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ANALYSIS OF HILFER FRACTIONAL VOLTERRA-FREDHOLM SYSTEM

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Abstract. In this manuscript, we study the sufficient conditions for existence and uniqueness results of solutions of impulsive Hilfer fractional Volterra-Fredholm integro-differential equations with integral boundary conditions. Fractional calculus and Banach contraction theorem used to prove the uniqueness of results. Moreover, we also establish Hyers-Ulam stability for this problem. An example is also presented at the end.

1. Introduction

Fractional integro-differential equation (FIDE) has recently given a natural foundation for mathematical modeling of many real-world events, particularly in the control, biological, and medical domains [5, 17, 20]. Theoretical and practical foundations have been built for the study of such situations. Many disciplines of physics and technological sciences use FIDEs and control issues. FIDEs [8, 9, 10, 12] are measured as an alternative model to nonlinear differential equations.

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Te concept of derivatives and integrals of any arbitrary real or complex order is called fractional calculus (FC), and initially, it was proposed in works by mathematicians such as L'Hopital, Liouville, Leibniz, Riemann, and Abel. The non-locality of fractional derivatives, a characteristic inherent to many complex systems, makes them important for modeling phenomena in numerous disciplines of engineering and science. A significant study had already been conducted in this area [17, 18, 19, 23, 27, 28]. Fractional derivatives provide more realistic representations of real-world behavior than ordinary derivatives when dealing with some phenomena because they consider the global evolution of the system rather than just local dynamics.

In [26], Podlubny has examined the methods and applications of fractional derivatives and fractional differential equations (FDEs). Many authors have investigated the applications of FDEs [1, 2, 3, 4, 11, 22, 24]. There has recently been a lot of attention drawn to the quadratic perturbation of nonlinear differential equations (DEs) also known as hybrid DEs. Due to the inclusion of several dynamic systems as special instances, research on hybrid differential equations is significant. Hilal and Kajouni [14] have discussed boundary value problems (BVPs) for hybrid fractional differential equations (HFDEs).

In [25], the authors have studied on the experimental applications of hybrid functions to FDEs. There are numerous applications of BVPs within the field of applied mathematics. For instance, concentration in chemical or biological issues and nonlinear sources produce nonlinear diffusion and the theory of thermal ignition of gases. In addition, BVPs with integral boundary conditions have numerous contributions of mathematical modeling to the heat conduction process, chemic conduction process, and hydrodynamics issues. Many authors have investigated FDEs with boundary conditions [7, 15, 16, 21, 30].

Vu et al. in [29] for the RFDE with impulses proved existence and uniqueness (EU) of the solution by using Banach and Schauder fixed point theorem. In [13] Harikrishnan et al. investigated stability and dynamical behavior of RFDEs involving Ψ-Hilfer FD. Dong et al. [6] showed the EU and Ulam stability for the following random fractional integro-differential equation (RFIDE) by using mean square sense Caputo FD

$$\mathcal{D}_0^{\alpha}\mathbb{X}(t) = \mathbb{F}(t, \mathbb{X}(t)) + \int_0^t \mathbb{G}(t, r, \mathbb{X}(r))dr, \quad t, r \in \mathbb{J}.$$

Inspired by the aforementioned works, we studied the BVP for impulsive Hilfer fractional Volterra-Fredholm IDEs:

$$\begin{aligned} &\mathcal{D}_{0^+}^{p,q}\eta(u) = \mathbb{F}(u, \eta(u)) + \int_0^u \mathbb{G}(u, r, \eta(r))dr + \int_0^T \mathbb{H}(u, r, \eta(r))dr, \\ &\qquad u, r \in \mathcal{J} : [0, T], \\ &\mathcal{J}^{1-v}\eta(u, \omega)\big|_{u=0} = \mu, \ v = p + q - pq, \end{aligned} \tag{1.1}$$

where $\mathcal{D}_{0^+}^{p,q}$ is Hilfer FD of order $p \in (0,1)$ and type $q \in [0,1], \mathfrak{J}^{1-v}$ is the $\mathcal{R}-\mathcal{L}$ fractional integral of order $1-v$. In addition, $\mu : J \to L_2(\Omega)$ is the random variable with $E\left(\mu^2\right) < \infty$.

Let $\mathbb{F}, \mathbb{G}, \mathbb{H}$ be m.s. continuous functions such that $\mathbb{F} : \mathcal{J} \times L_2(\Omega) \to L_2(\Omega)$ and $\mathbb{G}, \mathbb{H} : \mathbb{J} \times L_2(\Omega) \to L_2(\Omega)$, where $\mathbb{J} = \{(u,r) \in \mathcal{J} \times \mathcal{J}$ such that $r \le u\}$.

2. Supporting notes

Here, we define some spaces, definitions, which will be used throughout this paper. These definitions and results are taken from [14, 26]. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probabilistic space. Let $\eta(u,\omega) := \{\eta(u), u \in \mathcal{J} = [0,T]$ and $\omega \in \Omega\}$, $T > 0$ be a second-order stochastic process, that is, $E\left(\eta^2(u)\right) < \infty$ and $L_2(\Omega)$ represents the Banach space of random variables $\eta : \Omega \to \mathbb{R}$. Consider the following model:

Let $\mathcal{C}\left(\mathfrak{d}, L_2(\Omega)\right)$ be the Banach space of all continuous functions from $\mathcal{J} \times \Omega$ into $\mathbb{R}$ with norm

$$\|\eta\|_{\mathcal{C}} = \max_{u \in \mathcal{I}} \|\eta(u)\|_2, \text{ where } \|\eta(u)\|_2 = \left(E\left(\eta^2(u)\right)\right)^{\frac{1}{2}}.$$

Definition 2.1. ([26, 31]) The stochastic fractional integral of order $p > 0$ is defined as

$$\mathcal{J}^p \eta(s) = \frac{1}{\Gamma(p)} \int_0^s (s-\tau)^{p-1} \eta(\tau) d\tau.$$

Definition 2.2. ([26]) Let $p \in (0,1), q \in [0,1]$ and $\omega \in \Omega$. The Hilfer fractional derivative of order p and type q is defined as

$$\left(\mathcal{D}_0^{p,q} \eta\right)(s) = \left(\mathcal{J}_0^{q(1-p)} \frac{d}{dt} \mathfrak{J}_0^{(1-p)(1-q)} \eta\right)(s).$$

Let $p \in (0,1)$, $q \in [0,1]$ and $v = p + q - pq$.

(1) The operator $\left(\mathcal{D}_0^{p,q}\eta\right)(s)$ can be written as;

$$\left(\mathcal{D}_0^{p,q}\eta\right)(s) = \left(\mathcal{J}_0^{q(1-p)} \frac{d}{dt} \mathfrak{J}_0^{(1-v)} \eta\right)(s) = \left(\mathcal{J}_0^{q(1-p)} \mathcal{D}_0^{v} \eta\right)(s). \tag{2.1}$$

(2) The generalization of (2.1) for $q = 0$, coincides with the $\mathcal{R}-\mathcal{L}$ derivative $\mathcal{D}_0^{p,0} = \mathcal{D}_0^p$ and $q = 1$ with Caputo fractional derivative $\mathcal{D}_0^{p,1} = {}^c\mathcal{D}_0^p$.

(3) If $\left(\mathcal{D}^{q(1-p)}\eta\right)$ exists, then

$$\left(\mathcal{D}_0^{p,q} \mathcal{J}_0^p \eta\right)(s) = \left(\mathcal{J}_0^{q(1-p)} \mathcal{D}_0^{q(1-p)} \eta\right)(s).$$

(4) If $(\mathcal{D}^v\eta)$ exists, then

$$\left(\mathcal{J}_0^p\mathcal{D}_0^{p,q}\eta\right)(s) = \left(\mathcal{J}_0^v\mathcal{D}_0^v\eta\right)(s) = \eta(s) - \frac{\mathcal{J}^{1-v}\left(0^+\right)}{\Gamma(v)}s^{v-1}.$$

3. Uniqueness and existence results

In this section, the uniqueness of the solutions to Eq. (1.1) is presented. In the sequel, we need the following hypotheses:

(H_1) There exist positive constants $\mathcal{A}$, $\mathcal{B}$ and B such that

$$\|\mathbb{F}(u,\eta) - \mathbb{F}(u,\theta)\|_2 \le \mathcal{A}\|\eta-\theta\|_2,$$

$$\|\mathbb{G}(u,r,\eta) - \mathbb{G}(u,r,\theta)\|_2 \le \mathcal{B}\|\eta-\theta\|_2$$

and

$$\|\mathbb{H}(u,r,\eta) - \mathbb{H}(u,r,\theta)\|_2 \le B\|\eta-\theta\|_2.$$

(H_2) There exists positive constant $\mathcal{D}$ such that

$$\max\left\{\|\mathbb{F}(u,0)\|_2, \|\mathbb{G}(u,r,0)\|_2, \|\mathbb{H}(u,r,0)\|_2\right\} \le \mathcal{D}.$$

(H_3) For functions $\mathbb{F}$, $\mathbb{G}$ and $\mathbb{H}$, we have that

$$\|\mathbb{F}(u,\eta)\|_2 \le \sup\left\{\mathrm{f}(u)\phi_1\left(\|\eta\|_e\right)\right\},$$

$$\|\mathbb{G}(u,s,\eta)\|_2 \le \sup\left\{\mathrm{g}(u,s)\phi_2\left(\|\eta\|_e\right)\right\}$$

and

$$\|\mathbb{H}(u,s,\eta)\|_2 \le \sup\left\{\mathrm{h}(u,s)\phi_3\left(\|\eta\|_e\right)\right\},$$

where f, g, h, ϕ_1, ϕ_2 and ϕ_3 lies in $\mathcal{C}_{\mathfrak{c}}\left(\mathfrak{J}, L_2(\Omega)\right)$ are non-decreasing on $\mathcal{J}$.

(H_4) There exist positive constants $\mathcal{A}$, $\mathcal{B}$ and B such that

$$\frac{\mathcal{A}T^p}{\Gamma(1+p)} + \frac{\mathcal{B}T^{p+1}}{(p+1)\Gamma(p)} + \frac{BT^{p+1}}{(p+1)\Gamma(p)} < 1.$$

Lemma 3.1. *A function $\eta(u)$ is the solution of (1.1) if and only if $\eta(u)$ satisfied the random integral equation:*

$$\begin{aligned}\eta(u) &= \frac{\mu}{\Gamma(v)}u^{v-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\mathbb{F}(s,\eta(s))ds \\ &+\frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \mathbb{G}(s,r,\eta(r))drds \\ &+\frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^T \mathbb{H}(s,r,\eta(r))drds. \qquad (3.1)\end{aligned}$$

Theorem 3.2. *Assume that hypothesis* (H_1), (H_2) *and* (H_4) *are satisfied. If*

$$\mathcal{A} \leq \frac{\Gamma(1+p)}{6T^p}, \;\; \mathcal{B} \leq \frac{(p+1)\Gamma(p)}{6T^{p+1}} \;\; \textit{and} \;\; B \leq \frac{(p+1)\Gamma(p)}{6T^{p+1}}, \tag{3.2}$$

then (1.1) *has a unique solution.*

Proof. We divide the proof of this theorem in two steps.
Step 1. Define the operator $\mathcal{Q} : \mathcal{C}(\mathfrak{d}, L_2(\Omega)) \to \mathfrak{C}(\mathcal{J}, L_2(\Omega))$. Hence, $\eta(u)$ is the solution of (1.1), where the equivalent integral equation (3.1) can be written in the operator form

$$\begin{aligned}(Q\eta)(u) = \;& \tfrac{\mu}{\Gamma(v)} u^{v-1} + \tfrac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \mathbb{F}(s, \eta(s)) ds \\ & + \tfrac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \mathbb{G}(s, r, \eta(r)) dr ds \\ & + \tfrac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \mathbb{H}(s, r, \eta(r)) dr ds. \end{aligned} \tag{3.3}$$

Set $\mathcal{B}_\alpha = \{\eta \in L_2(\Omega) : \|\eta\|_e \leq \alpha\}$. Now, we will prove that $Q\mathcal{B}_\alpha \subset \mathcal{B}_\alpha$ for any $\eta \in \mathcal{C}(\mathfrak{J}, L_2(\Omega))$. We have that

$$\begin{aligned} &\Big\|(Q\eta)(u)\Big\|_2 \\ &\leq \Big\| \frac{\mu}{\Gamma(v)} u^{v-1} + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \mathbb{F}(s, \eta(s)) ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \mathbb{G}(s, r, \eta(r)) dr ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \mathbb{H}(s, r, \eta(r)) dr ds \Big\|_2 \\ &\leq \frac{\|\mu\|_2}{\Gamma(v)} u^{v-1} + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \left(\|\mathbb{F}(s, \eta(s)) - \mathbb{F}(s, 0)\|_2 + \|\mathbb{F}(s, 0)\|_2 \right) ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \|\mathbb{G}(s, r, \eta(r)) - \mathbb{G}(s, r, 0)\|_2 + \|\mathbb{G}(s, r, 0)\|_2 dr ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \|\mathbb{H}(s, r, \eta(r)) - \mathbb{H}(s, r, 0)\|_2 + \|\mathbb{H}(s, r, 0)\|_2 dr ds \\ &\leq \frac{\|\mu\|_2}{\Gamma(v)} u^{v-1} + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \left(\mathcal{A} \|\eta\|_2\right) + \|\mathbb{F}(s, 0)\|_2 ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \left(\int_0^s (\mathcal{B}\|\eta\|_2) + \|\mathbb{G}(s, r, 0)\|_2 dr \right) ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \left(\int_0^T (B\|\eta\|_2) + \|\mathbb{H}(s, r, 0)\|_2 dr \right) ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|\mu\|_2}{\Gamma(v)} u^{v-1} + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \left(\mathcal{A}\|\eta\|_2 + \mathcal{D}\right) ds \\
&\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \left(\mathcal{B}\|\eta\|_2 + \mathcal{D}\right) drds \\
&\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \left(B\|\eta\|_2 + \mathcal{D}\right) drds \\
&\leq \frac{\|\mu\|_2}{\Gamma(v)} u^{v-1} + \frac{(\mathcal{A}\alpha + \mathcal{D})u^p}{(1+p)\Gamma(p)} + \frac{(\mathcal{B}\alpha + \mathcal{D})u^{p+1}}{(1+p)\Gamma(p)} + \frac{(B\alpha + \mathcal{D})T^{p+1}}{(1+p)\Gamma(p)}.
\end{aligned}
$$

From the estimation, we have

$$
\left\|(Q\eta)(u)\right\|_2 \leq \frac{\|\mu\|_e}{\Gamma(v)} u^{v-1} + \frac{(\mathcal{A}\alpha + \mathcal{D})T^p}{(1+p)\Gamma(p)} + \frac{(\mathcal{B}\alpha + \mathcal{D})T^{p+1}}{(1+p)\Gamma(p)} + \frac{(B\alpha + \mathcal{D})T^{p+1}}{(1+p)\Gamma(p)} = \alpha.
$$

This proves that Q transform the ball

$$
\mathcal{B}_\alpha = \{\eta \in \mathcal{C}\left(\mathcal{J}, L_2(\Omega)\right) : \|\eta\|_{\mathrm{e}}\left(\mathcal{J}, L_2(\Omega)\right) \leq \alpha\}
$$

into itself, that is $Q\left(\mathcal{B}_\alpha\right) \subset \mathcal{B}_\alpha$.

Step 2. In this step, we are going to show that Q is contractive. For any $\eta, \theta \in \mathcal{B}_\alpha$, we get

$$
\begin{aligned}
&\|(Q\eta)(u) - (Q\theta)(u)\|_2 \\
&= \Big\| \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \mathbb{F}(s, \eta(s)) ds + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \mathbb{G}(s, r, \eta(r)) drds \\
&\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \mathbb{H}(s, r, \eta(r)) drds \\
&\quad - \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \mathbb{F}(s, \theta(s)) ds - \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \mathbb{G}(s, r, \theta(r)) drds \\
&\quad - \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \mathbb{H}(s, r, \theta(r)) drds \Big\|_2 \\
&\leq \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \|\mathbb{F}(s, \eta(s)) - \mathbb{F}(s, \theta(s))\|_2 ds \\
&\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \|\mathbb{G}(s, r, \eta(r)) - \mathbb{G}(s, r, \theta(r))\|_2 drds \\
&\quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \|\mathbb{H}(s, r, \eta(r)) - \mathbb{H}(s, r, \theta(r))\|_2 drds
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{\mathcal{A}u^p}{\Gamma(1+p)}\|\eta-\theta\|_{\mathrm{e}} + \frac{\mathcal{B}}{\Gamma(p)}\left(\frac{u^{p+1}}{p} - \frac{u^{p+1}}{p+1}\right)\|\eta-\theta\|_{\mathrm{e}} \\
&\quad + \frac{B}{\Gamma(p)}\left(\frac{T^{p+1}}{p} - \frac{T^{p+1}}{p+1}\right)\|\eta-\theta\|_{\mathrm{e}} \\
&\leq \left(\frac{\mathcal{A}T^p}{\Gamma(1+p)} + \frac{\mathcal{B}T^{p+1}}{(p+1)\Gamma(p)} + \frac{BT^{p+1}}{(p+1)\Gamma(p)}\right)\|\eta-\theta\|_{\mathrm{e}}.
\end{aligned}$$

Therefore, by assumption (H4), we imply that $\frac{\mathcal{A}T^p}{\Gamma(1+p)} + \frac{\mathcal{B}T^{p+1}}{(p+1)\Gamma(p)} + \frac{BT^{p+1}}{(p+1)\Gamma(p)} < 1$. Therefore, we deduced that Q is a contraction. Finally, applying Banach contraction theorem we deduce that there exists a unique solution of the problem (1.1). The proof is completed. □

Theorem 3.3. *Assume that the hypothesis* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* (H_3) *are satisfied. Then the problem* (1.1) *has at least one solution.*

Proof. By hypothesis (H_3), the functions $\mathbb{F}$, $\mathbb{G}$ and $\mathbb{H}$ are continuous. So, we can find constants L_1, L_2 and L_3 such that

$$\|\mathbb{F}(u,\eta)\|_2 \leq \sup\left\{\mathrm{f}(u)\phi_1\left(\|\eta\|_e\right)\right\} := L_1,$$

$$\|\mathbb{G}(u,s,\eta)\|_2 \leq \sup\left\{\mathrm{g}(u,s)\phi_2\left(\|\eta\|_e\right)\right\} := L_2$$

and

$$\|\mathbb{H}(u,s,\eta)\|_2 \leq \sup\left\{\mathrm{h}(u,s)\phi_2\left(\|\eta\|_e\right)\right\} := L_3.$$

Consider the operator, $\mathcal{P} : \mathcal{B}_\beta \to \mathcal{B}_\beta$ given by

$$\begin{aligned}
(\mathcal{P}\eta)(u) = {}& \frac{\mu}{\Gamma(v)}u^{v-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1} \\
&\times \left(\mathbb{F}(s,\eta(s))ds + \int_0^s \mathbb{G}(s,r,\eta(r))dr + \int_0^T \mathbb{H}(s,r,\eta(r))dr\right)ds,
\end{aligned}$$

where

$$\mathcal{B}_\beta := \left\{\eta \in \mathcal{C}\left(\mathcal{J}, L_2(\Omega)\right) : \|\eta-\mu\|_2 \leq \beta\right\}$$

such that

$$\beta \geq \frac{L_1u^p}{\Gamma(1+p)} + \frac{L_2u^{p+1}}{(1+p)\Gamma(p)} + \frac{L_3T^{p+1}}{(1+p)\Gamma(p)}.$$

Firstly, we see that the operator $\mathcal{P}$ maps into itself. For this we take any $u \in [0,T]$ and $\eta \in \mathcal{B}_\beta$, we get

$$
\begin{aligned}
\|(\mathcal{P}\eta)(u)\|_2 \leq & \frac{\|\mu\|_2}{\Gamma(v)} u^{v-1} + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \|\mathbb{F}(s,\eta(s))\|_2 ds \\
& + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \|\mathbb{G}(s,r,\eta(r))\|_2 dr ds \\
& + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \|\mathbb{H}(s,r,\eta(r))\|_2 dr ds \\
\leq & \frac{\|\mu\|_2}{\Gamma(v)} u^{v-1} + \frac{L_1 u^p}{\Gamma(1+p)} + \frac{L_2 u^{p+1}}{(1+p)\Gamma(p)} + \frac{L_3 T^{p+1}}{(1+p)\Gamma(p)} \\
\leq & \|\mu\|_2 + \frac{L_1 u^p}{\Gamma(1+p)} + \frac{L_2 u^{p+1}}{(1+p)\Gamma(p)} + \frac{L_3 T^{p+1}}{(1+p)\Gamma(p)}.
\end{aligned}
$$

Thus, we have

$$
\|(\mathcal{P}\eta)(u) - \mu\|_2 \leq \frac{L_1 u^p}{\Gamma(1+p)} + \frac{L_2 u^{p+1}}{(1+p)\Gamma(p)} + \frac{L_3 T^{p+1}}{(1+p)\Gamma(p)} \leq \beta.
$$

That is, $\mathcal{P}(\mathcal{B}_\beta)$ is uniformly bounded. This proves that $\mathcal{P}(\mathcal{B}_\beta) \subset \mathcal{B}_\beta$.

Now, we shall show that the operator $\mathcal{P}$ satisfies all the conditions of Schauder's theorem.

Step 1: $\mathcal{P}$ is continuous. Let η_n be a sequence such that $\eta_n \to \eta$ in $\mathcal{B}_\beta$.

$$
\begin{aligned}
& \|((\mathcal{P}\eta_n)(u) - (\mathcal{P}\eta)(u))\|_2 \\
& \leq \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \|\mathbb{F}(s,\eta_n(s)) - \mathbb{F}(s,\eta(s))\|_2 ds \\
& \quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s \|\mathbb{G}(s,r,\eta_n(r)) - \mathbb{G}(s,r,\eta(r))\|_2 dr ds \\
& \quad + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^T \|\mathbb{H}(s,r,\eta_n(r)) - \mathbb{H}(s,r,\eta(r))\|_2 dr ds.
\end{aligned}
$$

Since, $\mathbb{F}, \mathbb{G}$ and $\mathbb{H}$ are continuous functions, by the the Lebesgue dominated convergence theorem, we get

$$
\|(\mathcal{P}\eta_n)(u) - (\mathcal{P}\eta)(u)\| \to 0 \quad \text{as} \quad n \to \infty.
$$

Hence, $\mathcal{P}$ is continuous in $\mathcal{B}_\beta$.

Step 2: $\mathcal{P}(\mathcal{B}_\beta)$ is uniformly bounded. This is clear since $\mathcal{P}(\mathcal{B}_\beta) \subset \mathcal{B}_\beta$ is bounded.

Step 3: For any $u_1, u_2 \in [0,T]$, $u_1 < u_2$, we have

$$\begin{aligned}
&\|(\mathcal{P}\eta)(u_2) - (\mathcal{P}\eta)(u_1)\|_2 \\
&= \Big\| \frac{1}{\Gamma(p)} \int_0^{u_2} (u-s)^{p-1} \Big(\mathbb{F}(s,\eta(s)) + \int_0^s \mathbb{G}(s,r,\eta(r))dr \\
&\quad + \int_0^T \mathbb{H}(s,r,\eta(r))dr \Big) ds - \Big(\frac{1}{\Gamma(p)} \int_0^{u_1} (u-s)^{p-1} \Big(\mathbb{F}(s,\eta(s)) \\
&\quad + \int_0^s \mathbb{G}(s,r,\eta(r))drds + \int_0^T \mathbb{H}(s,r,\eta(r))dr \Big) ds \Big) \Big\|_2 \\
&\leq \frac{L_1\left(u_2^p - u_1^p\right)}{\Gamma(1+p)} + \frac{L_2\left({u_2}^{p+1} - u_1^{p+1}\right)}{(1+p)\Gamma(p)} + \frac{L_3\left({u_2}^{p+1} - u_1^{p+1}\right)}{(1+p)\Gamma(p)} \\
&\to 0 \quad \text{as} \quad u_2 \to u_1.
\end{aligned}$$

That is,

$$\|(\mathcal{P}\eta)(u_2) - (\mathcal{P}\eta)(u_1)\| \to 0.$$

This means that $(\mathcal{P}\eta)(u)$ is equi-continuous on $[0,U]$ and completely continuous. So, by Schauder's theorem together with the steps $1-3$ we obtain that the operator $\mathcal{P}$ has at least one fixed point in $\mathcal{B}_\beta$. This completes the proof. □

4. Stability results

In this section of the manuscript, we will present the stability results for the problem (1.1).

Definition 4.1. ([31]) A system is $\mathbb{UH}$ stable, if there exists a real number $\mathbb{C}_1$ such that for each ϵ and for each solution $\eta \in \mathfrak{C}(\mathcal{J}, L_2(\Omega))$ of the following inequality:

$$\left\| \mathcal{D}_0^{p,q}\eta(u) - \mathbb{F}(u,\eta(u)) - \int_0^u \mathbb{G}(u,r,\eta(r))dr - \int_0^T \mathbb{H}(u,r,\eta(r))dr \right\|_2 \leq \epsilon, \quad (4.1)$$

there exists a solution $\theta \in \mathfrak{C}(\mathcal{J}, L_2(\Omega))$ with $\|\eta - \theta\|_2 \leq \mathbb{C}_1\epsilon$.

Definition 4.2. ([31]) The problem (1.1) is $\mathbb{UHR}$ stable, if there exists a real number $\mathbb{C}_2$ such that for each ϵ and for each solution $\eta \in \mathcal{C}(\mathcal{J}, L_2(\Omega))$ of the following inequality:

$$\left\| \mathcal{D}_0^{p,q}\eta(u) - \mathbb{F}(u,\eta(u)) - \int_0^u \mathbb{G}(u,r,\eta(r))dr - \int_0^T \mathbb{H}(u,r,\eta(r))dr \right\|_2 \leq \epsilon\psi(u), \quad (4.2)$$

there exists a solution $\theta \in \mathcal{C}(\mathcal{J}, L_2(\Omega))$ with $\|\eta - \theta\|_2 \leq \mathbb{C}_2\epsilon\psi(u)$.

Theorem 4.3. *Under the hypotheses* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* (H_3) *we have that* $\mathbb{RFIDE}$ (1.1) *is* $\mathbb{UH}$ *stable.*

Proof. Let $\eta(u)$ be a solution of the inequality

$$\left\| \mathcal{D}_0^{p,q}\eta(u) - \mathbb{F}(u,\eta(u)) - \int_0^u \mathbb{G}(u,r,\eta(r))dr - \int_0^T \mathbb{H}(u,r,\eta(r))dr \right\|_2 \leq \epsilon. \quad (4.3)$$

Let $\theta(u)$ be the solution of the following equation

$$\mathcal{D}_0^{p,q}\theta(u) = \mathbb{F}(u,\theta(u)) + \int_0^u \mathbb{G}(u,r,\theta(r))dr + \int_0^T \mathbb{H}(u,r,\theta(r))dr \quad (4.4)$$

and

$$\mathfrak{J}_0^{1-v}\eta(u)\big|_{u=0} = \mu \text{ and } \mathfrak{J}_0^{1-v}\theta(u)\big|_{u=0} = \lambda.$$

Using Lemma 3.1 we obtain

$$\begin{aligned}\theta(u) =& \frac{\mu}{\Gamma(v)}(u)^{v-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\mathbb{F}(s,\theta(s))ds \\ &+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \mathbb{G}(s,r,\theta(r))drds \\ &+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^T \mathbb{H}(s,r,\theta(r))drds.\end{aligned}$$

By integration of (4.4), we have

$$\begin{aligned}&\left\| \mathcal{J}_0^p\mathcal{D}_0^{p,q}\eta(u) - \mathcal{J}_0^p\left(\mathbb{F}(u,\eta(u)) + \int_0^s \mathbb{G}(s,r,\eta(r))dr + \int_0^T \mathbb{H}(s,r,\eta(r))dr\right)\right\|_2 \\ &\leq \mathcal{J}_0^p\epsilon \\ &= \frac{\epsilon u^p}{\Gamma(1+p)}.\end{aligned}$$

On the other hand we obtain

$$\begin{aligned}&\|\eta(u)-\theta(u)\|_2 \\ &\quad\leq \left\| \eta(u) - \frac{\mu u^{1-v}}{\Gamma(v)} - \frac{1}{\Gamma(p)}\int_0^p (u-s)^{p-1} \right. \\ &\quad\quad \left. \times \left(\mathbb{F}(u,\theta(u)) + \int_0^s \mathbb{G}(s,r,\theta(r))dr + \int_0^T \mathbb{H}(s,r,\theta(r))dr\right) du \right\|_2\end{aligned}$$

$$
\begin{aligned}
&\leq \Big\| \eta(u) - \frac{\mu u^{1-v}}{\Gamma(v)} - \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \\
&\quad \times \left(\mathbb{F}(u,\eta(u)) + \int_0^s \mathbb{G}(u,r,\eta(r))dr + \int_0^T \mathbb{H}(u,r,\eta(r))dr \right) \\
&\quad + \frac{\mu u^{1-v}}{\Gamma(v)} + \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \\
&\quad \times \left(\mathbb{F}(u,\eta(u)) + \int_0^s \mathbb{G}(u,r,\eta(r))dr + \int_0^T \mathbb{H}(u,r,\eta(r))dr \right) \\
&\quad - \frac{\mu u^{1-v}}{\Gamma(v)} - \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \\
&\quad \times \left(\mathbb{F}(u,\theta(u)) + \int_0^s \mathbb{G}(u,r,\theta(r))dr + \int_0^T \mathbb{H}(u,r,\theta(r))dr \right) \Big\|_2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\eta(u)-\theta(u)\|_2 \leq & \frac{\epsilon u^p}{\Gamma(1+p)} + \frac{\mathcal{A}}{\Gamma(p)} \int_0^u (u-s)^{p-1} \|\eta(u)-\theta(u)\| ds \\
& + \int_0^u (u-s)^{p-1} \frac{\mathcal{A}}{\Gamma(p)} \int_0^u \frac{\mathcal{B}}{\mathcal{A}} \|\eta(r)-\theta(r)\| dr du \\
& + \int_0^u (u-s)^{p-1} \frac{\mathcal{A}}{\Gamma(p)} \int_0^T \frac{B}{\mathcal{A}} \|\eta(r)-\theta(r)\| dr du.
\end{aligned}
$$

Now, using Pachpatte's inequality (see [14]), we get

$$
\begin{aligned}
\|\eta(u)-\theta(u)\| &\leq \frac{\epsilon u^p}{\Gamma(1+p)} \Big[1 + \int_0^u \frac{\mathcal{A}}{\Gamma(p)} (u-s)^{p-1} \\
&\quad \times \left(\int_0^u \left(\frac{\mathcal{A}}{\Gamma(p)} (u-s)^{p-1} + \frac{\mathcal{B}}{\mathcal{A}} \right) ds + \int_0^T \frac{B}{\mathcal{A}} ds \right) du \Big] \\
&\leq \frac{u^p}{\Gamma(1+p)} \left[1 + \frac{\mathcal{A} u^p}{\Gamma^2(p)} \left(\frac{\mathcal{A} u^p}{\Gamma(1+p)} + \frac{\mathcal{B} u}{\mathcal{A}} + \frac{BT}{\mathcal{A}} \right) \right] \epsilon \\
&\leq \mathbb{C}_1 \epsilon.
\end{aligned}
$$

This provides that

$$
\mathbb{C}_1 = \frac{u^p}{\Gamma(1+p)} \left[1 + \frac{\mathcal{A} u^p}{\Gamma^2(p)} \left(\frac{\mathcal{A} u^p}{\Gamma(1+p)} + \frac{\mathcal{B} u}{\mathcal{A}} + \frac{BT}{\mathcal{A}} \right) \right].
$$

Finally, this implies that

$$
\|\eta(u)-\theta(u)\| \leq \mathbb{C}_1 \epsilon.
$$

Thus, we may conclude that (1.1) is $\mathbb{UH}$ stable. □

Theorem 4.4. *Under the hypotheses* $(\mathrm{H}_1), (\mathrm{H}_2)$*,* (H_3) *and there exists a constant* $\sigma_\psi > 0$ *such that*

$$\int_0^u \frac{(u-s)^{p-1}\psi(u)}{\Gamma(p)} du \leq \sigma_\psi \psi(u),$$

where ψ *is non decreasing. Then* (1.1) *is* $\mathbb{UHR}$ *stable.*

Proof. Let $\eta(u)$ be a solution for

$$\left\| \mathcal{D}_0^{p,q}\eta(u) - \mathbb{F}(u,\eta(u)) - \int_0^u \mathbb{G}(u,r,\eta(r))dr - \int_0^T \mathbb{H}(u,r,\eta(r))dr \right\|_2 \leq \epsilon\psi(u).$$

Let $\theta(u)$ be a unique solution of the equation

$$\mathcal{D}_0^{p,q}\theta(u) = \mathbb{F}(u,\theta(u)) + \int_0^u \mathbb{G}(u,r,\theta(r))dr + \int_0^T \mathbb{H}(u,r,\theta(r))dr$$

and

$$\mathfrak{J}_0^{1-v}\eta(u)\big|_{u=0} = \mu \text{ and } \mathfrak{J}_0^{1-v}\theta(u)\big|_{u=0} = \lambda.$$

So, by using Lemma 3.1, we obtain

$$\begin{aligned}
\theta(u) = \frac{\mu}{\Gamma(v)}(u)^{v-1} &+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\mathbb{F}(s,\theta(s))ds \\
&+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \mathbb{G}(s,r,\theta(r))drds \\
&+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^T \mathbb{H}(s,r,\theta(r))drds.
\end{aligned}$$

By integration of (4.3), we have

$$\begin{aligned}
&\left\| \mathcal{J}_0^p\mathcal{D}_0^{p,q}\eta(u) - \mathcal{J}_0^p\left(\mathbb{F}(u,\eta(u)) + \int_0^u \mathbb{G}(u,r,\eta(r))dr + \int_0^T \mathbb{H}(u,r,\eta(r))dr\right) \right\| \\
&\quad \leq \mathcal{J}_0^p\epsilon\psi(u) \\
&\quad \leq \epsilon\int_0^u \frac{(u-s)^{p-1}\psi(u)}{\Gamma(p)}du \\
&\quad \leq \epsilon\sigma_\psi\psi(u).
\end{aligned}$$

Following the same procedure used in Theorem 4.3, we have

$$\begin{aligned}
\|\eta(u)-\theta(u)\| \leq & \epsilon\sigma_\psi\psi(u) + \frac{\mathcal{A}}{\Gamma(p)}\int_0^u (u-s)^{p-1}\|\eta(u,\omega)-\theta(u,\omega)\|ds \\
&+ \int_0^u (u-s)^{p-1}\frac{\mathcal{A}}{\Gamma(p)}\int_0^u \frac{\mathcal{B}}{\mathcal{A}}\|\eta(r)-\theta(r)\|drdu \\
&+ \int_0^u (u-s)^{p-1}\frac{\mathcal{A}}{\Gamma(p)}\int_0^T \frac{B}{\mathcal{A}}\|\eta(r)-\theta(r)\|drdu.
\end{aligned}$$

Now, using Pachpatte's inequality (see [14]), we obtain

$$\begin{aligned}\|\eta(u)-\theta(u)\| &\leq \epsilon\sigma_\psi\psi(u)\left[1+\frac{\mathcal{A}u^p}{\Gamma^2(p)}\left(\frac{\mathcal{A}u^p}{\Gamma(1+p)}+\frac{\mathcal{B}u}{\mathcal{A}}+\frac{BT}{\mathcal{A}}\right)\right]\epsilon\\ &\leq \sigma_\psi\left[1+\frac{\mathcal{A}u^p}{\Gamma^2(p)}\left(\frac{\mathcal{A}u^p}{\Gamma(1+p)}+\frac{\mathcal{B}u}{\mathcal{A}}+\frac{BT}{\mathcal{A}}\right)\right]\epsilon\psi(u)\\ &\leq \mathbb{C}_2\epsilon\psi(u),\end{aligned}$$

where

$$\mathbb{C}_2=\sigma_\psi\left[1+\frac{\mathcal{A}u^p}{\Gamma^2(p)}\left(\frac{\mathcal{A}u^p}{\Gamma(1+p)}+\frac{\mathcal{B}u}{\mathcal{A}}+\frac{BT}{\mathcal{A}}\right)\right].$$

Therefore, the equation (1.1) is $\mathbb{UHR}$ stable. □

5. Example

Example 5.1. Consider the Hlifer fractional BVP:

$$\begin{aligned}\mathcal{D}_{0^+}^{\frac{2}{3},\frac{1}{2}}\eta(s) &= \frac{\tan\left(s^4\right)\nu^4e^{-25-s}}{\nu^4+\sin(s)+1}\eta(s)\\ &\quad+\int_0^s\frac{e^{\sin(s)}\left(1+\nu^2\right)}{1+e^{25}}\eta(r)dr+\int_0^1\frac{e^{\cos(s)}\left(s^2+\nu^2\right)}{1+e^{25}}\eta(r)dr,\\ \mathcal{J}^{1-\frac{7}{6}}\eta(s,\omega)\Big|_{s=0} &= \eta(\omega).\end{aligned}\tag{5.1}$$

From $\mathbb{RFDE}$ (5.1), we see that $p=\frac{2}{3}, q=\frac{1}{2}$ and $v=\frac{7}{6}$. Also, for $s\in[0,1]$ and $\eta\in L_2(\Omega)$, we can easily find $\mathcal{A}=\mathcal{B}=B=\mathcal{C}_1=\mathcal{C}_2=\frac{1}{e^{25}}$. From Theorem 3.2, we see that inequalities $\mathcal{A}\leq\Gamma\left(\frac{5}{3}\right)$, $\mathcal{B}\leq\frac{5}{3}\Gamma\left(\frac{2}{3}\right)$ and $B\leq\frac{5}{3}\Gamma\left(\frac{2}{3}\right)$ are satisfied. Hence (5.1) has a unique solution. Model (5.1) also satisfy the conditions of Theorem 3.3, so (5.1) has at least one solution.

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