in the famous Bernstein-inequality [4]. It states that, if $p(z)$ is a polynomial of degree n, then

$$\|p'\|_\infty \le n\|p\|_\infty. \tag{1.1}$$

Equality holds in (1.1) if and only if $p(z)$ has all its zeros at the origin. Inequality (1.1) can be obtained by letting $r \to \infty$ in the inequality

$$\|p'\|_r \le n\|p\|_r, \quad r > 0. \tag{1.2}$$

Inequality (1.2) was proved by Zygmund [38] for $r \ge 1$ and by Arrestov [1] for $0 < r < 1$.

If we restrict to the class of polynomials having no zeros in $|z| < 1$, then inequalities (1.1) and (1.2) can be respectively improved as

$$\|p'\|_\infty \le \frac{n}{2}\|p\|_\infty, \tag{1.3}$$

$$\|p'\|_r \le \frac{n}{\|1+z\|_r}\|p\|_r, \quad r > 0. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [19] whereas, inequality (1.4) was proved by De-Bruijn [8] for $r \ge 1$ and by Rahman and Schmeisser [26] for $0 < r < 1$.

On the other hand, in 1939 (see [37]), Turán obtained a lower bound for the maximum of $|p'(z)|$ on $|z| = 1$, by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le 1$, then

$$\|p'\|_\infty \ge \frac{n}{2}\|p\|_\infty. \tag{1.5}$$

Both inequalities (1.3) and (1.5) attain equality for the polynomial $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

Malik [20] generalized (1.3) by considering that if $p(z)$ has no zero in $|z| < k$, $k \geq 1$ and proved that

$$\|p'\|_\infty \leq \frac{n}{1+k}\|p\|_\infty. \tag{1.6}$$

Inequality (1.6) is best possible and equality holds for $p(z) = (z+k)^n$. Under the same hypothesis for the polynomial as above, Govil and Rahman [12] extended (1.6) into integral mean version by proving that

$$\|p'\|_r \leq \frac{n}{\|k+z\|_r}\|p\|_r, \quad r \geq 1. \tag{1.7}$$

Gardner and Weems [11], and Rather [27] independently proved that (1.7) also holds for $0 < r < 1$.

Govil et al. [13] refined (1.6) by involving certain coefficients for the polynomial of degree n having no zero in $|z| < k$, $k \geq 1$ by proving that

$$\|p'\|_\infty \leq n\left\{\frac{n|a_0| + k^2|a_1|}{n|a_0|(1+k^2) + 2k^2|a_1|}\right\}\|p\|_\infty. \tag{1.8}$$

Inequality (1.8) was extended to integral mean by Aziz and Rather [3]. They proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for each $r > 0$,

$$\|p'\|_r \leq \frac{n}{\|\delta_{k,1} + z\|_r}\|p\|_r, \tag{1.9}$$

where $\delta_{k,1} = \frac{n|a_0|k^2 + |a_1|k^2}{n|a_0| + k^2|a_1|}$.

On the other hand, as a generalization of (1.5), Malik [20] proved that if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\|p'\|_\infty \geq \frac{n}{1+k}\|p\|_\infty. \tag{1.10}$$

For the first time in 1984, Malik [21] extended inequality (1.5) proved by Turán [37] into integral mean version and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $r > 0$,

$$\|1+z\|_r\|p'\|_\infty \geq n\|p\|_r.$$

The result is sharp and equality holds for $p(z) = (z+1)^n$.

In 1988, Aziz [2] obtained the integral mean of inequality (1.10) and proved that if $p(z)$ has all its zeros in $|z| \leq k, k \leq 1$, then for each $r > 0$,

$$\|1+kz\|_r\|p'\|_\infty \geq n\|p\|_r. \tag{1.11}$$

Equality in (1.11) holds for the polynomial $p(z) = (\alpha z + \beta k)^n$, where $|\alpha| = |\beta|$.

As a refinement of (1.10), by involving certain coefficients of the polynomial, Govil et al. [13] proved that if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\|p'\|_\infty \geq n\left\{\frac{n\,|a_n| + |a_{n-1}|}{n|a_n|(1+k^2) + 2\,|a_{n-1}|}\right\}\|p\|_\infty. \tag{1.12}$$

Inequality (1.12) was extended to integral analogue by Aziz and Rather [3] by proving that if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for each $r > 0$,

$$n\left\|\frac{p}{p'}\right\|_r \leq \|1 + t_{k,1}z\|_r \tag{1.13}$$

and

$$\|p'\|_\infty \geq \frac{n}{\|1 + t_{k,1}z\|_r}\|p\|_r, \tag{1.14}$$

where $t_{k,1} = \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + |a_{n-1}|}$.

Several improvements, generalizations and extensions of the above inequalities which fundamentally estimates the bounds of $\left\{\frac{\|p'\|_\infty}{\|p\|_\infty}\right\}$ under prescribed restrictions on zeros of $p(z)$ are available in the literature. It is also desirable to know the dependence of the above mentioned ratio on the coefficients of the polynomial under consideration.

In this direction, Govil et al. [13] proved the following two results, where the first improves (1.6) and (1.8), and the second (1.10) and (1.12) by involving certain coefficients of the polynomial.

Theorem 1.1. ([13]) *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\|p'\|_\infty \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|}\|p\|_\infty, \tag{1.15}$$

where $\lambda = \frac{ka_1}{na_0}$ *and* $\mu = \frac{2k^2a_2}{n(n-1)a_0}$.

Theorem 1.2. ([13]) *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then*

$$\|p'\|_\infty \geq \frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|) + k(n-1)|\Omega - \omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|) + k(n-1)|\Omega - \omega^2|}\|p\|_\infty, \tag{1.16}$$

where $\omega = \frac{\bar{a}_{n-1}}{nk\bar{a}_n}$ *and* $\Omega = \frac{2\bar{a}_{n-2}}{n(n-1)k^2\bar{a}_n}$.

Krishnadas et al. [16] extended both Theorem 1.1 and Theorem 1.2 to integral versions, where the first result obtained not only generalizes Theorem 1.1 but also improves inequalities (1.7) and (1.9), and the second to integral analogue of Theorem 1.2 and improves inequalities (1.11) and (1.13), (1.14). Finally, by using Holder's inequality [14] the authors established a result which is a generalization of their second result.

Theorem 1.3. ([16]) *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then for each* $r > 0$,

$$\|p'\|_r \leq \frac{n}{\left\|A_{\lambda,\mu} + z\right\|_r} \|p\|_r,$$

where $A_{\lambda,\mu} = k\frac{|\lambda|(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+(1-|\lambda|)k^2}{(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+|\lambda|(1-|\lambda|)k^2}$, $\lambda = \frac{ka_1}{na_0}$ *and* $\mu = \frac{2k^2 a_2}{n(n-1)a_0}$.

Theorem 1.4. ([16]) *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then for each* $r > 0$,

$$n\left\|\frac{p}{p'}\right\|_r \leq \|1 + B_{\omega,\Omega} z\|_r,$$

where $B_{\omega,\Omega} = k\frac{|\omega|(1-|\omega|)+(n-1)|\Omega-\omega^2|k+(1-|\omega|)k^2}{(1-|\omega|)+(n-1)|\Omega-\omega^2|k+|\omega|(1-|\omega|)k^2}$, $\omega = \frac{\bar{a}_{n-1}}{nk\bar{a}_n}$ *and* $\Omega = \frac{2\bar{a}_{n-2}}{n(n-1)k^2\bar{a}_n}$.

Theorem 1.5. ([16]) *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then for* $s > 1$, $t > 1$ *with* $s^{-1} + t^{-1} = 1$ *and for each* $r > 0$,

$$\|p'\|_{tr} \geq \frac{n}{\|1 + B_{\omega,\Omega} z\|_{sr}} \|p\|_r, \tag{1.17}$$

where $B_{\omega,\Omega}$ *is as defined in Theorem* 1.4.

Next, by considering polynomials of degree $n \geq 3$, Soraisam et al. [33] proved the following two results, where the first improves Theorem 1.1, and the second Theorem 1.2 by involving $\min_{|z|=k} |p(z)|$.

Theorem 1.6. ([33]) *If* $p(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* $n\geq 3$ *having no zero in* $|z|<k$, $k\geq 1$, *then*

$$\|p'\|_\infty \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\|p\|_\infty - \frac{n}{k^n}\left(1-\frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\right)m \tag{1.18}$$

where $\lambda=\frac{ka_1}{na_0}$, $\mu=\frac{2k^2a_2}{n(n-1)a_0}$ *and* $m=\min_{|z|=k}|p(z)|$.

Theorem 1.7. ([33]) *If* $p(z)=\sum_{j=0}^{n} a_j z^j$, $a_0\neq 0$ *is a polynomial of degree* $n\geq 3$ *having all its zeros in* $|z|\leq k$, $k\leq 1$, *then*

$$\|p'\|_\infty \geq \frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+k(n-1)|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+k(n-1)|\Omega-\omega^2|} \times\left\{\|p\|_\infty+m\right\}, \tag{1.19}$$

where $\omega=\frac{\bar{a}_{n-1}}{nk\bar{a}_n}$, $\Omega=\frac{2\bar{a}_{n-2}}{n(n-1)k^2\bar{a}_n}$ *and* $m=\min_{|z|=k}|p(z)|$.

The improvement and generalization of the inequalities concerning complex polynomials is a widely studied topic, and for more information in this direction, we refer to the recently published papers [5], [7], [9], [17], [18], [22], [23], [24], [28], [30], [31], [32], [34] etc.

The present paper is mainly motivated by the desire to establish an integral version of inequalities (1.18) and (1.19), and an improvement of inequality (1.17) by involving the minimum modulus of the polynomial. The paper is organized as follows. In Section 2, we present some auxiliary results necessary in proving the main results. Then the main results in the integral setting and its proofs are given along with remarks and a corollary in Section 3.

2. Lemmas

The following lemmas are needed for the proof of the theorems.

Lemma 2.1. ([35]) *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z|<k$, $k\geq 1$, *then for* $|z|\leq k$, $|\xi|\leq k$, *where* ξ *is a real or complex number, we have*

$$(\xi-z)p'(z)+np(z)\neq 0.$$

Lemma 2.2. ([13]) *If $f(z)$ is analytic and $|f(z)| \le 1$ in $|z| \le 1$, then for $|z| \le 1$,*

$$|f(z)| \le \frac{(1-|a|)|z|^2 + |bz| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |bz| + (1-|a|)},$$

where $a = f(0)$, $b = f'(0)$. The example

$$f(z) = \frac{a + \frac{b}{1+a}z - z^2}{1 - \frac{b}{1+a}z - az^2}$$

shows that the estimate is sharp.

Lemma 2.3. ([3]) *If $p(z)$ is a polynomial of degree n and $q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$, then for each γ, $0 \le \gamma < 2\pi$ and for each $r > 0$,*

$$\int_0^{2\pi}\int_0^{2\pi} |q'(e^{i\theta}) + e^{i\gamma}p'(e^{i\theta})|^r d\theta d\gamma \le 2\pi n^r \int_0^{2\pi} |p(e^{i\theta})|^r d\theta.$$

Lemma 2.4. ([10]) *Let z be any complex and independent of γ, where γ is any real, then for each $r > 0$,*

$$\int_0^{2\pi} |1 + ze^{i\gamma}|^r d\gamma = \int_0^{2\pi} |e^{i\gamma} + |z||^r d\gamma.$$

Lemma 2.5. ([25]) *If $p(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$ is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \le 1.$$

Lemma 2.6. ([33]) *If $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$1 - \frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \ge 0,$$

where $\lambda = \frac{ka_1}{na_0}$ and $\mu = \frac{2k^2a_2}{n(n-1)a_0}$.

Lemma 2.7. ([16]) *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$ *and* $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, *then on* $|z| = 1$,

$$|q'(z)| \leq B_{\omega,\Omega}|p'(z)|,$$

where $B_{\omega,\Omega} = k\frac{|\omega|(1-|\omega|)+(n-1)|\Omega-\omega^2|k+(1-|\omega|)k^2}{(1-|\omega|)+(n-1)|\Omega-\omega^2|k+|\omega|(1-|\omega|)k^2}$, $\omega = \frac{\bar{a}_{n-1}}{nk\bar{a}_n}$ *and* $\Omega = \frac{2\bar{a}_{n-2}}{n(n-1)k^2\bar{a}_n}$ *such that* $|\omega| \leq 1$ *and* $(n-1)|\Omega - \omega^2| \leq 1 - |\omega|^2$.

3. Main results

In this paper, we consider polynomials of degree $n \geq 3$ and prove the following theorem which is an integral extension of Theorem 1.6 and is an improvement of Theorem 1.3 by involving $\min_{|z|=k} |p(z)|$. In fact, we prove:

Theorem 3.1. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* $n \geq 3$ *having no zero in* $|z| < k$, $k \geq 1$, *then for every complex number* α *with* $|\alpha| < \frac{1}{k^n}$ *and for each* $r > 0$,

$$\|A_{\lambda,\mu} + z\|_r \left\|p'(z) + \alpha mnz^{n-1}\right\|_r \leq n \left\|p(z) + m\alpha z^n\right\|_r, \tag{3.1}$$

where $A_{\lambda,\mu} = k\frac{|\lambda|(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+(1-|\lambda|)k^2}{(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+|\lambda|(1-|\lambda|)k^2}$, $\lambda = \frac{ka_1}{na_0}$, $\mu = \frac{2k^2a_2}{n(n-1)a_0}$ *and* $m = \min_{|z|=k} |p(z)|$.

Proof. Consider a new polynomial $P(z) = p(z) + m\alpha z^n$, where α is a complex number with $|\alpha| < \frac{1}{k^n}$, $m = \min_{|z|=k} |p(z)|$. Now, on $|z| = k$

$$|m\alpha z^n| < m\frac{1}{k^n}k^n = m \leq |p(z)|.$$

Then by Rouche's theorem [6], $p(z)$ and $P(z)$ must have same number of zeros in $|z| < k$ and hence $P(z)$ has no zero in $|z| < k$. And for $|z| < k$, $|\xi| < k$, where ξ is a complex number, by Lemma 2.1, we have

$$nP(z) + (\xi - z)P'(z) \neq 0,$$

that is,

$$nP(z) - zP'(z) \neq -\xi P'(z).$$

Consequently, for $|z| \leq k$

$$\left|\frac{P'(z)}{nP(z) - zP'(z)}\right| \leq \frac{1}{k}.$$

Hence if

$$f(z)=\frac{kP'(kz)}{nP(kz)-kzP'(kz)},$$

then $|f(z)|\leq 1$ for $|z|\leq 1$. Also

$$f(0)=\frac{ka_1}{na_0}=\lambda$$

and

$$f'(0)=(n-1)\left\{\frac{2k^2a_2}{n(n-1)a_0}-\left(\frac{ka_1}{na_0}\right)^2\right\}=(n-1)(\mu-\lambda^2).$$

Then for $|z|\leq 1$, we use Lemma 2.2 to conclude that

$$|f(z)|\leq\frac{(1-|\lambda|)|z|^2+(n-1)|\mu-\lambda^2||z|+|\lambda|(1-|\lambda|)}{|\lambda|(1-|\lambda|)|z|^2+(n-1)|\mu-\lambda^2||z|+(1-|\lambda|)}.$$

Thus in particular for $|z|=1$, we have

$$|P'(z)|\leq\frac{1}{k}\frac{(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+|\lambda|(1-|\lambda|)k^2}{|\lambda|(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+(1-|\lambda|)k^2}|nP(z)-zP'(z)|. \tag{3.2}$$

If $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$, then on $|z|=1$, $|nP(z)-zP'(z)|=|Q'(z)|$. Therefore, inequality (3.2) becomes

$$|P'(z)|\leq\frac{1}{k}\frac{(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+|\lambda|(1-|\lambda|)k^2}{|\lambda|(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+(1-|\lambda|)k^2}|Q'(z)|,$$

that is,

$$\left|\frac{Q'(z)}{P'(z)}\right|\geq k\frac{|\lambda|(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+(1-|\lambda|)k^2}{(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+|\lambda|(1-|\lambda|)k^2}.$$

For points $e^{i\theta}$, $0\leq\theta<2\pi$ for which $P'(e^{i\theta})\neq 0$, we obtain

$$\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}\right|\geq A_{\lambda,\mu}, \tag{3.3}$$

where $A_{\lambda,\mu}=k\frac{|\lambda|(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+(1-|\lambda|)k^2}{(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+|\lambda|(1-|\lambda|)k^2}$.

We have for every real γ and $L\geq l\geq 1$,

$$|L+e^{i\gamma}|\geq|l+e^{i\gamma}|.$$

Then, for every $r>0$ we have

$$\int_0^{2\pi}|L+e^{i\gamma}|^r d\gamma\geq\int_0^{2\pi}|l+e^{i\gamma}|^r d\gamma. \tag{3.4}$$

Applying Lemma 2.3 to $P(z)$, we have for each $r>0$,

$$\int_0^{2\pi}\int_0^{2\pi} |Q'(e^{i\theta})+e^{i\gamma}P'(e^{i\theta})|^r d\theta d\gamma \leq 2\pi n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \tag{3.5}$$

For points $e^{i\theta}$, $0\leq\theta<2\pi$ for which $P'(e^{i\theta})\neq 0$, we denote $L=\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}\right|$ and $l=A_{\lambda,\mu}$, then by (3.3) and Remark 3.3 we have $L\geq l\geq 1$ and for each $r>0$,

$$\begin{aligned}\int_0^{2\pi} |Q'(e^{i\theta})+e^{i\gamma}P'(e^{i\theta})|^r d\gamma &= |P'(e^{i\theta})|^r \int_0^{2\pi} \left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}+e^{i\gamma}\right|^r d\gamma \\ &= |P'(e^{i\theta})|^r \int_0^{2\pi} \left|\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}\right|+e^{i\gamma}\right|^r d\gamma \\ &\geq |P'(e^{i\theta})|^r \int_0^{2\pi} \left|A_{\lambda,\mu}+e^{i\gamma}\right|^r d\gamma.\end{aligned}$$

Integrating both sides with respect to θ from 0 to 2π, we have

$$\int_0^{2\pi}\int_0^{2\pi} |Q'(e^{i\theta})+e^{i\gamma}P'(e^{i\theta})|^r d\theta d\gamma \geq \left\{\int_0^{2\pi} \left|A_{\lambda,\mu}+e^{i\gamma}\right|^r d\gamma\right\}\left\{\int_0^{2\pi} |P'(e^{i\theta})|^r d\theta\right\}. \tag{3.6}$$

Combining inequalities (3.5) and (3.6), we get

$$\left\{\int_0^{2\pi} \left|A_{\lambda,\mu}+e^{i\gamma}\right|^r d\gamma\right\}\left\{\int_0^{2\pi} |P'(e^{i\theta})|^r d\theta\right\} \leq 2\pi n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta.$$

This is equivalent to

$$\begin{aligned}&\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|A_{\lambda,\mu}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}\left\{\frac{1}{2\pi}\int_0^{2\pi} |p'(e^{i\theta})+\alpha mne^{i(n-1)\theta}|^r d\theta\right\}^{\frac{1}{r}} \\ &\quad\leq n\left\{\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})+m\alpha e^{in\theta}|^r d\theta\right\}^{\frac{1}{r}}\end{aligned}$$

and hence the proof of Theorem 3.1 is completed. $\square$

Remark 3.2. Letting $r \to \infty$ on both sides of (3.1), we get

$$\{A_{\lambda,\mu}+1\}\max_{|z|=1}|p'(z)+\alpha mnz^{n-1}| \leq n\max_{|z|=1}|p(z)+m\alpha z^n|. \tag{3.7}$$

Let z_0 on $|z|=1$ be such that

$$\max_{|z|=1}|p'(z)| = |p'(z_0)|. \tag{3.8}$$

Now,

$$|p'(z_0)+\alpha mnz_0^{n-1}| \leq \max_{|z|=1}|p'(z)+\alpha mnz^{n-1}|. \tag{3.9}$$

In the left hand side of inequality (3.9) for suitable choice of the argument of α, we have

$$|p'(z_0)+\alpha mnz_0^{n-1}| = |p'(z_0)|+n|\alpha|m. \tag{3.10}$$

Using (3.10) and (3.8) in inequality (3.9), we have

$$\max_{|z|=1}|p'(z)|+n|\alpha|m \leq \max_{|z|=1}|p'(z)+\alpha mnz^{n-1}|. \tag{3.11}$$

Combining inequalities (3.11) and (3.7), we have

$$\{A_{\lambda,\mu}+1\}\left(\max_{|z|=1}|p'(z)|+n|\alpha|m\right) \leq n\max_{|z|=1}|p(z)+m\alpha z^n|. \tag{3.12}$$

Again, let z_1 on $|z|=1$ be such that

$$\begin{aligned}\max_{|z|=1}|p(z)+m\alpha z^n| &= |p(z_1)+m\alpha z_1^n| \\ &\leq |p(z_1)|+|\alpha|m \\ &\leq \max_{|z|=1}|p(z)|+|\alpha|m. \end{aligned} \tag{3.13}$$

Using inequality (3.13) in inequality (3.12), we have

$$\{A_{\lambda,\mu}+1\}\left(\max_{|z|=1}|p'(z)|+n|\alpha|m\right) \leq n\left(\max_{|z|=1}|p(z)|+|\alpha|m\right),$$

which on simplification and letting the limit as $|\alpha| \to \frac{1}{k^n}$, gives inequality (1.18) of Theorem 1.6.

Remark 3.3. Inequality (3.1) is an improvement of (1.7) proved by Govil and Rahman [12] for $r \geq 1$, and Gardner and Weems [11], and Rather [27] for $0 < r < 1$. It is sufficient to prove that $A_{\lambda,\mu} \geq k$, where $A_{\lambda,\mu}$ is defined in Theorem 3.1 and $k \geq 1$, which is equivalent to showing that

$$\frac{|\lambda|(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+(1-|\lambda|)k^2}{(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+|\lambda|(1-|\lambda|)k^2} \geq 1,$$

that is,

$$|\lambda|+k^2 \geq 1+|\lambda|k^2.$$

Therefore, we have $k^2 \geq 1$, which is true since $1-|\lambda| \geq 0$ (by Lemma 2.5) and $k \geq 1$.

Remark 3.4. Inequality (3.1) is also an improvement of (1.9) proved by Aziz and Rather [3]. For this, it is enough to show that $A_{\lambda,\mu} \geq \delta_{k,1}$, where $A_{\lambda,\mu}$ and $\delta_{k,1}$ are defined in Theorem 3.1 and inequality (1.9) respectively.

Since $\delta_{k,1} = \frac{n|a_0|k^2+|a_1|k^2}{n|a_0|+k^2|a_1|} = \frac{k+|\lambda|}{k+|\lambda|k^2}$, where λ is as defined Theorem 3.1, it is sufficient to show that

$$k\frac{|\lambda|(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+(1-|\lambda|)k^2}{(1-|\lambda|)+(n-1)|\mu-\lambda^2|k+|\lambda|(1-|\lambda|)k^2} \geq \frac{k+|\lambda|}{k+|\lambda|k^2},$$

which implies

$$(k-1)[(1-|\lambda|)\{k^3|\lambda|(k+1)+k^2|\lambda|^2+k^3+|\lambda|(k+1)\}$$
$$+k(n-1)|\mu-\lambda^2|\{k+|\lambda|(k^2+k+1)\}] \geq 0,$$

from which we eventually obtain

$$(1-|\lambda|)\{k^4|\lambda|+k^3(|\lambda|+1)+k^2|\lambda|^2+k|\lambda|+|\lambda|\}$$
$$+k(n-1)|\mu-\lambda^2|\{k^2|\lambda|+k(|\lambda|+1)+|\lambda|\} \geq 0,$$

which obviously holds due to the fact that $|\lambda| \leq 1$ (by Lemma 2.5).

Remark 3.5. Inequality (3.1) in its ordinary form obtained as in Remark 3.2 improves inequality (1.15) of Theorem 1.1 due to Govil et al. [13]. For this it is sufficient to show that

$$\left(1-\frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\right) \geq 0. \tag{3.14}$$

From Lemma 2.6, we have inequality (3.14).

Next, we obtain the following integral analogue of Theorem 1.7 which is an improvement of Theorem 1.4 as well.

Theorem 3.6. *If* $p(z) = \sum\limits_{j=0}^{n} a_j z^j$, $a_0 \neq 0$ *is a polynomial of degree* $n \geq 3$ *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then for every complex number* α *with* $|\alpha| < 1$ *and for each* $r > 0$,

$$n\left\|\frac{p(z)+m\alpha}{p'(z)}\right\|_r \leq \|1+B_{\omega,\Omega}z\|_r, \tag{3.15}$$

where $B_{\omega,\Omega} = k\frac{|\omega|(1-|\omega|)+(n-1)|\Omega-\omega^2|k+(1-|\omega|)k^2}{(1-|\omega|)+(n-1)|\Omega-\omega^2|k+|\omega|(1-|\omega|)k^2}$, $\omega = \frac{\bar{a}_{n-1}}{nk\bar{a}_n}$, $\Omega = \frac{2\bar{a}_{n-2}}{n(n-1)k^2\bar{a}_n}$ *and* $m = \min\limits_{|z|=k}|p(z)|$.

Proof. Consider the polynomial $P(z) = p(z) + m\alpha$, where α is a complex number with $|\alpha| < 1$, $m = \min_{|z|=k} |p(z)|$. Then by Rouches theorem [6] it follows that $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$. Since $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, by Gauss Lucas Theorem $P'(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$ and hence the polynomial

$$z^{n-1}\overline{P\left(\frac{1}{\bar{z}}\right)} = nQ(z) - zQ'(z), \tag{3.16}$$

where $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$ has all its zeros in $|z| \geq \frac{1}{k}$, $\frac{1}{k} \geq 1$. Further, since $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, we have by Lemma 2.7

$$|Q'(z)| \leq B_{\omega,\Omega}|P'(z)| \quad \text{on} \quad |z| = 1, \tag{3.17}$$

where $B_{\omega,\Omega} = k\frac{|\omega|(1-|\omega|)+(n-1)|\Omega-\omega^2|k+(1-|\omega|)k^2}{(1-|\omega|)+(n-1)|\Omega-\omega^2|k+|\omega|(1-|\omega|)k^2}$.

For $|z| = 1$, we also have

$$|P'(z)| = |nQ(z) - zQ'(z)|. \tag{3.18}$$

Using (3.18) in (3.17) we have on $|z| = 1$

$$|Q'(z)| \leq B_{\omega,\Omega}|nQ(z) - zQ'(z)|. \tag{3.19}$$

Thus, in view of (3.16) and (3.19), the function

$$\phi(z) = \frac{zQ'(z)}{B_{\omega,\Omega}\{nQ(z) - zQ'(z)\}}$$

is analytic in $|z| \leq 1$, $|\phi(z)| \leq 1$ on $|z| = 1$ and $\phi(0) = 0$. Therefore, the function $1 + B_{\omega,\Omega}\phi(z)$ is subordinate to the function $1 + B_{\omega,\Omega}z$ for $|z| \leq 1$. Hence, by a well-known property of subordination [15], we have for each $r > 0$

$$\int_0^{2\pi} \left|1 + B_{\omega,\Omega}\phi(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + B_{\omega,\Omega}e^{i\theta}\right|^r d\theta. \tag{3.20}$$

Now,

$$1 + B_{\omega,\Omega}\phi(z) = 1 + \frac{zQ'(z)}{nQ(z) - zQ'(z)} = \frac{nQ(z)}{nQ(z) - zQ'(z)},$$

which implies for $|z| = 1$,

$$\begin{aligned} |nQ(z)| &= |1 + B_{\omega,\Omega}\phi(z)||nQ(z) - zQ'(z)| \\ &= |1 + B_{\omega,\Omega}\phi(z)||P'(z)|. \end{aligned}$$

Since $|P(z)| = |Q(z)|$ on $|z| = 1$, we have from the proceeding inequality

$$n|P(z)| = |1 + B_{\omega,\Omega}\phi(z)||P'(z)| \quad \text{on} \quad |z| = 1, \tag{3.21}$$

that is,

$$n\left|\frac{P(z)}{P'(z)}\right| = |1+B_{\omega,\Omega}\phi(z)| \quad \text{on} \quad |z|=1.$$

Then for each $r>0$ and $0\le\theta<2\pi$, we have

$$n^r\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^r d\theta = \int_0^{2\pi}\left|1+B_{\omega,\Omega}\phi(e^{i\theta})\right|^r d\theta,$$

which on using (3.20) gives

$$n^r\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^r d\theta \le \int_0^{2\pi}\left|1+B_{\omega,\Omega}e^{i\theta}\right|^r d\theta,$$

which implies

$$n\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{p(e^{i\theta})+m\alpha}{p'(e^{i\theta})}\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+B_{\omega,\Omega}e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}}$$

and this completes the proof. □

Since $|p'(e^{i\theta})|\le\|p'\|_\infty$ for $0\le\theta<2\pi$, the following corollary immediately follows.

Corollary 3.7. *If* $p(z)=\sum\limits_{j=0}^{n}a_jz^j$, $a_0\ne 0$ *is a polynomial of degree* $n\ge 3$ *having all its zeros in* $|z|\le k$, $k\le 1$, *then for every complex number* α *with* $|\alpha|<1$ *and for each* $r>0$,

$$\|p'\|_\infty \ge \frac{n}{\|1+B_{\omega,\Omega}z\|_r}\|p(z)+m\alpha\|_r, \tag{3.22}$$

where $B_{\omega,\Omega}$ *and* m *are as defined in Theorem* 3.6.

Remark 3.8. Inequality (3.22) is an improvement of (1.11) proved by Aziz [2]. It is sufficient to prove that $B_{\omega,\Omega}\le k$, where $B_{\omega,\Omega}$ is defined in Theorem 3.6 and $k\le 1$, which is equivalent to showing that

$$\frac{|\omega|(1-|\omega|)+(n-1)|\Omega-\omega^2|k+(1-|\omega|)k^2}{(1-|\omega|)+(n-1)|\Omega-\omega^2|k+|\omega|(1-|\omega|)k^2}\le 1,$$

that is,

$$|\omega|+k^2\le 1+|\omega|k^2.$$

Therefore, we have $k^2\le 1$, which is true since $1-|\omega|\ge 0$ (by Lemma 2.7) and $k\le 1$.

Remark 3.9. Inequality (3.15) is an improvement of (1.13) proved by Aziz and Rather [3]. For this, it is enough to show that $B_{\omega,\Omega} \leq t_{k,1}$, where $B_{\omega,\Omega}$ and $t_{k,1}$ are defined in Theorem 3.6 and inequality (1.13) respectively.

Since $t_{k,1} = \frac{n|a_n|k^2+|a_{n-1}|}{n|a_n|+|a_{n-1}|} = \frac{k+|\omega|}{k+|\omega|k^2}$, where ω is as defined Theorem 3.6, it is sufficient to show that

$$k\frac{|\omega|(1-|\omega|)+(n-1)|\mu-\omega^2|k+(1-|\omega|)k^2}{(1-|\omega|)+(n-1)|\mu-\omega^2|k+|\omega|(1-|\omega|)k^2} \leq \frac{k+|\omega|}{k+|\omega|k^2},$$

which implies

$$(k-1)[(1-|\omega|)\{k^3|\omega|(k+1)+k^2|\omega|^2+k^3+|\omega|(k+1)\}$$
$$+k(n-1)|\mu-\omega^2|\{k+|\omega|(k^2+k+1)\}] \leq 0,$$

from which we eventually obtain

$$(1-|\omega|)\{k^4|\omega|+k^3(|\omega|+1)+k^2|\omega|^2+k|\omega|+|\omega|\}$$
$$+k(n-1)|\mu-\omega^2|\{k^2|\omega|+k(|\omega|+1)+|\omega|\} \leq 0,$$

which obviously holds due to the fact that $|\omega| \leq 1$ (by Lemma 2.7).

Remark 3.10. Due to the fact that $B_{\omega,\Omega} \leq t_{k,1}$, inequality (3.22) of Corollary 3.7 is a refinement of inequality (1.14) due to Aziz and Rather [3].

Remark 3.11. Corollary 3.7 is a generalization of Theorem 1.7. We know by definition

$$\frac{n}{\|1+B_{\omega,\Omega}z\|_\infty} = n\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+B_{\omega,\Omega}e^{i\theta}\right|^r d\theta\right\}^{-\frac{1}{r}} = \frac{n}{1+B_{\omega,\Omega}},$$

its R.H.S. further simplifies to

$$\frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+k(n-1)|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+k(n-1)|\Omega-\omega^2|}.$$

Thus, letting $r\to\infty$, (3.22) reduces to

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+k(n-1)|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+k(n-1)|\Omega-\omega^2|} \times \max_{|z|=1}|p(z)+m\alpha|. \tag{3.23}$$

Suppose z_0 on $|z|=1$ be such that $\max_{|z|=1}|p(z)| = |p(z_0)|$. Then, in particular

$$\max_{|z|=1}|p(z)+m\alpha| \geq |p(z_0)+m\alpha|. \tag{3.24}$$

Now we can choose the argument of α suitably such that

$$|p(z_0)+m\alpha| = |p(z_0)|+|\alpha|m. \tag{3.25}$$

Using (3.25) in (3.24), we have

$$\max_{|z|=1}|p(z)+m\alpha| \geq |p(z_0)|+|\alpha|m. \tag{3.26}$$

On combining (3.23) and (3.26), and taking the limit as $|\alpha|\to 1$ gives inequality (1.19) of Theorem 1.7.

Remark 3.12. Inequality (3.22) in its ordinary form obtained as in Remark 3.11 improves inequality (1.16) of Theorem 1.2 due to Govil et al. [13] by involving $\min\limits_{|z|=k}|p(z)|$. For this it is sufficient to show that

$$\frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+k(n-1)|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+k(n-1)|\Omega-\omega^2|} \geq 0,$$

which is equivalent to showing

$$1-|\omega| \geq 0,$$

which is satisfied by Lemma 2.7.

Finally, we use Holder's inequality [14] to establish the following result which is a generalization of Theorem 3.6 and is an improvement of Theorem 1.5 as well by involving $\min\limits_{|z|=k}|p(z)|$.

Theorem 3.13. *If $p(z)=\sum\limits_{j=0}^{n}a_jz^j$, $a_0\neq 0$ is a polynomial of degree $n\geq 3$ having all its zeros in $|z|\leq k$, $k\leq 1$, then for every complex number α with $|\alpha|<1$, for $s>1$, $t>1$ with $s^{-1}+t^{-1}=1$ and for each $r>0$,*

$$\|p'\|_{tr} \geq \frac{n}{\|1+B_{\omega,\Omega}z\|_{sr}}\|p(z)+m\alpha\|_r,$$

where $B_{\omega,\Omega}$ and m are as defined in Theorem 3.6.

Proof. Proceeding similarly as in the proof of Theorem 3.6, we have from (3.21)

$$n|P(z)| = |1+B_{\omega,\Omega}\phi(z)||P'(z)|.$$

Then for each $r>0$ and $0\leq\theta<2\pi$, we have

$$n^r\int_0^{2\pi}|P(e^{i\theta})|^r d\theta = \int_0^{2\pi}\left|1+B_{\omega,\Omega}\phi(e^{i\theta})\right|^r|P'(e^{i\theta})|^r d\theta.$$

Applying Holder's inequality [14] to the above inequality, we have for $s > 1$, $t > 1$ with $s^{-1} + t^{-1} = 1$ and for each $r > 0$

$$n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq \left\{ \int_0^{2\pi} \left|1 + B_{\omega,\Omega}\phi(e^{i\theta})\right|^{sr} d\theta \right\}^{\frac{1}{s}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{tr} d\theta \right\}^{\frac{1}{t}},$$

which implies

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} \left|1 + B_{\omega,\Omega}\phi(e^{i\theta})\right|^{sr} d\theta \right\}^{\frac{1}{sr}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{tr} d\theta \right\}^{\frac{1}{tr}}.$$

Using (3.20) in the above inequality, we have

$$n \left\{ \int_0^{2\pi} |p(e^{i\theta}) + m\alpha|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} \left|1 + B_{\omega,\Omega} e^{i\theta}\right|^{sr} d\theta \right\}^{\frac{1}{sr}} \left\{ \int_0^{2\pi} |p'(e^{i\theta})|^{tr} d\theta \right\}^{\frac{1}{tr}},$$

which is the desired conclusion of the theorem. $\square$

Remark 3.14. Letting $t \to \infty$ (so that $s \to 1$) in Theorem 3.13, we obtain Corollary 3.7. Thus, taking the limit as $t \to \infty$ (so that $s \to 1$) or $s \to \infty$ (so that $t \to 1$) and then letting $r \to \infty$, in view of Remark 3.10, Theorem 3.13 reduces to (1.19) of Theorem 1.7.

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