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CONVERGENCE THEOREMS FOR GENERALIZED α-NONEXPANSIVE MAPPINGS IN UNIFORMLY HYPERBOLIC SPACES

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Abstract. In this paper, we establish strong and Δ-convergence theorems for new iteration process namely S-R iteration process for a generalized α-nonexpansive mappings in a uniformly convex hyperbolic space and also we show that our iteration process is faster than other iteration processes appear in the current literature's. Our results extend the corresponding results of Ullah et al. [5], Imdad et al. [16] in the setting of uniformly convex hyperbolic spaces and many more in this direction.

1. Introduction

In this paper, $\mathbb{N}$ denotes the set of all positive integers, while $\Omega(T)$ denotes the set of all fixed point of T. Let C be a nonempty subset of normed space X and mapping $T : C \to C$ is said to be

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(1) *nonexpansive*, if
$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C,$
(2) *quasi nonexpansive*, if
$\|Tx - p\| \leq \|x - p\|, \ \text{for all } x \in C \text{ and } p \in \Omega(T).$

On the other hand, an existence of a solution for an operator equation is established then in many cases, such solution cannot be obtained by using ordinary analytical methods. To overcome such cases, one needs the approximate value of this solution. To do this, we first rearrange the operator equation in the form of fixed point equation $Tx = x$, where T, the fixed point mapping, may be nonlinear. A solution x^* of the problem $Tx = x$, is called a fixed point of the mapping T. Now we apply the most appropriate iterative algorithm on the fixed point equation, and the limit of the sequence generated by this most appropriate algorithm is in fact the value of the desired fixed point for the fixed point equation and the solution for the operator equation.

In 1920 [11], Banach was proved a remarkable know as Banach Fixed Point Theorem it states that every contraction mapping in a complete metric space has a unique fixed point, generated by the sequence $w_{n+1} = Tw_n$ (Picard iterates).

Since for the class of nonexpansive mappings, Picard iterates do not always converge to a fixed point of a certain nonexpansive mapping, we, therefore use some other iterative processes involving different steps and set of parameters. Among the other things, Mann [25], Ishikawa [17], Noor [26], S iteration of Agarwal et al. ([3], [4]), SP iteration of Phuengrattana and Suantai [28], S iteration of Karahan and Ozdemir [19], Normal-S [30], Picard-Mann hybrid [21], Krasnoselskii-Mann [2], Abbas et al. [1], Thakur et al. [33] and Picard-S [18] are the most studied iterative processes. In 2018, Ullah and Arshad introduced M iteration process [34] for Suzuki mappings and proved that it converges faster than all of these iteration processes.

Recently, Ali and Ali [8] introduced the novel iteration process, namely, F iterative scheme for generalized contractions as follows:

$$\begin{cases} w_1 \in C, \\ u_n = T((1-\alpha_n)w_n + \alpha_n T w_n), \\ v_n = T u_n, \\ w_{n+1} = T v_n, \quad n \geq 1. \end{cases} \tag{1.1}$$

In this paper, we establish strong and Δ-convergence theorems for S-R iteration process generated by generalized α-nonexpansive mappings in a uniformly convex Banach space and also show the numerical efficiency of our established results, we provide a new example of generalized α-nonexpansive mappings and we prove that our iteration process is more efficient than many other iterative schemes. Our results extend the corresponding results appear in the current literature's in the setting of uniformly convex hyperbolic spaces.

2. Preliminaries

Let (X, d) be a metric space and C be a nonempty subset of X. In 2008, Suzuki [31] introduce a class of single valued mappings called Suzuki-generalized nonexpansive mappings (or condition (C)) satisfying a condition:

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \implies d(Tx,Ty) \leq d(x,y) \text{ for all } x,y \in C.$$

It is obvious that, every mapping satisfying in condition (C) with a fixed point is quasi-nonexpansive mapping in fact the converse is not true (see Example 2.1. and 2.2., [16], [31]).

In 2011, Aoyama and Kohsaka [9] introduced the class of α-nonexpansive mappings in Banach spaces and obtained fixed point theorems for such mappings ([7], [22]). Ariza-Puiz et al. in [10] showed that the concept of α-nonexpansive is trivial for $\alpha < 0$. It is obvious that every nonexpansive mapping is α-nonexpansive and also every α-nonexpansive mapping with a fixed point is quasi-nonexpansive. In general condition (C) and $\alpha-$nonexpansive mapping are not continuous mappings (see [16] and [31]).

In 2017, Pant and Shukla [27] proved that the notion of of generalized α-nonexpansive maps is weaker than the notion of maps endowed with condition (C).

Let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be generalized α-nonexpansive, if there exists and $\alpha \in [0, 1)$ such that for all $x, y \in C$,

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \implies d(Tx,Ty) \leq \alpha d(x,Ty) + \alpha d(Ty,x) + (1-2\alpha)d(x,y). \tag{2.1}$$

The following results for generalized α-nonexpansive mappings can be found in [27].

Proposition 2.1. ([27]) *Let (X, d) be a metric space and C be a nonempty closed subset of X. Let $T : C \to C$ be a generalized α-nonexpansive for $\alpha \in [0, 1)$ the following hold:*

(i) *Every mapping satisfying condition (C) is a generalized α-nonexpansive mapping, but the converse is not true.*
(ii) *$F(T)$ is closed. Moreover, if C is strictly convex and C is convex, then $F(T)$ is convex.*

(iii) *If T is generalized nonexpansive then for every choice of $x, y \in C$,*

$$d(y, Tx) \leq \left(\frac{3+\alpha}{1-\alpha}\right) d(y, Ty) + d(x, y).$$

Throughout, in this paper we work in the setting of hyperbolic spaces introduced by Kohlenbach [23].

A *hyperbolic space* (X, d, W) is a metric space (X, d) together with a convexity mapping $W : X^2 \times [0, 1] \to X$ satisfying

(W_1) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha) d(u, y)$;
(W_2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;
(W_3) $W(x, y, \alpha) = W(y, x, 1-\alpha)$;
(W_4) $d(W(x, z, \alpha), W(y, w, 1-\alpha)) \leq (1-\alpha) d(x, y) + \alpha d(z, w)$,
for x, y, w and z in X and $\alpha, \beta \in [0, 1]$.

A metric space is said to be a convex metric space in the sense of Takahashi [32], if a triple (X, d, W) satisfy only W_1. The concept of hyperbolic spaces in [23] is more restrictive than the hyperbolic type introduced by Goebel and Kirk [13] since W_1 and W_2 together are equivalent to (X, d, W) being a space of hyperbolic type in [13]. But it is slightly more general than the hyperbolic space defined in Reich and Shafrir [29] (see [15]). This class of metric spaces in [23] covers all normed linear spaces, R-trees in the sense of Tits, the Hilbert ball with the hyperbolic metric (see [6]), Cartesian products of Hilbert balls, Hadamard manifolds (see [29]), and $CAT(0)$ spaces in the sense of Gromov [12]. A thorough discussion of hyperbolic spaces and a detailed treatment of examples can be found in [23] (see also [13, 14, 29]).

If $x, y \in X$ and $\lambda \in [0, 1]$, then we use the notation $(1-\lambda)x \bigoplus \lambda y$ for $W(x, y, \lambda)$. The following holds even for the more general setting of convex metric space [32]: for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(x, (1-\lambda)x \oplus \lambda y) = \lambda d(x, y) \quad \text{and} \quad d(y, (1-\lambda)x \oplus \lambda y) = (1-\lambda) d(x, y).$$

As consequence, we have

$$1\ x \oplus 0\ y = x,\ 0x \oplus 1\ y = y$$

and

$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

A hyperbolic space (X, d, W) is a uniformly convex [24], if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists $\delta \in (0, 1]$ such that for all $a, x, y \in X$,

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta) r,$$

provided $d(x, a) \leq r, d(y, a) \leq r$ and $d(x, y) \geq \epsilon r$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$, which providing such a $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$, is called as a modulus of uniform convexity. We call the function η is monotone if it decreases with r (for fixed ϵ), that is $\eta(r_2,\epsilon) \le \eta(r_1,\epsilon)$ for all $r_2 \ge r_1 > 0$.

In [24], Leustean proved that $CAT(0)$ spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta(r_1,\epsilon) = \frac{\epsilon^2}{8}$ quadratic in ϵ. Thus, the class of uniformly convex hyperbolic spaces is a natural generalization of both uniformly convex Banach spaces and $CAT(0)$ spaces.

Now, we give the concept of Δ-convergence and some of its basic properties.

Let C be a nonempty subset of metric space (X,d) and $\{x_n\}$ be any bounded sequence in X while $diam(C)$ denote the diameter of C. Consider a continuous functional $r_a(.,\{x_n\}) : X \to \mathbb{R}^+$ defined

$$r_a(.,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x),\ \ x \in X.$$

The infimum of $r_a(.,\{x_n\})$ over C is said to be a asymptotic radius of $\{x_n\}$ with respect to C and is denoted by $r_a(C,\{x_n\})$. A point $z \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to C if

$$r_a(z,\{x_n\}) = \inf\{r_a(x,\{x_n\}) : x \in C\},$$

the set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $AC(C,\{x_n\})$. This set may be empty, a singleton, or certain infinitely many points. If the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(z,\{x_n\}) = r_a(\{x_n\})$ and $AC(X,\{x_n\}) = AC(\{x_n\})$, respectively. We know that for $x \in X, r_a(x,\{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$. It is known that every bounded sequence has a unique asymptotic center with respect to each closed convex subset in uniformly convex Banach spaces and even $CAT(0)$ spaces.

The following lemma is due to Leustean [24] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.2. ([24, Proposition 3.3]) *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity g. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Recall that, a sequence $\{x_n\}$ in X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write Δ-lim $x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.3. ([20]) *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{t_n\}$ be a sequence* *in* $[a, b]$ *for some* $a, b \in (0, 1)$*. If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that*

$$\limsup_{n\to\infty} d(x_n, x) \le c,\ \limsup_{n\to\infty} d(y_n, x) \le c,$$
$$\limsup_{n\to\infty} d(W(x_n, y_n, t_n), x) \le c,$$

for some $c \ge 0$*, then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.4. *Let (X, d) be a complete uniformly convex hyperbolic space with monotone modulus of convexity* η*,* C *be a nonempty closed convex subset of* X *and* $T : C \to C$ *be a generalised* α*-nonexpansive mapping for some* $\alpha \in [0, 1)$ *on C. Suppose* $\{x_n\}$ *is bounded sequence in* C *such that*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Then T *has a fixed point.*

Proof. Since $\{x_n\}$ is bounded sequence in X, by Lemma 2.2, has unique asymptotic center in C, that is, $AC(C, \{x_n\}) = \{x_n\}$ is singleton and

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since T is generalised α-nonexpansive for all $x, y \in$ C such that

$$d(x_nTx_n) \le \left(\frac{3+\alpha}{1-\alpha}\right) d(x_n, Tx_n) + d(x_n, x).$$

Taking lim sup as $n \to \infty$ on both side of the above inequality, we get

$$\limsup_{n\to\infty} d(x_nTx_n) \le \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{n\to\infty} d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x),$$

using the fact that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have

$$\begin{aligned} r_a(Tx, \{x_n\}) &= \limsup_{n\to\infty} d(x_nTx_n) \\ &= \limsup_{n\to\infty} d(x_n, x) \\ &= r_a(x, \{x_n\}). \end{aligned}$$

By using the uniqueness of asymptotic center, we have T has fixed point, that is $\Omega(T) \ne \phi$. ☐

3. Main results

In this section, we deal strong convergence and Δ-convergence theorem for iterative sequence which is faster than the sequence defined by [5].

Let C be a nonempty subset of a hyperbolic space X, and $T : C \to C$ be a generalized α-nonexpansive mapping. Let $\{w_n\}$ be sequence generated by

$$\begin{cases} x_n = W(w_n, Tw_n, \alpha_n), \\ u_n = W(Tx_n, 0, 0), \\ v_n = W(Tu_n, 0, 0), \\ w_{n+1} = W(Tv_n, 0, 0), \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is in $[\epsilon, 1-\epsilon]$ for all $n \in \mathbb{N}$ and $\epsilon \in (0,1)$. The (3.1) is know as S-R iteration process.

Now we define the Fejer monotone sequence and its properties.

Definition 3.1. ([16, Lemma 3.1]) Let C be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *Fejer monotone* with respect to C, if for all $x \in C$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

Let C be a nonempty subset of X and T be quasi-nonexpansive such that $\Omega(T) \neq \emptyset$, $x_0 \in C$. Then the Picard iteration

$$x_{n+1} = Tx_n,$$

is Fajer monotone with respect to $\Omega(T)$. The following proposition crucial role in the proof of our result.

Proposition 3.2. ([16, Proposition 3.1]) *Let $\{w_n\}$ be a sequence in $\emptyset \neq C \subset X$. Suppose that $\{w_n\}$ is Fejer monotone with respect to C. Then we have the followings:*

(1) *$\{w_n\}$ is bounded.*
(2) *The sequence $\{w_n\}$ is decreasing and converges to $p \in \Omega(T)$.*

Lemma 3.3. *Let C be a nonempty closed convex subset of a uniformly convex hyperbolic space and $T : C \to C$ be a mapping which satisfies generalized α-nonexpansive mapping. If $\{w_n\}$ is a sequence defined by (3.1), then $\{w_n\}$ is Fejer monotone to $\Omega(T)$.*

Proof. We may take any $p \in \Omega(T)$, then generalized α-nonexpansive mapping beging quasi-nonexpansive. Then by (3.1), we have

$$\begin{aligned} d(w_{n+1}, p) &= d(W(Tv_n, 0, 0), p) \\ &\leq d(Tv_n, p) \\ &\leq d(v_n, p). \end{aligned} \tag{3.2}$$

Again using (3.1) and (3.2), we have

$$\begin{aligned} d(v_n, p) &= d(W(Tu_n, 0, 0), p) \\ &\leq d(Tu_n, p) \\ &\leq d(u_n, p). \end{aligned} \tag{3.3}$$

Again using (3.1) and (3.3), we have

$$\begin{aligned} d(u_n, p) &= d(W(Tx_n, 0, 0), p) \\ &\leq d(Tx_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.4}$$

Using (3.1) and (3.4), we have

$$\begin{aligned} d(x_n, p) &= d(W(w_n.Tw_n, \alpha_n), p) \\ &\leq (1-\alpha_n)d(w_n, p) + \alpha_n d(Tw_n, p) \\ &\leq (1-\alpha_n)d(w_n, p) + \alpha_n d(w_n, p) \\ &\leq d(w_n, p). \end{aligned} \tag{3.5}$$

Consequently, from (3.2), (3.3) and (3.4), we have

$$d(w_{n+1}, p) \leq d(w_n, p) \tag{3.6}$$

for all $p \in \Omega(T)$. Hence $\{w_n\}$ is Fajer monotone with respect to $\Omega(T)$. □

The following theorem gives the necessary and sufficient condition for the existence of fixed points for any given generalized α-nonexpansive mapping in a uniformly convex hyperbolic space.

Lemma 3.4. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and $T : C \to C$ be a generalized α-nonexpansive mapping. If $\{w_n\}$ is a sequence generated by* (3.1)*, then $\Omega(T) \neq \phi$ if and only if the sequence $\{w_n\}$ is bounded and $\lim_{n\to\infty} d(w_n, Tw_n) = 0$.*

Proof. Suppose that the fixed point set $\Omega(T)$ is nonempty. Then, by Lemma 3.2, $\{w_n\}$ is Fejer monotone with respect to $\Omega(T))$ and hence by Proposition 3.2, we have $\lim_{n\to\infty} d(w_n, p)$ exists and $\{w_n\}$ is bounded. Suppose that this limit is equal to some constant k, that is, $\lim_{n\to\infty} d(w_n, p) = k \geq 0$.

If $k = 0$, then it is obvious $\lim_{n\to\infty} d(w_n, Tw_n) = 0$. Now let $k > 0$, since T is a generalized α-nonexpansive mapping with $p \in \Omega(T)$, we have

$$\limsup_{n\to\infty} d(Tw_n, p) \leq \limsup_{n\to\infty} d(w_n, p) \leq k. \tag{3.7}$$

From (3.1) and (3.7), we have

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, p) &\leq \limsup_{n\to\infty} \Big(d(W(w_n, Tw_n, \alpha_n), p) \Big) \\ &\leq \limsup_{n\to\infty} d(w_n, p) \\ &= k. \end{aligned} \tag{3.8}$$

Again from (3.1), we have

$$\begin{aligned} \liminf_{n\to\infty} d(w_{n+1}, p) &\leq \liminf_{n\to\infty} d(v_n, p) \\ &\leq \liminf_{n\to\infty} d(u_n, p) \\ &\leq \liminf_{n\to\infty} d(x_n, p) \\ &\leq k \\ &\leq \liminf_{n\to\infty} d(x_n, p). \end{aligned} \tag{3.9}$$

Hence, from (3.7) and (3.9), we get

$$\lim_{n\to\infty} d(x_n, p) = k. \tag{3.10}$$

Hence, it follows from Lemma 2.3, we have $\lim_{n\to\infty} d(w_n, Tw_n) = 0$.

Conversely, suppose the sequence $\{w_n\}$ is bounded and

$$\lim_{n\to\infty} d(w_n, Tw_n) = 0.$$

Hence, it holds all the assumption of Lemma 2.4, so we have $Tx = x$, that is, $\Omega(T) \neq \emptyset$. □

Theorem 3.5. *Suppose X is a uniformly convex hyperbolic space and C is a nonempty closed convex subset of X with monotone modulus of uniform convexity η and the mapping $T : C \to C$ is the generalized α-nonexpansive mapping. If the sequence $\{w_n\}$ is defined by* (3.1)*, then the sequence $\{w_n\}$ is Δ-convergent to a fixed point of T.*

Proof. From Lemma 3.4, we see that $\{w_n\}$ is a bounded sequence, therefore $\{w_n\}$ has a Δ-convergent subsequence. Now we prove that every Δ-convergent subsequence of $\{w_n\}$ has unique Δ-limit $F(t)$. For this, let a and b be Δ-limits of the subsequences $\{a_n\}$ and $\{b_n\}$ of $\{w_n\}$ respectively. By Lemma 2.2, we have $AC(C, \{a_n\})=\{a\}$ and $AC(C, \{b_n\})=\{b\}$. By Lemma 3.4, we have $\lim_{n\to\infty} d(a_n, Ta_n) = 0$.

We claim that a and b are the fixed points of T and it is unique.

By Lemma 2.4, a and b are the fixed points of T. Now we show that $a = b$. If not, then by uniqueness of asymptotic center

$$\begin{aligned}\limsup_{n\to\infty} d(w_n, a) &= \limsup_{n\to\infty} d(a_n, a) \\ &< \limsup_{n\to\infty} d(a_n, b) \\ &= \limsup_{n\to\infty} d(w_n, b) \\ &= \limsup_{n\to\infty} d(b_n, b) \\ &< \limsup_{n\to\infty} d(b_n, a) \\ &= \limsup_{n\to\infty} d(w_n, a),\end{aligned}$$

which is a contradiction. Hence $a = b$, the sequence $\{w_n\}$ is Δ-convergent to a fixed point of T. □

Theorem 3.6. *Suppose X is any complete uniformly convex hyperbolic space and C be a nonempty closed convex subset of X with monotone modulus of uniform convexity η and the mapping $T : C \to C$ is the generalized α-nonexpansive mapping. If the sequence $\{w_n\}$ is defined by* (3.1)*, converges strongly to some fixed point of T if and only if*

$$\liminf_{n\to\infty} D(w_n, \Omega(T)) = 0,$$

where $D(w_n, \Omega(T)) = \inf_{w\in\Omega(T)} d(w_n, w)$.

Proof. Necessary part is obvious, we have to prove only sufficient part. First we prove that the fixed point set $\Omega(T)$ is closed, let $\{w_n\}$ be a sequence in $\Omega(T)$ which converges to some point $p \in C$. As

$$\begin{aligned}\lambda d(w_n, Tw_n) &= 0 \\ &\le d(w_n, p),\end{aligned}$$

since the mapping $T : C \to C$ is the generalized α-nonexpansive mapping, we have

$$\begin{aligned}d(w_n, Tp) &= d(Tw_n, Tp) \\ &\le d(x_n, p).\end{aligned}$$

By taking the limit of both sides we obtain

$$\begin{aligned}\lim_{n\to\infty} d(w_n, Tp) &\le \lim_{n\to\infty} d(x_n, p) \\ &= 0.\end{aligned}$$

In view of the uniqueness of the limit, we have $p = Tp$, so that $\Omega(T)$ is closed. Suppose

$$\liminf_{n\to\infty} D(w_n, \Omega(T)) = 0.$$

Then, from (3.5)

$$D(w_{n+1}, \Omega(T)) \le D(w_n, \Omega(T)),$$

it follows from the Lemma 3.3 and Proposition 3.2 that

$$\lim_{n\to\infty} d(w_n, \Omega(T))$$

exists. Hence we know that

$$\lim_{n\to\infty} D(w_n, \Omega(T)) = 0.$$

Consider a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that

$$d(w_{n_k}, p_k) < \frac{1}{2^k},$$

for all $k \ge 1$, where p_k is in $\Omega(T)$. By Lemma 3.3, we have

$$\begin{aligned} d(w_{n_{k+1}}, p_k) &\le d(w_{n_k}, p_k) \\ &< \frac{1}{2^k}, \end{aligned}$$

which implies that

$$\begin{aligned} d(p_{k+1}, p_k) &\le d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence. Since $\Omega(T)$ is closed, $\{p_k\}$ is a convergent sequence.

Let $\lim_{k\to\infty} p_k = p$. Then we know that $\{w_n\}$ converges to p. In fact, since

$$\begin{aligned} d(w_{n_k}, p) &\le d(w_{n_k}, p_k) + d(p_k, p) \\ &\to 0 \quad \text{as} \quad k \to \infty, \end{aligned}$$

we have

$$\lim_{k\to\infty} d(w_{n_k}, p) = 0.$$

Since $\lim_{n\to\infty} d(w_n, p)$ exists, the sequence $\{w_n\}$ is convergent to p. □

4. Comparison table and Graph

	SR - Iteration	F - Iteration
1	7.9	7.9
2	7.0646875	7.0646875
3	7.00464941	7.00464941
4	7.00033418	7.00033418
5	7.00002402	7.00002402
6	7.00000173	7.00000173
7	7.00000012	7.00000012
8	7	7.00000001
9	7	7
10	7	7
11	7	7
12	7	7
13	7	7
14	7	7
15	7	7

Figure 1. Comparison Table between SR iteration and F iteration.

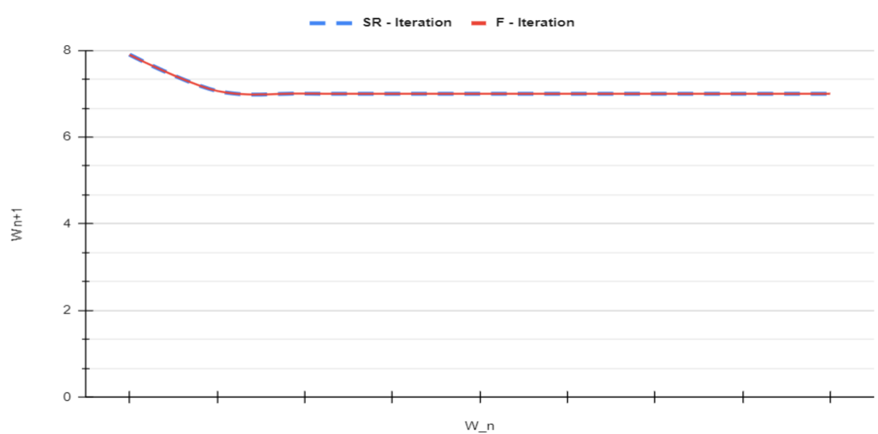

Figure 2. Graph.

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