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RESULTS ON THE HADAMARD-SIMPSON'S INEQUALITIES

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Abstract. It is well known that inequalities enable us to analyze and solve complex problems with precision and efficiency. The inequalities provide powerful tools for establishing bounds, optimizing solutions, and deepening our understanding of mathematical concepts, paving the way for advancements in areas such as optimization, analysis, and probability theory. In this paper, we present some properties for Hadamard-Simpsons type inequalities in the classic integral and Riemann-Liouville fractional integral. We use the convexity of the given function and its first derivative.

1. Introduction

The function $f : [a, b] \to \mathbb{R}$ is said to be convex if for each $x_1, x_2 \in [a, b]$

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2), \quad \lambda \in [0,1].$$

The Hermite-Hadamard-type inequality (HHIs) gives an estimation of mean value for a convex function f in a specific interval [2]. It states that on a convex function $f : [a, b] \to \mathbb{R}$, where $a < b$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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The inequalities of HHIs are proved in many ways ([2,6,11,13]). One of the ways is based on the error relation of trapezoidal and midpoint numerical integrations. If $f \in C^2([a,b])$, the trapezoidal integration rule follows [4]:

$$\int_a^b f(x)\,dx - \frac{b-a}{2}\left(f(a)+f(b)\right) = -\frac{(b-a)^3}{12} f''(\xi), \quad a<\xi<b.$$

If f is a convex function on (a,b), then $f'' > 0$ and so the right hand side inequality of HHIs is proved. Moreover, if $f \in C^2([a,b])$, the midpoint integration rule follows [4]:

$$\int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) = \frac{(b-a)^3}{24} f''(\xi), \quad a<\xi<b.$$

Again if f is a convex function on (a,b), then $f'' > 0$ and so the left hand side inequality of HHI is proved.

Some extension of the right hand side inequality of (1.1) can be given in a straightforward way. For example, the following inequality is a corollary of the Hermite-Hadamard-type inequality.

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\le \frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}. \end{aligned}$$

According to the Simpson's integration rule, one can find an interesting inequality. If $f \in C^4([a,b])$, the Simpson's integration rule follows [4]:

$$\begin{aligned} &\int_a^b f(x)\,dx - \frac{1}{6}(b-a)\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) \\ &= -\frac{(b-a)^5}{2880} f^{(4)}(\xi), \qquad a<\xi<b. \end{aligned}$$

Now, if f'' is a convex function on (a,b), then $f^{(4)} > 0$ and so the following inequality holds:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}.$$

If f and f'' are convex functions on (a,b), then

$$\begin{aligned} \left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\le \frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}, \end{aligned}$$

that are called Hadamard-Simpson type inequalities (HSIs).

An extension of HHI [9,15] and HSIs [5] is on the Riemann-Liouville (R-L) fractional integrals. If f is a function in $L_1([a,b])$ in which $0<a<b$, then the left and right R-L fractional integrals of f from order α denoted respectively by $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ are defined by [10]

$$J^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt, \qquad x\geq a$$

and

$$J^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt, \qquad x\leq b,$$

where

$$\Gamma(\alpha)=\int_0^{\infty} x^{\alpha-1}e^{-x}dx, \qquad \alpha>0$$

is the Gamma function.

Note that when $\alpha=1$, we have $\Gamma(1)=1$ and from there $J^{\alpha}_{a^+}f(x)$ and $J^{\alpha}_{b^-}f(x)$ will be classic integral.

There are many researches that state the properties of Hermite-Hadamard type inequalities for R-L fractional integrals [5,8,14]. For example, again consider $f:[a,b]\to\mathbb{R}$ in which $a\geq 0$ and also f is a positive and convex function in $L_1([a,b])$. Then, the following inequality for R-L fractional integrals of positive order α holds [1]:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

This inequality explicitly estimates the mean value of the R-L fractional integral on the three points a,b and $\frac{a+b}{2}$. Other properties of HHIs for fractional calculus can be found in [3,7,12].

In [11], some theorems have been given for upper bound of

$$\frac{b-a}{2}\left(f(a)+f(b)\right)-\int_a^b f(x)\,dx$$

and

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right].$$

In this paper, we find some sharp upper bounds for

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{1.2}$$

and

$$\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{3(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(\frac{a+b}{2}\right)\right.$$
$$\left.+2\ J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a)+2J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f\left(\frac{a+b}{2}\right)\right]. \tag{1.3}$$

2. Results on the Hadamard-Simpson's inequalities

In this section, we present some results on the upper bound of (1.2). These upper bounds are based on the convexity of the first derivative of the given function.

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differential mapping on (a,b) such that $f^{'}\in L^{1}([a,b])$. Then, the following equality holds:*

$$\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx$$
$$=\frac{b-a}{12}\left(\int_{0}^{1}(2-3t)\ f^{'}\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt\right.$$
$$\left.+\int_{0}^{1}(1-3t)\,f^{'}\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)dt\right). \tag{2.1}$$

Proof. By taking the change of variable $x=ta+(1-t)\left(\frac{a+b}{2}\right)$ on

$$\int_{0}^{1}(2-3t)\ f^{'}\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt$$

and $x=t\left(\frac{a+b}{2}\right)+(1-t)\,b$ on

$$\int_{0}^{1}(1-3t)\,f^{'}\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)dt$$

and using the integration by part, the equality (2.1) is derived. □

In the following, we present a theorem for the upper bound of (1.2) when f' is a convex function on (a,b).

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differential mapping on (a,b) such that f' is convex on (a,b). Then, the following inequality holds:*

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\frac{b-a}{12}\left(\frac{8}{27}\left|f^{'}(a)\right|+\frac{29}{27}\left|f^{'}\left(\frac{a+b}{2}\right)\right|+\frac{8}{27}\left|f^{'}(b)\right|\right).$$

Proof. By Lemma 2.1,

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{12}\left(\left|\int_0^1 (2-3t)\ f'\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt\right|\right.$$
$$\left.+\left|\int_0^1 (1-3t)\,f'\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)dt\right|\right). \tag{2.2}$$

Taking the convexity of f' into account, we have

$$\left|\int_0^1 (2-3t)\ f'\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt\right|$$
$$\leq \int_0^1 |2-3t|\ \left|f'\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)\right|dt$$
$$\leq \int_0^1 |2-3t|\ \left|tf'(a)+(1-t)\,f'\left(\frac{a+b}{2}\right)\right|dt$$
$$\leq \left|f'(a)\right|\int_0^1 t\,|2-3t|dt+\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1 (1-t)\,|2-3t|dt$$
$$=\frac{8}{27}\left|f'(a)\right|+\frac{29}{54}\left|f'\left(\frac{a+b}{2}\right)\right| \tag{2.3}$$

and similarly, we have

$$\left|\int_0^1 (1-3t)\,f'\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)dt\right|$$
$$\leq \int_0^1 |1-3t|\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\right|dt$$
$$\leq \int_0^1 |1-3t|\left|tf'\left(\frac{a+b}{2}\right)+(1-t)\,f'(b)\right|dt$$
$$=\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1 t\,|1-3t|\,dt+\left|f'(b)\right|\int_0^1 (1-t)\,|1-3t|\,dt$$
$$=\frac{29}{54}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{8}{27}\left|f'(b)\right|. \tag{2.4}$$

Now, substituting (2.3) and (2.4) into (2.2) establishes the proof. □

3. Hadamard-Simpson inequalities for fractional integrals

In this section, we generalize the results of Section 2 for R-L fractional integrals.

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differential mapping on (a,b) such that $f^{'}\in L^{1}([a,b])$. Then, the following equality holds:*

$$\begin{aligned}
&\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{3(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(\frac{a+b}{2}\right)\right.\\
&\quad\left.+2\,J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a)+2J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}f(b)+\,J_{b^{-}}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\\
&=\frac{b-a}{12}\left(\int_{0}^{1}\left[2(1-t)^{\alpha}-t^{\alpha}\right]\,f^{'}\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt\right.\\
&\quad\left.+\int_{0}^{1}\left[(1-t)^{\alpha}-2t^{\alpha}\right]\,f^{'}\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)dt\right).
\end{aligned}$$

Proof. Integrating by part implies that

$$\begin{aligned}
I_{1}&=\int_{0}^{1}(1-t)^{\alpha}\,f^{'}\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt\\
&=\left[(1-t)^{\alpha}\times\frac{2}{a-b}f\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)\right]\Bigg|_{t=0}^{t=1}\\
&\quad+\frac{2}{a-b}\int_{0}^{1}\alpha(1-t)^{\alpha-1}f\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt\\
&=\frac{2f\left(\frac{a+b}{2}\right)}{b-a}-\frac{2\alpha}{b-a}\int_{\frac{a+b}{2}}^{a}\left(\frac{2(a-x)}{a-b}\right)^{\alpha-1}\frac{2f(x)}{a-b}\,dx\\
&=\frac{2f\left(\frac{a+b}{2}\right)}{b-a}-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a).
\end{aligned}$$

In the computation of I_1, we have used the following change of variable

$$x=ta+(1-t)\left(\frac{a+b}{2}\right).$$

Similarly, using integration by part yields that

$$\begin{aligned}
I_2 &= \int_0^1 t^{\alpha}\ f^{'}\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt\\
&= \left.\frac{2t^{\alpha}f\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)}{a-b}\right|_{t=0}^{t=1} - \alpha\int_0^1 t^{\alpha-1}\frac{2f\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)}{a-b}dt\\
&= -\frac{2f(a)}{b-a}+\frac{2\alpha}{b-a}\int_{\frac{a+b}{2}}^{a}\left(\frac{a+b-2x}{b-a}\right)^{\alpha-1}\frac{2f(x)}{a-b}\ dx\\
&= -\left[\frac{2f(a)}{b-a}-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}J_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)\right].
\end{aligned}$$

By similar operation one can write

$$\begin{aligned}
I_3 &= \int_0^1 (1-t)^{\alpha}\ f^{'}\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)dt\\
&= \frac{2f(b)}{b-a}-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}J_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_0^1 t^{\alpha}\ f^{'}\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)dt\\
&= -\left[\frac{2f\left(\frac{a+b}{2}\right)}{b-a}-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)\right].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\frac{1}{12}\left(2I_1-I_2+I_3-2I_4\right)\\
&\quad= \frac{1}{12}\Bigg(\int_0^1\left[2(1-t)^{\alpha}-t^{\alpha}\right]\ f^{'}\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)dt\\
&\qquad+\int_0^1\left[(1-t)^{\alpha}-2t^{\alpha}\right]\ f^{'}\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)dt\Bigg). \qquad (3.1)
\end{aligned}$$

Now, considering (3.1) and the equalities of $I_1,\ I_2, I_3, I_4$ the Lemma 3.1 is proved. □

NOTE: In the particular case $\alpha = 1$, Lemma 3.1 gives the same Lemma 2.1.

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a differential mapping on (a, b) such that f' is a convex function on (a, b). Then, the following inequality holds for R-L fractional integrals:*

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{3(b-a)^{\alpha}}\right.$$
$$\left.\times\left[J_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{12}\left(P\left|f'(a)\right|+Q\left|f'\left(\frac{a+b}{2}\right)\right|+R\left|f'(b)\right|\right),$$

where

$$P=4\left[\frac{(1-t_*)^{\alpha+2}}{\alpha+2}-\frac{(1-t_*)^{\alpha+1}}{\alpha+1}\right]+\frac{2}{\alpha+1}-2\frac{t_*^{\alpha+2}}{\alpha+2}-\frac{1}{\alpha+2},$$

$$Q=2\frac{t_*^{\alpha+2}}{\alpha+2}-4\frac{(1-t_*)^{\alpha+2}}{\alpha+2}-2\frac{t_*^{\alpha+1}}{\alpha+1}-2\frac{(1-s_*)^{\alpha+2}}{\alpha+2}+4\frac{s_*^{\alpha+2}}{\alpha+2}-4\frac{s_*^{\alpha+1}}{\alpha+1}+\frac{3}{\alpha+1}$$

and

$$R=-2\frac{(1-s_*)^{\alpha+1}}{\alpha+1}+2\frac{(1-s_*)^{\alpha+2}}{\alpha+2}-4\frac{s_*^{\alpha+2}}{\alpha+2}+\left(\frac{1}{\alpha+1}+\frac{1}{\alpha+2}\right)$$

with

$$t_*=\frac{2^{\frac{1}{\alpha}}}{1+2^{\frac{1}{\alpha}}},\qquad s_*=\frac{1}{2^{\frac{1}{\alpha}}+1}.$$

Proof. In view of Lemma 3.1 and convexity of f', we have

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{3(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)\right.\right.$$
$$\left.\left.+2\,J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)+2J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{12}\left(\int_0^1\left|2(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)\right|dt\right.$$
$$\left.+\int_0^1\left|(1-t)^{\alpha}-2t^{\alpha}\right|\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|dt\right)$$
$$\leq\frac{b-a}{12}\int_0^1\left(\left|2(1-t)^{\alpha}-t^{\alpha}\right|\left(t\left|f'(a)\right|+(1-t)\left|f'\left(\frac{a+b}{2}\right)\right|\right)\right.$$
$$\left.+\left|(1-t)^{\alpha}-2t^{\alpha}\right|\left((1-t)\left|f'\left(\frac{a+b}{2}\right)\right|+t\left|f'(b)\right|\right)\right)dt.\tag{3.2}$$

Putting $g(t)=2(1-t)^{\alpha}-t^{\alpha}$, we observe that $g'(t)<0$ for $t\in(0,1)$. So, g is a continuous and decreasing function. Then, g has a unique root in $(0,1)$. It is readily seen that

$$t_* = \frac{2^{\frac{1}{\alpha}}}{1+2^{\frac{1}{\alpha}}},$$

is the root of g. Then,

$$\begin{aligned}
I_1 &= \int_0^1 t\,|2(1-t)^{\alpha} - t^{\alpha}|dt \\
&= \int_0^{t_*} t\,(2(1-t)^{\alpha} - t^{\alpha})\,dt + \int_{t_*}^1 t\,(-2(1-t)^{\alpha} + t^{\alpha})dt \\
&= 4\left[-\frac{(1-t_*)^{\alpha+1}}{\alpha+1} + \frac{(1-t_*)^{\alpha+2}}{\alpha+2}\right] - 2\frac{t_*^{\alpha+2}}{\alpha+2} + \left(-\frac{1}{\alpha+2} + \frac{2}{\alpha+1}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 &= \int_0^1 (1-t)\,|2(1-t)^{\alpha} - t^{\alpha}|dt \\
&= \int_0^{t_*} (1-t)\,(2(1-t)^{\alpha} - t^{\alpha})\,dt + \int_{t_*}^1 (1-t)\,(-2(1-t)^{\alpha} + t^{\alpha})dt \\
&= 2\frac{t_*^{\alpha+2}}{\alpha+2} - 2\frac{t_*^{\alpha+1}}{\alpha+1} - 4\frac{(1-t_*)^{\alpha+2}}{\alpha+2} + \left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2}\right).
\end{aligned}$$

Similarly, $h\,(t) = (1-t)^{\alpha} - 2t^{\alpha}$ is a continuous and decreasing function. Moreover,

$$s_* = \frac{1}{2^{\frac{1}{\alpha}}+1},$$

is the root of h in $[0,1]$.

$$\begin{aligned}
I_3 &= \int_0^1 (1-t)\,|(1-t)^{\alpha} - 2t^{\alpha}|dt \\
&= \int_0^{s_*} (1-t)\,((1-t)^{\alpha} - 2t^{\alpha})\,dt + \int_{s_*}^1 (1-t)\,(-(1-t)^{\alpha} + 2t^{\alpha})dt \\
&= 4\frac{s_*^{\alpha+2}}{\alpha+2} - 4\frac{s_*^{\alpha+1}}{\alpha+1} - 2\frac{(1-s_*)^{\alpha+2}}{\alpha+2} + \left(-\frac{1}{\alpha+2} + \frac{2}{\alpha+1}\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
I_4 &= \int_0^1 t\,|(1-t)^{\alpha} - 2t^{\alpha}|dt \\
&= \int_0^{s_*} t\,((1-t)^{\alpha} - 2t^{\alpha})\,dt + \int_{s_*}^1 t\,(-(1-t)^{\alpha} + 2t^{\alpha})dt \\
&= -4\frac{s_*^{\alpha+2}}{\alpha+2} - 2\frac{(1-s_*)^{\alpha+1}}{\alpha+1} + 2\frac{(1-s_*)^{\alpha+2}}{\alpha+2} + \left(\frac{1}{\alpha+2} + \frac{1}{\alpha+1}\right).
\end{aligned}$$

Now, by substituting $I_j, \quad j = 1, 2, 3, 4$ into (3.2) and taking

$$P = I_1, \quad Q = I_2 + I_3, \quad R = I_4,$$

then, we can get the desired result. $\square$

NOTE: By taking $\alpha = 1$ in Theorem 3.2, we reach the same Theorem 2.

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