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ERROR BOUNDS FOR NONLINEAR MIXED VARIATIONAL-HEMIVARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this article, we considered a class of nonlinear variational hemivariational inequality problems and investigated a gap function and regularized gap function for the problems. We discussed the global error bounds for such inequalities in terms of gap function and regularized gap functions by utilizing the Clarke generalized gradient, relaxed monotonicity, and relaxed Lipschitz continuous mappings. Finally, as applications, we addressed an application to non-stationary non-smooth semi-permeability problems.

1. Introduction

The theory of variational inequality problems was first introduced by Stampacchia [19] for modeling problems arising from mechanics to study the regularity problem for partial differential equations. Thus, the variational inequality problem can be considered as a central problem in optimization and

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nonlinear analysis to study various complementarity and equilibrium problems occurring in operation research, mechanics, mathematical programming, we often naturally meet the variational inequality problem for finding $x \in \Omega$ such that

$$\langle N(x), y - x\rangle_X \geq 0, \ \forall y \in \Omega, \tag{1.1}$$

where Ω is a nonempty closed convex subset of a normed space X, $N : X \to X^*$ is a given operator, and $\langle \cdot, \cdot\rangle_X$ denotes the duality pairing between X and its dual X^*.

It is well known that the variational inequality (1.1) can be solved by transforming it into an equivalent optimization problem with so-called merit function $\pi(\cdot;\alpha) : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\pi(x;\alpha) = \sup\{\langle N(x), x - z\rangle_X - \alpha\|x - z\|_X^2 \mid z \in \Omega\} \text{ for } x \in \Omega, \tag{1.2}$$

where α is a nonnegative parameter.

Remark 1.1. (i) If $\alpha = 0$, then (1.2) was first studied in [3].
(ii) If $\alpha > 0$, then (1.2) was studied in [7].

The function $\pi(\cdot, 0)$ is usually known as the gap function, and the function $\pi(\cdot, \alpha)$ for $\alpha > 0$ is a regularized gap function. Furthermore, we see that for all $\alpha > 0$, the function $\pi(\cdot, \alpha)$ is nonnegative on Ω, and $\pi(x^*, \alpha) = 0$ whenever x^* satisfies the variational inequality (1.1), *see* [8].

The theory of variational hemivariational inequalities is known as a generalization of variational inequalities and hemivariational inequalities to the case involving both convex and nonconvex potentials, and based on the notion of the Clarke generalized gradient for locally Lipschitz continuous functions, *see,* [1, 6, 12, 13, 14, 15, 16, 17].

The theory of gap functions is a very effective tool to investigate the conditions of existence, a method for a solution, and conditions of equilibrium for optimization-related problems to simplify the computational aspects. The concept of the regularized function has been introduced by Yamashita and Fukushima in [21]. They also suggested the so-called error bounds for variational inequalities *via* the regularized gap functions. The concept of error bounds is referred to as an upper estimate of the distance between an arbitrary feasible point and a particular problem's solution set. Such error estimates have played a crucial role in the convergence analysis of iterative algorithms for solving variational inequalities, *see,* [2, 9, 10, 20].

The Chang *et al.* [4, 5] introduce the mixed set-valued vector inverse quasi-variational inequality problem and to obtain error bounds for mixed set-valued vector inverse quasi-variational inequality problems in terms of the residual gap function, the regularized gap function, and the D-gap function, *see* [11].

In this paper, we consider a class of nonlinear mixed variationalhemivariational inequality problems and suggest the gap function and regularized gap function. We treated the gap functions for the Minty version of these inequalities by utilizing the relaxed monotone mapping and relaxed Lipschitz continuous mapping. Next, we studied the gap function, regularized gap function and Moreau-Yosida type regularized gap function and provided two new global error bounds for the nonlinear mixed variational hemivariational inequality problems. Finally, we illustrate the abstract results of a nonsmooth semipermeability obstacle problem described by a nonlinear mixed variational-hemivariational inequality problem for which we deliver global error bounds.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ be a real Banach space and $\langle\cdot,\cdot\rangle_X$ be the duality pairing between X and its dual X^*.

Definition 2.1. ([5]) A function $F : X \to \mathbb{R}\cup\{+\infty\}$ is said to be

(a) convex, if $F(tx+(1-t)y) \le tF(x)+(1-t)F(y)$, for all $x, y \in X$, $t \in [0,1]$.

(b) lower semicontinuous (l.s.c.) at $x \in X$, if for any sequence $\{x_n\} \subset X$ such that
$$x_n \to x,$$
it holds
$$F(x) \le \liminf F(x_n).$$

(c) upper semicontinuous (u.s.c.) at $x \in X$, if for any sequence $\{x_n\} \subset X$ such that
$$x_n \to x,$$
it holds
$$\limsup F(x_n) \le F(x).$$

(d) l.s.c (resp. u.s.c.) on X, if F is l.s.c (resp. u.s.c.) at every $x \in X$.

Definition 2.2. ([6]) Let $g : X \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function. The convex subdifferential $\partial_c g : X \to X^*$ of g is defined by
$$\partial_c g(x) = \left\{x^* \in X^* \middle| \langle x^*, y-x\rangle_X \le g(y)-g(x),\ \forall\ y \in X\right\},\ \forall\ x \in X.$$

An element $x^* \in \partial_c g(x)$ is a subgradient of g at $x \in X$.

Definition 2.3. ([16]) A function $F : X \to \mathbb{R}$ is said to be locally Lipschitz continuous, if for every $x \in X$, there exist a neighbourhood $\mathcal{U}$ of x and a constant $L_x > 0$ such that
$$|F(z_1) - F(z_2)| \le L_x\|z_1 - z_2\|_X,\ \forall\ z_1, z_2 \in \mathcal{U}.$$

Let $F : X \to \mathbb{R}$ be a locally Lipschitz continuous function. The Clark generalized directional derivative of F at the point $x \in X$ in the direction $y \in X$ dened by

$$F^{\circ}(x;y) = \limsup_{z \to x, t \to 0^+} \frac{F(z+ty) - F(z)}{t}.$$

The generalized gradient of F at $x \in X$ is a subset of X defined by

$$\partial F(x) = \left\{ x^* \in X^* | F^{\circ}(x;y) \geq \langle x^*, y \rangle_X, \ \forall \ y \in X \right\}.$$

We can easily prove the following lemma from the definition of the Clark generalized directional derivative of F.

Lemma 2.4. *Let $F : X \to \mathbb{R}$ be a locally Lipschitz continuous function. Then the following assumptions are satisfied:*

(a) *For each $x \in X$, the function $X \ni y \rightarrowtail F^{\circ}(x;y) \in \mathbb{R}$ is finite, subadditive, positively homogeneous and*

$$|F^{\circ}(x;y)| \leq L_x \|y\|_X, \ \forall \ y \in X,$$

where $L_x > 0$ is a Lipschitz constant of F near x.

(b) *The function $X \times X \ni (x,y) \rightarrowtail F^{\circ}(x;y) \in \mathbb{R}$ is upper semicontinuous.*

(c) *For every $x, y \in X$, it holds*

$$F^{\circ}(x;y) = \max\{\langle \zeta, y \rangle_X \mid \zeta \in \partial F(x)\}.$$

Definition 2.5. ([11]) An operator $N : X \times X \to X^*$ is said to be pseudomonotone, if N is a bounded operator and for every sequence $\{x_n\} \subseteq X$ converging weakly to $x \in X$ such that

$$\limsup \langle N(x_n, x_n), x_n - x \rangle \leq 0,$$

we have

$$\langle N(x,x), x - y \rangle \leq \liminf \langle N(x_n, x_n), x_n - y \rangle, \ \forall \ y \in X.$$

Let Ω be a nonempty closed convex subset of a reflexive Banach space X. Let $N : \Omega \times \Omega \to X^*$, $\varphi : \Omega \times \Omega \to \mathbb{R}$ and $J : X \to \mathbb{R}$ be the functions, and $f \in X$. Then we consider the following nonlinear mixed variational hemivatiational inequality problem for finding $x \in \Omega$ such that

$$\langle N(x,x) - f, y - x \rangle_X + \varphi(x,y) - \varphi(x,x) + J^{\circ}(x; y-x) \geq 0, \ \forall \ y \in \Omega, \quad (2.1)$$

together with the following hypotheses:

(A) $N : X \times X \to X^*$ is satisfying

  (i) N is pseudomonotone.

(ii) N is relaxed monotone with respect to the first variable with constant $\alpha_N > 0$ such that

$$\langle N(y_1, x) - N(y_2, x), y_1 - y_2\rangle_X \geq -\alpha_N \|y_1 - y_2\|_X^2, \ \forall\ y_1, y_2 \in X. \tag{2.2}$$

(iii)) N is relaxed Lipschitz continuous with respect to the second variable with constant $\beta_N > 0$ such that

$$\langle N(x, y_1) - N(x, y_2), y_1 - y_2\rangle_X \leq -\beta_N \|y_1 - y_2\|_X^2, \ \forall\ y_1, y_2 \in X. \tag{2.3}$$

(B) $\varphi : \Omega \times \Omega \to \mathbb{R}$ is such that
  (i) for each $x \in \Omega$, $\varphi(x, \cdot) : \Omega \to \mathbb{R}$ is convex and lower semicontinuous.
  (ii) for all $x_1, x_2, y_1, y_2 \in \Omega$, there exists $\alpha_\varphi > 0$ such that

$$\varphi(x_1, y_2) - \varphi(x_1, y_1) + \varphi(x_2, y_1) - \varphi(x_2, y_2) \leq \alpha_\varphi \|x_1 - x_2\|_X \|y_1 - y_2\|_X. \tag{2.4}$$

(C) $J : X \to \mathbb{R}$ is a locally Lipschitz function such that
  (i) $\|\partial J(y)\|_{X^*} \leq \lambda_0 + \lambda_1 \|y\|_X$, for all $y \in X$ and $\lambda_0, \lambda_1 \geq 0$.
  (ii) there exists $\alpha_J \geq 0$ such that

$$J^\circ(y_1; y_2 - y_1) + J^\circ(y_2; y_1 - y_2) \leq \alpha_J \|y_1 - y_2\|_X^2, \ \forall\ y_1, y_2 \in \Omega. \tag{2.5}$$

(D) Ω is a nonempty, closed and convex subset of X and

$$f \in X. \tag{2.6}$$

3. Main results

First, we prove the existing result of equation (2.1).

Theorem 3.1. *Assume that (A)-(D) hold and the following condition is satisfied*

$$\alpha_j + \alpha_\varphi + \alpha_N - \beta_N < 1. \tag{3.1}$$

Then (2.1) has the unique solution. Moreover, x solves (2.1) if and only if it solves the following Minty nonlinear mixed variational-hemivariational inequality problem for finding $x \in \Omega$ such that

$$\langle N(y, y) - f, y - x\rangle_X + \varphi(x, y) - \varphi(x, x) + J^\circ(y; y - x) \geq 0, \ \forall\ y \in \Omega. \tag{3.2}$$

Proof. Let $x \in \Omega$ be the unique solution of (2.1). We note that (C(ii)) is equivalent to the following relaxed monotonicity condition of the generalized gradient

$$\langle \partial J(y) - \partial J(x), y - x\rangle_X \geq -\alpha_j \|y - x\|_X^2, \ \forall\ y, x \in X. \tag{3.3}$$

Next, from (3.1), (3.3), and (A(ii))-(A(iii)), we have

$$\langle N(y, y) - N(x, x), y - x\rangle_X + \langle \zeta_y - \zeta_x, y - x\rangle_X \geq (-\alpha_N + \beta_N - \alpha_j)\|y - x\|_X^2, \tag{3.4}$$

for all $x,y\in\Omega,\ \zeta_y\in\partial J(y),\zeta_x\in\partial J(x)$.

Let $y\in\Omega$ be arbitrary. From (3.4), Lemma 2.4(c) and from the concept of generalized gradient, we have

$$\begin{aligned}\langle N(y,y)-f,y-x\rangle_X+\varphi(x,y)-\varphi(x,x)+J^\circ(y;y-x)&\\ \geq\langle N(y,y)-f+\zeta_y,y-x\rangle_X+\varphi(x,y)-\varphi(x,x)&\\ \geq\langle N(x,x)-f+\zeta_x,y-x\rangle_X+\varphi(x,y)&\\ -\varphi(x,x)+(-\alpha_N+\beta_N-\alpha_j)\|y-x\|_X^2&\\ \geq\langle N(x,x)-f+\zeta_x,y-x\rangle_X+\varphi(x,y)-\varphi(x,x)&\\ =\langle N(x,x)-f,y-x\rangle_X+\varphi(x,y)-\varphi(x,x)+J^\circ(x;y-x)&\\ \geq 0,&\end{aligned}$$

for all $\zeta_y\in\partial J(y),\zeta_x\in\partial J(x)$. Therefore,

$$J^\circ(x;y-x)=\langle\zeta_x,y-x\rangle_X,\ \forall y\in\Omega,\zeta_x\in\partial J(x).$$

Hence, $x\in\Omega$ is a unique solution of (3.2).

Now we prove the uniqueness of (2.1). Let x_1,x_2 be the two solutions of (2.1). Then

$$\langle N(x_1,x_1)-f,y-x_1\rangle_X+\varphi(x_1,y)-\varphi(x_1,x_1)+J^\circ(x_1;y-x_1)\geq 0,\ \forall\, y\in\Omega,\ (*)$$

$$\langle N(x_2,x_2)-f,y-x_2\rangle_X+\varphi(x_2,y)-\varphi(x_2,x_2)+J^\circ(x_2;y-x_2)\geq 0,\ \forall\, y\in\Omega.\quad(**)$$

Putting $y=x_2$ in $(*)$ and $y=x_1$ in $(**)$, and adding those inequalities, we have

$$\begin{aligned}\langle N(x_1,x_1)-N(x_2,x_2),x_2-x_1\rangle_X+\varphi(x_1,x_2)-\varphi(x_1,x_1)+\varphi(x_2,x_1)-\varphi(x_2,x_2)\\ +J^\circ(x_1;x_2-x_1)+J^\circ(x_2;x_1-x_2)\geq 0.\end{aligned}$$

Using the (A(ii)), (A(iii)), (B(ii)) and (C(ii)), we have

$$-(\alpha_j+\alpha_\varphi+\alpha_N-\beta_N\|x_1-x_2\|^2\geq 0,$$

it implies that

$$\|x_1-x_2\|=0.$$

Hence, $x_1=x_2$ and the solution is unique.

Conversely, let $x\in\Omega$ be a solution of (3.2). For $y\in\Omega$ and $t\in(0,1)$, we denote $y_t=ty+(1-t)x\in\Omega$. Inserting y_t into (3.2), we have

$$\begin{aligned}0&\leq t\langle N(y_t,y_t)-f,y-x\rangle_X+\varphi(x,y_t)-\varphi(x,x)+J^\circ(y_t;y_t-x)\\ &\leq t\langle N(y_t,y_t)-f,y-x\rangle_X+t\varphi(x,y)-t\varphi(x,x)+tJ^\circ(y_t;y-x).\end{aligned}$$

Now, from the convexity of

$$y\rightarrowtail\varphi(x,y)$$

and the positive homogeneity of

$$y \rightarrowtail J^{\circ}(x;y),$$

we have

$$\langle N(y_t, y_t) - f, y - x\rangle_X + \varphi(x,y) - \varphi(x,x) + J^{\circ}(y_t; y - x) \geq 0. \tag{3.5}$$

Since N is pseudomonotone, it is demicontinuous, *see* ([13], Theorem 3.69). Passing to the upper limit as $t \to 0^+$ in (3.5), we have

$$\begin{aligned}
\langle N(x,x) - f, y - x\rangle_X &+ \varphi(x,y) - \varphi(x,x) + J^{\circ}(x; y - x) \\
&\geq \limsup_{t\to 0^+} \langle N(y_t, y_t) - f, y - x\rangle_X + \varphi(x,y) \\
&\quad - \varphi(x,x) + \limsup_{t\to 0^+} J^{\circ}(y_t; y - x) \\
&\geq \limsup_{t\to 0^+} \{\langle N(y_t, y_t) - f, y - x\rangle_X + \varphi(x,y) \\
&\quad - \varphi(x,x) + J^{\circ}(y_t; y - x)\} \geq 0, \ \forall y \in \Omega.
\end{aligned}$$

Therefore, from Lemma 3.1(b), we conclude that $x \in \Omega$ is a solution to (2.1) and the proof is completed. □

Definition 3.2. ([3]) A function $\pi : \Omega \to \mathbb{R}$ is said to be a gap function for (2.1), if it satisfies the following assertions:

(a) $\pi(x) \geq 0, \ \forall x \in \Omega.$

(b) $x^* \in \Omega$ is such that

$$\pi(x^*) = 0$$

if and only if x^* is a solution of (2.1).

Consider the functions $\Lambda^f, \ \Lambda^f_* : \Omega \to \mathbb{R}$ defined by for all $x \in \Omega$,

$$\Lambda^f(x) = \sup_{y\in\Omega} \left\{\langle N(x,x) - f, x - y\rangle_X + \varphi(x,x) - \varphi(x,y) - J^{\circ}(x; y - x)\right\}, \tag{3.6}$$

$$\Lambda^f_*(x) = \sup_{y\in\Omega} \left\{\langle N(y,y) - f, x - y\rangle_X + \varphi(x,x) - \varphi(x,y) - J^{\circ}(y; y - x)\right\}. \tag{3.7}$$

The following lemma shows that the functions Λ^f and Λ^f_* are gap functions for (2.1).

Lemma 3.3. *Assume that the assumptions of Theorem* 3.1 *hold. Then, the functions* Λ^f *and* Λ^f_* *dened by* (3.6) *and* (3.7) *are two gap functions for* (2.1).

Proof. First of all, we prove that Λ^f is a gap function for (2.1). It is not difficult to demonstrate analogously that the function Λ^f_* is also a gap function for (2.1). We will review the two conditions of Definition 3.2.

(a) In fact, it is obvious that

$$\Lambda^f(x) \geq 0, \forall\, x \in \Omega.$$

Since this property has been retained for all $x \in \Omega$,

$$\begin{aligned}\Lambda^f(x) &\geq \langle N(x,x) - f, x - x\rangle_X + \varphi(x,x) - \varphi(x,x) - J^\circ(x; x-x)\\ &= -j^\circ(x;0)\\ &= 0. \end{aligned}\tag{3.8}$$

(b) Suppose that $x^* \in \Omega$ is such that

$$\Lambda^f(x^*) = 0,$$

that is,

$$\sup_{y\in\Omega}\{\langle N(x^*,x^*) - f, x^* - y\rangle_X + \varphi(x^*,x^*) - \varphi(x^*,y) - J^\circ(x^*; y - x^*)\} = 0. \tag{3.9}$$

This together with the fact

$$\langle N(x^*,x^*) - f, x^* - x^*\rangle_X + \varphi(x^*,x^*) - \varphi(x^*,x^*) - J^\circ(x^*; x^* - x^*) = 0$$

implies that (3.9) is equivalent to

$$\langle N(x^*,x^*) - f, y - x^*\rangle_X + \varphi(x^*,y) - \varphi(x^*,x^*) - J^\circ(x^*; y - x^*) \geq 0,\ \forall\, y \in \Omega.$$

Therefore, x^* is a solution of (2.1) if and only if

$$\Lambda^f(x^*) = 0.$$

□

Let $\gamma > 0$ be a fixed parameter. We consider the following functions $\Lambda^{f,\gamma}, \Lambda_*^{f,\gamma} : \Omega \to \mathbb{R}$ defined by for all $x \in \Omega$,

$$\Lambda^{f,\gamma}(x) = \sup_{y\in\Omega}\left\{\langle N(x,x) - f, x - y\rangle_X + \varphi(x,x) - \varphi(x,y) - J^\circ(x; y - x) - \frac{1}{2\gamma}\|x - y\|_X^2\right\}, \tag{3.10}$$

$$\Lambda_*^{f,\gamma}(x) = \sup_{y\in\Omega}\left\{\langle N(y,y) - f, x - y\rangle_X + \varphi(x,x) - \varphi(x,y) - J^\circ(y; y - x) - \frac{1}{2\gamma}\|x - y\|_X^2\right\}. \tag{3.11}$$

The functions defined by (3.10) and (3.11) are called the regularized gap functions for (2.1).

Theorem 3.4. *Suppose the assertions of Theorem* 3.1 *hold. Then, for any* $\gamma > 0$*, the functions* $\Lambda^{f,\gamma}$ *and* $\Lambda_*^{f,\gamma}$ *are two gap functions for* (2.1).

Proof. Now, we prove that $\Lambda^{f,\gamma}$ is a gap function for (2.1). Applying the analogous techniques, it is not difficult to show that $\Lambda_*^{f,\gamma}$ is also a gap function for (2.1). We will verify the two assumptions of Definition 3.2.

(a) For fixed $\gamma > 0$, it is trivial that for each $x \in \Omega$ and holds

$$\Lambda^{f,\gamma}(x) \geq 0.$$

Therefore, for all $x \in \Omega$,

$$\begin{aligned}\Lambda^{f,\gamma}(x) &= \langle N(x,x) - f, x - x\rangle_X + \varphi(x,x) - \varphi(x,x) \\ &\quad - J^\circ(x; x - x) - \frac{1}{2\gamma}\|x - x\|_X^2 \\ &= -J^\circ(x; 0) \\ &= 0.\end{aligned}$$

(b) Assume that $x^* \in \Omega$ is such that

$$\Lambda^{f,\gamma}(x^*) = 0,$$

and

$$\sup_{y\in\Omega}\Big\{\langle N(x^*,x^*) - f, x^* - y\rangle_X + \varphi(x^*,x^*) - \varphi(x^*,y) \\ - J^\circ(x^*; y - x^*) - \frac{1}{2\gamma}\|x^* - y\|_X^2\Big\} = 0.$$

This implies that

$$\begin{aligned}\langle N(x^*,x^*) - f, y - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,y) + J^\circ(x^*; y - x^*) \\ \geq -\frac{1}{2\gamma}\|x^* - y\|_X^2, \ \forall y \in \Omega. \qquad (3.12)\end{aligned}$$

For any $z \in \Omega$ and $t \in (0,1)$, we put $y = y_t = (1-t)x^* + tz \in \Omega$ in (3.12) to obtain

$$\begin{aligned} t\langle N(x^*,x^*) - f, z - x^*\rangle_X - t\varphi(x^*,x^*) + t\varphi(x^*,z) + tJ^\circ(x^*; z - x^*) \\ &\geq \langle N(x^*,x^*) - f, y_t - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,y_t) + J^\circ(x^*; y_t - x^*) \\ &\geq -\frac{1}{2\gamma}\|x^* - y_t\|_X^2 \\ &= -\frac{t^2}{2\gamma}\|x^* - z\|_X^2,\end{aligned}$$

where $y \rightarrowtail \varphi(x,y)$ and $y \rightarrowtail J^\circ(x;y)$. Hence, we have

$$\begin{aligned}\langle N(x^*,x^*) - f, z - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,z) - J^\circ(x^*; z - x^*) \\ \geq -\frac{t}{2\gamma}\|x^* - z\|_X^2, \ \forall z \in \Omega.\end{aligned}$$

If $t \to 0^+$, then

$$\langle N(x^*,x^*) - f, z - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,z) + J^\circ(x^*; z - x^*) \geq 0, \ \forall z \in \Omega.$$

Hence, x^* is a solution of (2.1).

Conversely, suppose that $x^* \in \Omega$ is a solution of (2.1). Then

$$\langle N(x^*, x^*) - f, y - x^* \rangle_X - \varphi(x^*, x^*) + \varphi(x^*, y) + J^\circ(x^*; y - x^*) \geq 0, \ \forall y \in \Omega.$$

This ensures that

$$\sup_{y \in \Omega} \left\{ \langle N(x^*, x^*) - f, x^* - y \rangle_X + \varphi(x^*, x^*) - \varphi(x^*, y) - J^\circ(x^*; y - x^*) - \frac{1}{2\gamma} \|x^* - y\|_X^2 \right\} \leq 0.$$

The latter combined with the fact

$$\Lambda^{f,\gamma}(x) \geq 0, \ \forall x \in \Omega,$$

then, we conclude that

$$\Lambda^{f,\gamma}(x^*) = 0,$$

and the proof is completed. □

Later on, we will prove that the regularized gap functions $\Lambda^{f,\gamma}$ and $\Lambda_*^{f,\gamma}$ are lower semicontinuous.

Lemma 3.5. *Assume that the assumptions of Theorem* 3.1 *are satisfied. If* $\varphi : \Omega \times \Omega \to \mathbb{R}$ *is continuous, then, for each* $\gamma > 0$, *the functions* $\Lambda^{f,\gamma}$ *and* $\Lambda_*^{f,\gamma}$ *are both lower semicontinuous.*

Proof. We can prove that $\Lambda^{f,\gamma}$ is lower semicontinuous for each $\gamma > 0$. It is not difficult to use a similar argument to verify that $\Lambda_*^{f,\gamma}$ has the same property.

Consider the function $\hat{\Lambda}^{f,\gamma} : \Omega \times \Omega \to \mathbb{R}$ dened by

$$\hat{\Lambda}^{f,\gamma}(x, y) = \langle N(x, x) - f, x - y \rangle_X + \varphi(x, x) - \varphi(x, y) - J^\circ(x; y - x) - \frac{1}{2\gamma} \|x - y\|_X^2.$$

Since the operator $N : X \times X \to X^*$ is demicontinuous being pseudomonotone, the function

$$x \rightarrowtail \langle N(x, x), x \rangle_X$$

is continuous. The latter together with the lower semicontinuity of

$$(x, y) \rightarrowtail -J^\circ(x; y),$$

and the continuity of

$$(x, y) \rightarrowtail \varphi(x, y) \text{ and } x \rightarrowtail \|x\|_X$$

guarantees that

$$x \rightarrowtail \hat{\Lambda}^{f,\gamma}(x, y)$$

is lower semicontinuous for all $y \in \Omega$.

Next, we see that

$$\Lambda^{f,\gamma}(x) = \sup_{y \in \Omega} \hat{\Lambda}^{f,\gamma}(x, y), \ \forall \ x \in \Omega.$$

Let $\{x_n\} \subset \Omega$ be such that

$$x_n \to x \quad \text{as} \quad n \to \infty.$$

Then, we have

$$\begin{aligned}
\liminf_{n\to\infty} \Lambda^{f,\gamma}(x_n) &= \liminf_{n\to\infty} \sup_{y\in\Omega} \hat{\Lambda}^{f,\gamma}(x_n, y) \\
&\geq \liminf_{n\to\infty} \hat{\Lambda}^{f,\gamma}(x_n, z) \\
&\geq \hat{\Lambda}^{f,\gamma}(x, z), \quad \forall z \in \Omega.
\end{aligned}$$

Passing to supremum on $z \in \Omega$ for the above inequality, given

$$\begin{aligned}
\liminf_{n\to\infty} \Lambda^{f,\gamma}(x_n) &\geq \sup_{z\in\Omega} \hat{\Lambda}^{f,\gamma}(x, z) \\
&= \Lambda^{f,\gamma}(x).
\end{aligned}$$

Therefore, the function $\Lambda^{f,\gamma}$ is lower semicontinuous and the proof is completed. $\square$

Let $\gamma, \rho > 0$ be two parameters. Moreover, let us consider the following functions:

$$\daleth_{\Lambda^{f,\gamma,\rho}},\ \daleth_{\Lambda_*^{f,\gamma,\rho}} : \Omega \to \mathbb{R}$$

defined by

$$\daleth_{\Lambda^{f,\gamma,\rho}}(x) = \inf_{z\in\Omega} \left\{ \Lambda^{f,\gamma}(z) + \rho \|x - z\|_X^2 \right\}, \quad \forall x \in \Omega, \tag{3.13}$$

$$\daleth_{\Lambda_*^{f,\gamma,\rho}}(x) = \inf_{z\in\Omega} \left\{ \Lambda_*^{f,\gamma}(z) + \rho \|x - z\|_X^2 \right\}, \quad \forall x \in \Omega. \tag{3.14}$$

The $\daleth_{\Lambda^{f,\gamma,\rho}}$ and $\daleth_{\Lambda_*^{f,\gamma,\rho}}$ are the Moreau-Yosida regularized gap functions. Subsequently, we will verify that these functions are two gap functions for (2.1).

Theorem 3.6. *Assume that the assumptions of Lemma* 3.5 *are satisfied. The two gap functions for* (2.1) *are* $\daleth_{\Lambda^{f,\gamma,\rho}}$ *and* $\daleth_{\Lambda_*^{f,\gamma,\rho}}$*, for all* $\gamma, \rho > 0$.

Proof. We can show that the gap function for (2.1) is $\daleth_{\Lambda^{f,\gamma,\rho}}$. An analogous proof can be made that $\daleth_{\Lambda_*^{f,\gamma,\rho}}$ is also a gap function for (2.1).

(a) Recall that $\Lambda^{f,\gamma,\rho}$ is a gap function for (2.1), so for any $\gamma, \rho > 0$ is fixed,

$$\Lambda^{f,\gamma,\rho}(x) \geq 0,\ \forall\ x \in \Omega.$$

In consequence,

$$\daleth_{\Lambda^{f,\gamma,\rho}}(x) \geq 0,\ \forall\ x \in \Omega.$$

(b) Suppose that $x \in \Omega$ is a solution of (2.1). Theorem 3.4 show that

$$\Lambda^{f,\gamma,\rho}(x^*) = 0.$$

Moreover, the inequality

$$\begin{aligned} \daleth_{\Lambda^{f,\gamma,\rho}}(x^*) &= \inf_{z\in\Omega} \left\{ \Lambda^{f,\gamma}(z) + \rho\|x^* - z\|_X^2 \right\} \\ &\leq \Lambda^{f,\gamma}(x^*) + \rho\|x^* - x^*\|_X^2 \\ &= 0 \end{aligned}$$

with

$$\daleth_{\Lambda^{f,\gamma,\rho}}(x^*) \geq 0$$

imply that

$$\daleth_{\Lambda^{f,\gamma,\rho}}(x^*) = 0.$$

Conversely, let $x^* \in \Omega$ be such that

$$\daleth_{\Lambda^{f,\gamma,\rho}}(x^*) = 0,$$

and

$$\inf_{z\in\Omega} \left\{ \Lambda^{f,\gamma}(z) + \rho\|x^* - z\|_X^2 \right\} = 0.$$

Therefore, there exists a minimizing sequence $\{z_n\}$ in Ω such that

$$0 \leq \Lambda^{f,\gamma}(z_n) + \rho\|x^* - z_n\|_X^2 < \frac{1}{n}. \tag{3.15}$$

It is obvious that

$$\Lambda^{f,\gamma}(z_n) \to 0$$

and

$$\|u^* - z_n\|_X \to 0 \quad \text{as} \quad n \to \infty,$$

implies

$$z_n \to x^* \quad \text{as} \quad n \to +\infty.$$

From Lemma 3.5 and the nonnegativity of $\Lambda^{f,\gamma}$, we have

$$0 \leq \Lambda^{f,\gamma}(x^*) \leq \liminf_{n\to+\infty} \Lambda^{f,\gamma}(z_n) = 0. \tag{3.16}$$

Thus

$$\Lambda^{f,\gamma}(x^*) = 0,$$

because the gap function $\Lambda^{f,\gamma}$. Therefore, x^* is a solution of (2.1), and the proof is completed. $\square$

4. The global error bounds

In this section, we look at two global error bounds for (2.1), one for the regularized gap functions $\Lambda^{f,\gamma,\rho}$ and the other for the Moreau-Yosida regularized gap function $\daleth_{\Lambda^{f,\gamma,\rho}}$. These global error estimates measure the distance between any admissible point and the unique solution of (2.1).

Theorem 4.1. *Let $x^* \in \Omega$ be the unique solution of* (2.1) *and $\gamma > 0$ be such that*

$$\beta_N - \alpha_N - \alpha_\varphi - \alpha_J > \frac{1}{2\gamma}. \tag{4.1}$$

Assume that the assertions of Theorem 3.1 *hold. Then, for each $x \in \Omega$, we have*

$$\|x - x^*\|_X \leq \sqrt{\frac{\Lambda^{f,\gamma}(x)}{\beta_N - \alpha_N - \alpha_\varphi - \alpha_J - \frac{1}{2\gamma}}}. \tag{4.2}$$

Proof. Let $x^* \in \Omega$ be the unique solution of (2.1), that is,

$$\langle N(x^*, x^*) - f, y - x^*\rangle_X + \varphi(x^*, y) - \varphi(x^*, x^*) + J^\circ(x^*; y - x^*) \geq 0, \ \forall y \in \Omega. \tag{4.3}$$

For any $x \in \Omega$ fixed, we put $y = x$ in (4.3), we obtain

$$\langle N(x^*, x^*) - f, x - x^*\rangle_X + \varphi(x^*, x) - \varphi(x^*, x^*) + J^\circ(x^*; x - x^*) \geq 0. \tag{4.4}$$

By virtue of the denition of $\Lambda^{f,\gamma}$, one has

$$\Lambda^{f,\gamma}(x) \geq \langle N(x,x) - f, x - x^*\rangle_X + \varphi(x,x) - \varphi(x,x^*) - J^\circ(x; x^* - x) - \frac{1}{2\gamma}\|x - x^*\|_X^2. \tag{4.5}$$

It follows from the assumptions (A(ii)), (A(iii)), (B(ii)), and (C(ii)), we have

$$\begin{aligned}
&\langle N(x,x) - f, x - x^*\rangle_X + \varphi(x,x) - \varphi(x,x^*) - J^\circ(x; x^* - x) - \frac{1}{2\gamma}\|x - x^*\|_X^2 \\
&\quad \geq \langle N(x^*, x^*) - f, x - x^*\rangle_X + \varphi(x^*, x) - \varphi(x^*, x^*) + J^\circ(x^*; x - x^*) \\
&\qquad + (\beta_N - \alpha_N - \alpha_j - \alpha_\varphi - \frac{1}{2\gamma})\|x - x^*\|_X^2 \\
&\quad \geq \left(\beta_N - \alpha_N - \alpha_j - \alpha_\varphi - \frac{1}{2\gamma}\right)\|x - x^*\|_X^2,
\end{aligned} \tag{4.6}$$

where the last inequality is obtained by using (4.4). Combining (4.5) and (4.6), we have

$$\Lambda^{f,\gamma}(x) \geq \left(\beta_N - \alpha_N - \alpha_j - \alpha_\varphi - \frac{1}{2\gamma}\right)\|x - x^*\|_X^2. \tag{4.7}$$

Hence, the desired inequality (4.2) is valid. $\square$

Theorem 4.2. *Let $x^* \in \Omega$ be the unique solution of* (2.1) *and $\gamma > 0$ be such that*

$$\beta_N - \alpha_N - \alpha_j - \alpha_\varphi \geq \frac{1}{2\gamma}. \tag{4.8}$$

Assume that the assumptions of Theorem 3.1 *hold. Then, for each $x \in \Omega$ and all $\rho > 0$, we have*

$$\|x - x^*\|_X \leq \sqrt{\frac{2\beth_{\Lambda^{f,\gamma,\rho}}(x)}{\min\left\{\beta_N - \alpha_N - \alpha_\varphi - \alpha_J - \frac{1}{2\gamma}, \rho\right\}}}. \tag{4.9}$$

Proof. Let $x^* \in \Omega$ be the unique solution of (2.1). By the definition of the function

$$\begin{aligned}
\beth_{\Lambda^{f,\gamma,\rho}}(x) &= \inf_{z\in\Omega}\left\{\Lambda^{f,\gamma}(z) + \rho\|x - z\|_X^2\right\} \\
&\geq \inf_{z\in\Omega}\left\{\left(\beta_N - \alpha_N - \alpha_J - \alpha_\varphi - \frac{1}{2\gamma}\right)\|x^* - z\|_X^2 + \rho\|x - z\|_X^2\right\} \\
&\geq \min\left\{\beta_N - \alpha_N - \alpha_J - \alpha_\varphi - \frac{1}{2\gamma}, \rho\right\}\inf_{z\in\Omega}\left\{\|x^* - z\|_X^2 + \|x - z\|_X^2\right\} \\
&\geq \frac{1}{2}\min\left\{\beta_N - \alpha_N - \alpha_J - \alpha_\varphi - \frac{1}{2\gamma}, \rho\right\}\|x - x^*\|_X^2, \ \forall x \in \Omega,
\end{aligned}$$

we have

$$\|x - x^*\|_X \leq \sqrt{\frac{2\beth_{\Lambda^{f,\gamma,\rho}}(x)}{\min\left\{\beta_N - \alpha_N - \alpha_\varphi - \alpha_J - \frac{1}{2\gamma}, \rho\right\}}}, \ \forall x \in \Omega,$$

which completes the proof. □

5. Applications

The object of this section is to investigate a boundary value problem with the generalized gradient and obstacle effect, which illustrates the applicability of the abstract results.

Let $\Re$ be a bounded domain with a Lipschitz continuous boundary Γ in $\mathbb{R}^d$ ($d = 2, 3$). The boundary is divided into two mutually disjoint measurable parts, Γ_1 and Γ_2, with the result that meas $(\Gamma_1) > 0$.

Consider the following nonlinear mixed boundary value problem with constraints. For finding a function $x : \Re \to \mathbb{R}$ such that

$$-div\ a(u, \nabla x) + \partial g(u, x) \ni f(u), \text{in } \Re, \tag{5.1}$$

where ∂g and $\partial_c F$ denote the generalized gradient and the convex subdifferential of the functions $g : \Re \times \mathbb{R} \to \mathbb{R}$ and $F : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$, *respectively* with respect to their second variables, while the conormal derivative

$$\frac{\partial x}{\partial \nu_a} = (a(u, \nabla x), \boldsymbol{\nu})_{\mathbb{R}^d}$$

represents the heat flux through the part Γ_2, and $\boldsymbol{\nu}$ stands for the outward unit normal on Γ. The function x represents the electric potential, the function $a = a(u, \nabla x)$ is the dielectric coefficient, and $f = f(u)$ is a given source term. The material which occupies $\Re$ is non-isotropic and heterogeneous, and thus a effectively depends on u.

$$x(u) \leq \psi(u), \text{ in } \Re, \tag{5.2}$$

represents an additional unilateral constraint for the solution,

$$x = 0, \text{ on } \Gamma_1 \tag{5.3}$$

$$-\frac{\partial x}{\partial \nu_a} \in \kappa(x)\partial_c F(u, x), \text{ on } \Gamma_2. \tag{5.4}$$

We remark that in general there is no function F such that

$$\partial \tilde{F} = \kappa \partial_c F.$$

This means that if $g \equiv 0$, then the weak form of (5.1), stated in (5.2) below, reduces to quasi-variational inequality.

We need the following standard functional space. Let X be defined by

$$X = \left\{ y \in H^1(\Re) \middle|\ y = 0 \text{ on } \Gamma_1 \right\}.$$

Since meas $(\Gamma_1) > 0$, the space X is endowed with the inner product and corresponding norm given by

$$\langle x, y \rangle = \int_{\Re} (\nabla x(u), \nabla y(u))_{\mathbb{R}^d}\, du$$

and

$$\|y\|_X = \left(\int_{\Re} \|\nabla y(u)\|^2_{\mathbb{R}^d} du \right)^{\frac{1}{2}}, \ \forall x, y \in X.$$

Let $\gamma_0 : X \to L^2(\Gamma)$ be the trace operator and Ω be the admissible set defined by

$$\Omega = \left\{ y \in X \middle|\ y(u) \leq \psi(u) \text{ for a.e. } u \in \Re \right\}.$$

For the unique solvability of (4.1), we suggest the following hypotheses:

(A) $a : \Re \times \mathbb{R}^d \to \mathbb{R}^d$ is such that

(i) $a(\cdot, w)$ is measurable on $\Re$ for all $w \in \mathbb{R}^d$ with

$$a(u, 0) = 0 \text{ for a.e.} u \in \Re. \tag{5.5}$$

(ii) $a(u,\cdot)$ is continuous on $\mathbb{R}^d$ for a.e. $u \in \Re$.

(iii) for all $w \in \mathbb{R}^d$, a.e. $u \in \Re$ with $\alpha_a > 0$, we have

$$\|a(u,w)\|_{\mathbb{R}^d} \leq \alpha_a(1+\|w\|_{\mathbb{R}^d}). \tag{5.6}$$

(iv) for all $w_1, w_2 \in \mathbb{R}^d$ and a.e. $u \in \Re$ with $\alpha_a > 0$, we have

$$(a(u,w_1) - a(u,w_2)) \cdot (w_1 - w_2) \geq -\alpha_a \|w_1 - w_2\|^2_{\mathbb{R}^d}. \tag{5.7}$$

(v) for all $w_1, w_2 \in \mathbb{R}^d$ and a.e. $u \in \Re$ with $\beta_a > 0$, we have

$$(a(u,w_1) - a(u,w_2)) \cdot (w_1 - w_2) \leq -\beta_a \|w_1 - w_2\|^2_{\mathbb{R}^d}. \tag{5.8}$$

(B) $g : \Re \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $g(\cdot, r)$ is measurable on $\Re$ for all $r \in \mathbb{R}$ and there exists $\tilde{e} \in L^2(\Re)$ such that

$$g(\cdot, \tilde{e}(\cdot)) \in L^1(\Re). \tag{5.9}$$

(ii) $g(u,\cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $u \in \Re$.

(iii) there exist $\bar{\lambda}_0, \bar{\lambda}_1 \geq 0$ such that

$$\left|\partial g(u,r)\right| \leq \bar{\lambda}_0 + \bar{\lambda}_1 |r|, \ \forall r \in \mathbb{R} \ \text{ and a.e. } u \in \Re. \tag{5.10}$$

(iv) there exists $\alpha_g \geq 0$ such that

$$g^\circ(u, r_1; r_2 - r_1) + g^\circ(u, r_2; r_1 - r_2) \leq \alpha_g |r_1 - r_2|^2, \tag{5.11}$$

for all $r_1, r_2 \in \mathbb{R}$ and a.e. $u \in \Re$.

(C) $F : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $F(\cdot, r)$ is measurable on Γ_2 for all $r \in \mathbb{R}$.

(ii) $F(u,\cdot)$ is convex on $\mathbb{R}$ for a.e. $u \in \Re$.

(iii) there exists $L_F > 0$ such that

$$\left|F(u,r_1) - F(u,r_2)\right| \leq L_F |r_1 - r_2|, \tag{5.12}$$

for all $r_1, r_2 \in \mathbb{R}$ and a.e. $u \in \Gamma_2$.

(D) $\kappa : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $\kappa(\cdot, r)$ is measurable on Γ_2 for all $r \in \mathbb{R}$.

(ii) there exists $L_\kappa > 0$ such that

$$\left|\kappa(u,r_1) - \kappa(u,r_2)\right| \leq L_\kappa |r_1 - r_2|, \tag{5.13}$$

for all $r_1, r_2 \in \mathbb{R}$ and a.e. $u \in \Gamma_2$.

(iii) $\kappa(u, 0) = 0$ for a.e. $u \in \Re$.

(E) $\psi \in X$ and

$$f \in L^2(\Re). \tag{5.14}$$

Now, using the standard method based on the Green theorem, *see* [18], we have the following variational formulation of (5.1) for finding $x \in \Omega$ such that

$$\int_{\Re} (a(u, \nabla x), \nabla(y-x))_{\mathbb{R}^d}\, du + \int_{\Gamma_2} (\kappa(x)F(u,y) - \kappa(x)F(u,x))\, d\Gamma + \int_{\Re} g^{\circ}(u,x;y-x)du \geq \int_{\Re} f(y-x)du,\ \forall y \in \Omega. \tag{5.15}$$

Theorem 5.1. *Assume that the assumptions **(A)-(E)** are satisfied. If the inequality holds*

$$\beta_a - \alpha_a - \alpha_g - L_F L_\kappa \|\gamma_0\|^2 \geq 0, \tag{5.16}$$

then (5.15) *has a unique solution* $x^* \in \Omega$.

Proof. Consider the operator $N : X \times X \to X^*$ and the functions $\varphi : \Omega\times\Omega \to \mathbb{R}$ and $J : X \to \mathbb{R}$ defined by

$$\begin{aligned} \langle N(x,x), y\rangle_X &= \int_{\Re} (a(u,\nabla x), \nabla y)_{\mathbb{R}^d} du, \\ \varphi(x,y) &= \int_{\Gamma_2} \kappa(x)F(y)d\Gamma, \\ J(y) &= \int_{\Re} g(u,y)du,\ \forall x,y \in X. \end{aligned}$$

It is simple to demonstrate that all conditions of Theorem 3.1 are met with

$$\alpha_N = \alpha_a,\ \beta_N = \beta_a,\ \alpha_J = \alpha_g, \curlywedge_0 = \bar{\curlywedge}_0, \curlywedge_1 = \bar{\curlywedge}_1 \text{ and } \alpha_\varphi = L_F L_\kappa \|\gamma_0\|^2.$$

From Theorem 3.1, we have

$$J^{\circ}(x;y) \leq \int_{\Re} g^{\circ}(x;y)du,\ \forall x,y \in X.$$

Therefore, we can conclude that (5.15) admits a solution. Moreover, the condition (5.16) guarantees that (5.15) is uniquely solvable.

Next, for any parameter $\gamma > 0$, we introduce the function $\tilde{\Lambda}^{f,\gamma} : \Omega \to \mathbb{R}$ defined by

$$\tilde{\Lambda}^{f,\gamma}(x) = \sup_{y\in\Omega} \Big\{ \int_{\Re} a(u,\nabla x)\cdot\nabla(x-y)du + \int_{\Gamma_2} (\kappa(x)F(u,x) - \kappa(x)F(u,y))\, d\Gamma - \int_{\Re} f(x-y)du - \int_{\Re} g^{\circ}(u,x;y-x)du - \frac{1}{2\gamma}\|x-y\|_X^2 \Big\}. \tag{5.17}$$

□

The following error estimates are obtained directly from the Theorems 3.4-3.6, 4.1-4.2 and Theorem 5.1.

Theorem 5.2. *Let $x^* \in \Omega$ be the unique solution of* (5.15). *Under the hypotheses of Theorem* 5.1, *we have*

(i) *for each $\gamma > 0$ and $f \in L^2(\Re)$, $\tilde{\Lambda}^{f,\gamma} : \Omega \to \mathbb{R}$ is a regularized gap function for* (5.15).

(ii) *If $\gamma > 0$ is such that*

$$\beta_a - \alpha_a - \alpha_g - L_F L_\kappa \|\gamma_0\|^2 > \frac{1}{2\gamma}. \tag{5.18}$$

Then for each $x \in \Omega$, it holds

$$\|x - x^*\|_X \leq \sqrt{\frac{\tilde{\Lambda}^{f,\gamma}(x)}{\beta_a - \alpha_a - \alpha_g - L_F L_\kappa \|\gamma_0\|^2 - \frac{1}{2\gamma}}}. \tag{5.19}$$

Theorem 5.3. *Let $x^* \in \Omega$ be the unique solution of* (5.15). *Under the hypotheses of Theorem* 5.1, *we have*

(i) *for any $\gamma, \rho > 0$, the function $\tilde{\mathfrak{I}}_{\tilde{\Lambda}^{f,\gamma,\rho}} : \Omega \to \mathbb{R}$ defined by*

$$\tilde{\mathfrak{I}}_{\tilde{\Lambda}^{f,\gamma,\rho}}(x) = \inf_{z \in \Omega} \left\{ \tilde{\Lambda}^{f,\gamma}(z) + \rho \|x - z\|_X^2 \right\} \tag{5.20}$$

is the Moreau-Yosida regularized gap function for (5.15).

(ii) *for any $\rho > 0$, if $\gamma > 0$ is such that*

$$\beta_a - \alpha_a - \alpha_g - L_F L_\kappa \|\gamma_0\|^2 > \frac{1}{2\gamma}. \tag{5.21}$$

Then for each $x \in \Omega$ the following bounds holds

$$\|x - x^*\|_{\mathbb{X}} \leq \sqrt{\frac{2\tilde{\mathfrak{I}}_{\tilde{\Lambda}^{f,\gamma,\rho}(x)}}{\min\left\{\beta_a - \alpha_a - \alpha_g - L_F L_\kappa \|\gamma_0\|^2 - \frac{1}{2\gamma}, \rho\right\}}}. \tag{5.22}$$

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