Bull. Korean Math. Soc. **61** (2024), No. 1, pp. 217–227
https://doi.org/10.4134/BKMS.b230088
pISSN: 1015-8634 / eISSN: 2234-3016

C^*-ALGEBRA OF LOCAL CONJUGACY EQUIVALENCE RELATION ON STRONGLY IRREDUCIBLE SUBSHIFT OF FINITE TYPE

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Abstract. Let G be an infinite countable group and A be a finite set. If $\Sigma \subseteq A^G$ is a strongly irreducible subshift of finite type and $\mathcal{G}$ is the local conjugacy equivalence relation on Σ. We construct a decreasing sequence $\mathcal{R}$ of unital C^*-subalgebras of $C(\Sigma)$ and a sequence of faithful conditional expectations $\mathcal{E}$ defined on $C(\Sigma)$, and obtain a Toeplitz algebra $\mathcal{T}(\mathcal{R}, \mathcal{E})$ and a C^*-algebra $C^*(\mathcal{R}, \mathcal{E})$ for the pair $(\mathcal{R}, \mathcal{E})$. We show that $C^*(\mathcal{R}, \mathcal{E})$ is $*$-isomorphic to the reduced groupoid C^*-algebra $C_r^*(\mathcal{G})$.

1. Introduction

In [7], Ruelle constructed C^*-algebras from the equivalence relation given by homoclinicity satisfying the "Condition C" in expansive dynamical systems of countable groups actions on metrizable spaces by homeomorphisms. Roughly speaking, the above homoclinicity with the "Condition C" means that homoclinic property of two points in the systems can be extended to a "uniform local homoclinicity", and this restriction ensures that the homoclinic equivalence relation under certain topology is étale. As a generalization of this strong version of homoclinic property, Thomsen introduced in [8] the notion of local conjugacy relation in a relatively expansive system and constructed the corresponding equivalence relation C^*-algebra which is called the Ruelle algebra or the homoclinic algebra (associated with the expansive system). It is very interesting to characterize the structure of this kind of algebra. The homoclinic algebra associated with two-sided shift system of a shift space $X \subseteq A^{\mathbb{Z}}$ over a finite A is isomorphic to the Krieger algebra of the shift space X, and for positively expansive group endomorphisms the homoclinic algebra is an AT-algebra ([8]).

Received February 15, 2023; Accepted June 14, 2023.
2020 *Mathematics Subject Classification.* 46L05, 46L85.
Key words and phrases. Local conjugacy equivalence relation, Toeplitz algebras, reduced groupoid C^*-algebra.
This work was financially supported by the NSF of China (Grant No. 12271469,11971419).

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For a finite set A and an infinite countable group G, the canonical shift action of G on each subshift $\Sigma \subseteq A^G$ forms a classic expansive dynamical system. The second author of this paper showed in [4] the homoclinicity and local conjugacy of two points in a strongly irreducible subshift Σ of finite type are consistent. He also proved that the homoclinic algebra associated with this kind of subshift is a minimal AF-algebra.

In [3], Exel and Renault considered a class of examples of approximately proper equivalence relation $\mathcal{R}$ and showed that the associated groupoid C^*-algebra $C_r^*(\mathcal{R})$ is isomorphic to the canonical quotient algebra $C^*(\mathcal{R}, \mathcal{E})$ of the Toeplitz algebra $\mathcal{T}(\mathcal{R}, \mathcal{E})$ associated to a pair $(\mathcal{R}, \mathcal{E})$, where $\mathcal{E}$ is a sequence of suitable conditional expectations. They mainly discussed the tail-equivalence relation on a Bratteli diagram and directly proved that the associated C^*-algebra is isomorphic to the AF-algebra of the diagram. In this paper, we will characterize the structure of the reduced groupoid C^*-algebra associated to the local conjugacy equivalence relation $\mathcal{G}$ on a strongly irreducible subshift of finite type Σ. For this kind of equivalence relation, we construct a decreasing sequence $\mathcal{R}$ of unital C^*-subalgebras of $C(\Sigma)$ and a sequence of faithful conditional expectations $\mathcal{E}$ defined on $C(\Sigma)$. We provide a very specific and different structure to prove that $C^*(\mathcal{R}, \mathcal{E})$ is $*$-isomorphic to the reduced groupoid C^*-algebra $C_r^*(\mathcal{G})$, even though this can be induced from Exel-Renault's result.

Now we recall some notions. For a finite set A and an infinite countable group G, let A^G be the set of all maps u from G into A. Under the topology of pointwise convergence, A^G is a compact metrizable space. Let α be the shift action of G on A^G defined by $gu(h) = u(g^{-1}h)$ for $g, h \in G$ and $u \in A^G$. A closed G-invariant subset of A^G is called a subshift. A subshift Σ is said to be of finite type (we abbreviate it as SFT) if there is a finite subset Ω of G and a subset $\mathcal{P} \subseteq A^\Omega$ such that $\Sigma = \{u \in A^G : (gu)|_\Omega \in \mathcal{P} \text{ for all } g \in G\}$, where $v|_S$ is the restriction of v in A^G to a subset S of G. The full shift, the empty shift and the golden mean shift are of finite type, but the even shift is not of finite type ([5, 9]). A subshift Σ is said to be strongly irreducible if there is a finite subset Δ of G such that Σ is Δ-irreducible, in the sense that, for given any two finite subsets Ω_1 and Ω_2 of G such that $(\Omega_1\Delta) \cap \Omega_2 = \emptyset$, and any two elements $u_1, u_2 \in \Sigma$, there is a $u \in \Sigma$ such that u coincides with u_1 on Ω_1 and with u_2 on Ω_2. Moreover, if Σ is strongly irreducible, then we can choose a finite subset Ω of G and $\mathcal{P} \subseteq A^\Omega$ satisfying that $e \in \Omega = \Omega^{-1}$ and Σ is Ω-irreducible. In particular, A^G is a strongly irreducible subshift of finite type.

Two elements x and y in X are said to be locally conjugate if, for a metric d on X compatible with the topology, there exist open neighborhoods U and V of x and y in X, respectively, and a homeomorphism $\gamma : U \to V$ such that $\gamma(x) = y$ and $\lim_{g\to\infty} d(gz, g\gamma(z)) = 0$ uniformly for $z \in U$. The triple (U, V, γ) is called a local conjugacy from x to y, or just a local conjugacy for short when it is not necessary to emphasize to the points x and y. Note that local conjugacy is an equivalence relation on X which is independent of the choice of the metric d. Endowed with $\mathcal{G}$ the topology whose base consists of all open subsets of the form $\{(z, \gamma z) : z \in U\}$ for local conjugacies (U, V, γ), $\mathcal{G}$ is a separable, locally compact, Hausdorff and r-discrete equivalence relation on X ([7]). Two points x and y in X are homoclinic in the sense that, for a given metric d on X compatible with the topology, $\lim_{g\to\infty} d(gx, gy) = 0$. Clearly, local conjugacy implies homoclinicity. There are two homoclinic, but not locally conjugate points in the even subshift of $\{0,1\}^{\mathbb{Z}}$ ([8]).

2. The Toeplitz algebra arising from the local conjugacy equivalence relation

Let Σ be an infinite strongly irreducible subshift of finite type defined by a finite subset Ω of G. Let $\{G_n\}_{n\geq 1}$ be an increasing sequence of finite subsets of G such that $G_1 = \Omega^3$, $G_n\Omega^3 \subset G_{n+1}$ for $n \geq 1$ and $\cup_{n=1}^{\infty} G_n = G$. For convenience, we let $G_0 = \emptyset$. For $u, v \in A^G$, let

$$k(u,v) = \begin{cases} 1, & \text{if there is } g \in G_1 \\ & \text{with } u(g) \neq v(g); \\ \sup\{n \geq 2 : u(g) = v(g) \text{ for } g \in G_{n-1}\}, & \text{otherwise,} \end{cases}$$

$$d(u,v) = \frac{1}{k(u,v)},$$

where we use the usual convenience $\frac{1}{\infty} = 0$. Then d is a metric on A^G compatible with the pointwise convergence topology.

Remark 2.1. For $u, v \in A^G$, $d(u,v) = 1$ if and only if there exists $g \in G_1$ such that $u(g) \neq v(g)$. Moreover, $d(u,v) \leq \frac{1}{k}$ for $k \geq 2$ if and only if $u(g) = v(g)$ for all $g \in G_{k-1}$.

The following property for strongly irreducible subshifts comes from [1, 4].

Lemma 2.2. *Let F be a finite subset of G and let $\partial F = F\Omega^2 \setminus F$. For $u_1, u_2 \in \Sigma$ such that u_1 coincides with u_2 on ∂F, let $u \in A^G$ be defined by $u(g) = u_1(g)$ for $g \in F\Omega$ and $u(g) = u_2(g)$ for $g \in G \setminus F\Omega$. Then $u \in \Sigma$.*

In particular, given $u_1, u_2 \in \Sigma$ with $u_1|_{G_{k+1}\setminus G_k} = u_2|_{G_{k+1}\setminus G_k}$ for some $k \geq 1$, let $u \in A^G$ be defined by $u(g) = u_1(g)$ for $g \in G_{k+1}$, and $u(g) = u_2(g)$ for $g \in G \setminus G_{k+1}$. Then $u \in \Sigma$.

In [4], Hou proved that two elements $u, v \in \Sigma$ are locally conjugate if and only if they are homoclinic if and only if there exists an integer $n \geq 0$ such that u coincides with v on $G \setminus G_n$.

Let $\mathcal{G} = \{(x, y) \in \Sigma \times \Sigma : x \text{ and } y \text{ are locally conjugate}\}$ be the local conjugacy equivalence relation on Σ. Let $\mathcal{G}_n = \{(u, v) \in \Sigma \times \Sigma : u|_{G\setminus G_n} = v|_{G\setminus G_n}\}$ for $n \geq 0$. Then $\mathcal{G}_0 = \{(u, u) : u \in \Sigma\} \cong \Sigma$, $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ for each $n \geq 0$, and $\mathcal{G} = \cup_{n\geq 0}\mathcal{G}_n$. Each $\mathcal{G}_n$ is a proper equivalence relation on Σ, and under the relative topology of $\Sigma \times \Sigma$, $\mathcal{G}_n$ is an open sub-equivalence relation of $\mathcal{G}_{n+1}$ for all $n \geq 0$. In [4], Hou showed that the inductive limit topology τ on $\mathcal{G}$ coincides with the topology τ_0 given by local conjugatcy, so $\mathcal{G}$ is an AF-groupoid in Renault's sense ([6]).

For $u \in \Sigma$ and $n \geq 0$, let $\mathcal{G}_n(u)$ and $\mathcal{G}(u)$ be the equivalence class of u in $\mathcal{G}_n$ and $\mathcal{G}$, respectively. Then each $\mathcal{G}_n(u)$ is finite, so $\mathcal{G}(u) = \cup_{n=0}^{\infty} \mathcal{G}_n(u)$ is countable. In this paper, we denote by ${}^{\#}S$ the cardinality of a set S.

For each $n \geq 0$, let

$$\mathcal{R}_n = \{f \in C(\Sigma) : f \text{ is constant on each equivalence class } \mathcal{G}_n(u)\}.$$

Then $\mathcal{R}_0 = C(\Sigma)$ and $\{\mathcal{R}_n\}_{n \geq 0}$ is a decreasing sequence of unital C^*-subalgebras of $C(\Sigma)$.

Definition. For each $n \geq 0$, let

$$E_n(f)(u) = \frac{1}{{}^{\#}\mathcal{G}_n(u)} \sum_{w \in \mathcal{G}_n(u)} f(w)$$

for each $f \in C(\Sigma)$ and $u \in \Sigma$.

Lemma 2.3. *Each E_n is a conditional expectation from $C(\Sigma)$ onto $\mathcal{R}_n$ for $n \geq 0$, and $E_n E_m = E_m E_n = E_m$ for $n \leq m$.*

Proof. Clearly, $E_0 = I$, so we can assume that $n \geq 1$.

We first claim that the function ${}^{\#}\mathcal{G}_n(u)$, thus its inverse $\frac{1}{{}^{\#}\mathcal{G}_n(u)}$, are continuous on Σ. In fact, for a sequence $\{u_m\}$ converging to u in Σ, there is a positive integer N such that $d(u_m, u) \leq \frac{1}{n+2}$, thus $u_m|_{G_{n+1}} = u|_{G_{n+1}}$ for all $m \geq N$. Let $m \geq N$ be arbitrary. For each $w \in \mathcal{G}_n(u_m)$, we have $w|_{G \setminus G_n} = u_m|_{G \setminus G_n}$, so $w|_{G_{n+1} \setminus G_n} = u|_{G_{n+1} \setminus G_n}$. From Lemma 2.2, there exists a unique $\widetilde{w} \in \Sigma$ such that $\widetilde{w}|_{G_{n+1}} = w|_{G_{n+1}}$ and $\widetilde{w}|_{G \setminus G_{n+1}} = u|_{G \setminus G_{n+1}}$, thus $\widetilde{w} \in \mathcal{G}_n(u)$. On the other hand, for each $\widetilde{v} \in \mathcal{G}_n(u)$, we have $\widetilde{v}|_{G \setminus G_n} = u|_{G \setminus G_n}$, so $\widetilde{v}|_{G_{n+1} \setminus G_n} = u_m|_{G_{n+1} \setminus G_n}$. From Lemma 2.2, there exists a unique $v \in \Sigma$ such that $v|_{G_{n+1}} = \widetilde{v}|_{G_{n+1}}$ and $v|_{G \setminus G_{n+1}} = u_m|_{G \setminus G_{n+1}}$, thus $v \in \mathcal{G}_n(u_m)$. Hence we have established a bijection from $\mathcal{G}_n(u_m)$ onto $\mathcal{G}_n(u)$, defined by $w \in \mathcal{G}_n(u_m) \mapsto \widetilde{w} \in \mathcal{G}_n(u)$, thus ${}^{\#}\mathcal{G}_n(u_m) = {}^{\#}\mathcal{G}_n(u)$ for all $m \geq N$.

Given $f \in C(\Sigma)$, for $\epsilon > 0$, there exists an integer $k \geq (n+1)$ such that, for $w, \widetilde{w} \in \Sigma$ with $d(w, \widetilde{w}) \leq \frac{1}{k+1}$, i.e., $w|_{G_k} = \widetilde{w}|_{G_k}$, we have $|f(w) - f(\widetilde{w})| < \epsilon$. Let $\{u_m\}$ be a sequence in Σ converging to u. We can choose an integer N such that $d(u_m, u) \leq \frac{1}{k+1}$ $(\leq \frac{1}{n+2})$ for each $m \geq N$. For each $m \geq N$, from the proof of above paragraph, we have ${}^{\#}\mathcal{G}_n(u_m) = {}^{\#}\mathcal{G}_n(u)$ and a bijection φ from $\mathcal{G}_n(u_m)$ onto $\mathcal{G}_n(u)$, defined by $\varphi : w \in \mathcal{G}_n(u_m) \mapsto \widetilde{w} \in \mathcal{G}_n(u)$. One can check that $w|_{G_k} = \widetilde{w}|_{G_k}$, thus $d(w, \widetilde{w}) \leq \frac{1}{k+1}$ for each $w \in \mathcal{G}_n(u_m)$. Hence, for $m \geq N$, we have

$$|E_n(f)(u_m) - E_n(f)(u)| \leq \frac{1}{{}^{\#}\mathcal{G}_n(u_m)} \sum_{w \in \mathcal{G}_n(u_m)} |f(w) - f(\widetilde{w})| \leq \epsilon,$$

so $E_n(f) \in C(\Sigma)$.

Clearly, $E_n(f)(u) = E_n(f)(v)$ for $f \in C(\Sigma)$ and $(u, v) \in \mathcal{G}_n$, so that $E_n(f) \in \mathcal{R}_n$. One can check that E_n is a conditional expectation from $C(\Sigma)$ onto $\mathcal{R}_n$. For two nonnegative integers n and m with $n < m$, since $\mathcal{R}_m \subseteq \mathcal{R}_n$

and E_k is a conditional expectation from $C(\Sigma)$ onto $\mathcal{R}_k$ for each k, we have $E_n(E_m(f)) = E_m(f)$ for each $f \in C(\Sigma)$, so $E_nE_m = E_m$.

For a subset F of G, let $\pi_F : \Sigma \to A^F$ be the restriction mapping defined by $\pi_F(u) = u|_F$ for $u \in \Sigma$, and let $\pi_F(\Sigma)$ be the range of π_F. Let $u \in \Sigma$. For $\eta \in \pi_{G_m \setminus G_n}(\Sigma)$ and $\xi \in \pi_{G_n}(\Sigma)$, we let $u_{\xi,\eta} \in A^G$ be defined by $u_{\xi,\eta}|_{G_n} = \xi$, $u_{\xi,\eta}|_{G_m \setminus G_n} = \eta$ and $u_{\xi,\eta}|_{G \setminus G_m} = u|_{G \setminus G_m}$. For $\eta \in \pi_{G_m \setminus G_n}(\Sigma)$, we set $\Sigma^u_\eta = \{\xi \in \pi_{G_n}(\Sigma) : u_{\xi,\eta} \in \Sigma\}$. For $\xi \in \Sigma^u_\eta$, one can check that $w \in \mathcal{G}_n(u_{\xi,\eta})$ if and only if there exists $\xi' \in \Sigma^u_\eta$ such that $w = u_{\xi',\eta}$. Hence, for $f \in C(\Sigma)$, one gets that

$$\sum_{\xi \in \Sigma^u_\eta} E_n(f)(u_{\xi,\eta}) = \sum_{\xi \in \Sigma^u_\eta} \frac{1}{\#\mathcal{G}_n(u_{\xi,\eta})} \sum_{\xi' \in \Sigma^u_\eta} f(u_{\xi',\eta}) = \sum_{\xi \in \Sigma^u_\eta} f(u_{\xi,\eta}).$$

Also since $v \in \mathcal{G}_m(u)$ if and only if there exist $\eta \in \pi_{G_m \setminus G_n}(\Sigma)$ and $\xi \in \Sigma^u_\eta$ such that $v = u_{\xi,\eta}$, we have

$$\begin{aligned} E_mE_n(f)(u) &= \frac{1}{\#\mathcal{G}_m(u)} \sum_{v \in \mathcal{G}_m(u)} E_n(f)(v) \\ &= \frac{1}{\#\mathcal{G}_m(u)} \sum_{\eta \in \pi_{G_m \setminus G_n}(\Sigma)} \sum_{\xi \in \Sigma^u_\eta} E_n(f)(u_{\xi,\eta}) \\ &= \frac{1}{\#\mathcal{G}_m(u)} \sum_{\eta \in \pi_{G_m \setminus G_n}(\Sigma)} \sum_{\xi \in \Sigma^u_\eta} f(u_{\xi,\eta}) \\ &= \frac{1}{\#\mathcal{G}_m(u)} \sum_{v \in \mathcal{G}_m(u)} f(v) = E_m(f)(u), \end{aligned}$$

thus $E_mE_n = E_m$. □

Given a unital C^*-algebra $\mathcal{A}$, let $\mathcal{R} = \{\mathcal{R}_n\}_{n \geq 0}$ be a decreasing sequence of unital C^*-subalgebras of $\mathcal{A}$ with $\mathcal{R}_0 = \mathcal{A}$. Assume that $\mathcal{E} = \{E_n\}_{n \geq 0}$ is a sequence of faithful conditional expectations defined on $\mathcal{A}$ with $E_n(\mathcal{A}) = \mathcal{R}_n$ and $E_{n+1}E_n = E_{n+1}$ for all $n \geq 0$. Recall that the Toeplitz algebra, $\mathcal{T}(\mathcal{R}, \mathcal{E})$, of the pair $(\mathcal{R}, \mathcal{E})$, is the universal C^*-algebra generated by $\mathcal{A}$ and a sequence $\{\hat{e}_n\}_{n \geq 0}$ of projections subject to the relations:

(i) $\hat{e}_0 = I$, $\hat{e}_{n+1}\hat{e}_n = \hat{e}_n\hat{e}_{n+1}$ for each n;
(ii) $\hat{e}_n a \hat{e}_n = E_n(a)\hat{e}_n$ for each n and $a \in \mathcal{A}$.

By [2, 3.7], the natural map from $\mathcal{A}$ into $\mathcal{T}(\mathcal{R}, \mathcal{E})$ is injective, so $\mathcal{A}$ can be regarded as a unital C^*-subalgebra of $\mathcal{T}(\mathcal{R}, \mathcal{E})$.

Let $\widehat{\mathcal{K}}_n$ be the closed linear span of the set $\{a\hat{e}_n b : a, b \in \mathcal{A}\}$ in $\mathcal{T}(\mathcal{R}, \mathcal{E})$ for each $n \geq 0$, and let $\mathcal{J}$ be the so-called redunancy ideal, which is generated by the following set

$$\left\{ \sum_{k=0}^{n} k_i : 0 \leq n, k_i \in \widehat{\mathcal{K}}_i, 0 \leq i \leq n, \ \sum_{k=0}^{n} k_i x = 0 \text{ for each } x \in \widehat{\mathcal{K}}_n \right\}.$$

The C^*-algebra $C^*(\mathcal{R},\mathcal{E})$ of the pair $(\mathcal{R},\mathcal{E})$, is defined to be the quotient of $\mathcal{T}(\mathcal{R},\mathcal{E})$ by the redunancy ideal $\mathcal{J}$. Let $q:\mathcal{T}(\mathcal{R},\mathcal{E})\to C^*(\mathcal{R},\mathcal{E})$ be the quotient mapping. From [2, 3.7], the restriction of q to $\mathcal{A}$ is injective, so we can identify $a\in\mathcal{A}$ with $q(a)\in C^*(\mathcal{R},\mathcal{E})$, thus $\mathcal{A}$ is a unital C^*-subalgebra of $C^*(\mathcal{R},\mathcal{E})$. Write $q(\hat{e}_n)=e_n$ and $\mathcal{K}_n=q(\widehat{\mathcal{K}}_n)$ for each n. Then $\mathcal{K}_n$ is the closed linear span of $\mathcal{A}e_n\mathcal{A}$ for each n and $C^*(\mathcal{R},\mathcal{E})$ is generated by $\mathcal{A}$ and $\{e_n\}_{n\geq 0}$. If all E_n are of index-finite type, then $\{\mathcal{K}_n\}_{n\geq 0}$ are increasing and $C^*(\mathcal{R},\mathcal{E})$ is the norm-closure of $\bigcup_{n=0}^{\infty}\mathcal{K}_n$ ([2]).

Definition. Let $\mathcal{A}=C(\Sigma)$, $\mathcal{R}=\{\mathcal{R}_n\}_{n\geq 0}$ and $\mathcal{E}=\{E_n\}_{n\geq 0}$. Let $\mathcal{T}(\mathcal{R},\mathcal{E})$ and $C^*(\mathcal{R},\mathcal{E})$ be the C^*-algebras for the pair $(\mathcal{R},\mathcal{E})$.

3. Isomorphism of $C^*(\mathcal{R},\mathcal{E})$ and $C_r^*(\mathcal{G})$

As before, let $\pi_F(\Sigma)$ be the range of the restriction mapping π_F from Σ onto A^F for a subset F of G. Recall that $\mathcal{G}_0=\{(u,u):u\in\Sigma\}$ and $\mathcal{G}_n=\{(u,v)\in\Sigma\times\Sigma : u|_{G\setminus G_n}=v|_{G\setminus G_n}\}$ for each $n\geq 1$. Let $\Lambda_0=\{(\xi,\xi):\xi\in\pi_{G_1}(\Sigma)\}$ and

$$\Lambda_k=\left\{(\xi,\eta)\in\pi_{G_{k+1}}(\Sigma)\times\pi_{G_{k+1}}(\Sigma):\xi|_{G_{k+1}\setminus G_k}=\eta|_{G_{k+1}\setminus G_k}\right\}\quad\text{for } k\geq 1.$$

Clearly, $\Lambda_k\subseteq\pi_{G_{k+1}}(\Sigma)\times\pi_{G_{k+1}}(\Sigma)$ is an equivalence relation on $\pi_{G_{k+1}}(\Sigma)$ for each $k\geq 0$. By Lemma 2.2, for $k\geq 0$ and $\xi,\eta\in\pi_{G_{k+1}}(\Sigma)$, we have (ξ,η) is in Λ_k if and only if there is $(u,v)\in\mathcal{G}_k$ with $u|_{G_{k+1}}=\xi$ and $v|_{G_{k+1}}=\eta$. For $\xi\in\pi_{G_{k+1}(\Sigma)}$, we let $\Lambda_k(\xi)$ be the equivalence class of ξ under Λ_k for each $k\geq 0$.

Lemma 3.1. *For $u\in\Sigma$, we have ${}^{\#}\mathcal{G}_k(u)={}^{\#}\Lambda_k(u|_{G_{k+1}})$ for each $k\geq 0$.*

Proof. If $k=0$, we have ${}^{\#}\mathcal{G}_0(u)={}^{\#}\Lambda_0(u|_{G_1})=1$.

Let $k\geq 1$. If $w,v\in\mathcal{G}_k(u)$, we have $w|_{G_{k+1}\setminus G_k}=u|_{G_{k+1}\setminus G_k}=v|_{G_{k+1}\setminus G_k}$, so we have a mapping $\varphi:\mathcal{G}_k(u)\to\Lambda_k(u|_{G_{k+1}})$, given by $\varphi(v)=v|_{G_{k+1}}$. For $\xi\in\Lambda_k(u|_{G_{k+1}})$, choose $v\in\Sigma$ such that $v|_{G_{k+1}}=\xi$ and $\xi|_{G_{k+1}\setminus G_k}=u|_{G_{k+1}\setminus G_k}$. From Lemma 2.2, the function $w\in A^G$, defined by $w|_{G\setminus G_{k+1}}=u|_{G\setminus G_{k+1}}$ and $w|_{G_{k+1}}=v|_{G_{k+1}}=\xi$ is in Σ, so that $w\in\mathcal{G}_k(u)$. One can check that φ is injective, so it is a bijection. Hence ${}^{\#}\mathcal{G}_k(u)={}^{\#}\Lambda_k(u|_{G_{k+1}})$. $\square$

Let E be a faithful conditional expectation from $\mathcal{A}$ onto its unital C^*-subalgebra $\mathcal{B}$. A finite subset $\{u_1,u_2,\ldots,u_n\}$ of $\mathcal{A}$ is called a quasi-basis for E if $x=\sum_{i=1}^n u_iE(u_i^*x)=\sum_{i=1}^n E(xu_i^*)u_i$ for all $x\in\mathcal{A}$. A faithful conditional expectation E is called to be of index-finite type if there is a quasi-basis for it [10].

Lemma 3.2. *Let $n\geq 0$ be given. For $\xi\in\pi_{G_{n+1}}(\Sigma)$, let $\Sigma_\xi=\{u\in\Sigma : u|_{G_{n+1}}=\xi\}$, I_ξ be the characteristic functional of Σ_ξ, and $\varphi_\xi=\sqrt{{}^{\#}\Lambda_n(\xi)}\,I_\xi$. Then $\{\varphi_\xi:\xi\in\pi_{G_{n+1}}(\Sigma)\}$ is a quasi-basis for E_n, thus E_n is of index-finite type.*

Proof. One can check that Σ_ξ is an open and closed subset of Σ, so that I_ξ and φ_ξ are in $C(\Sigma)$ for each $\xi \in \pi_{G_{n+1}}(\Sigma)$. Since E_0 is the identity map on $\mathcal{A} = C(\Sigma)$, we can assume that $n \geq 1$. For $f \in C(\Sigma)$ and $u \in \Sigma$, from Lemma 3.1, we have

$$\begin{aligned}
&\sum_{\xi \in \pi_{G_{n+1}}(\Sigma)} \varphi_\xi E_n(\varphi_\xi^* f)(u) \\
&= \sqrt{{}^\#\Lambda_n(u|_{G_{n+1}})}\; E_n(\varphi_{u|_{G_{n+1}}} f)(u) \\
&= \sqrt{{}^\#\Lambda_n(u|_{G_{n+1}})} \frac{1}{{}^\#\mathcal{G}_n(u)} \sum_{v \in \Lambda_n(u)} \sqrt{{}^\#\Lambda_n(u|_{G_{n+1}})} I_{u|_{G_{n+1}}}(v) f(v) \\
&= \frac{{}^\#\Lambda_n(u|_{G_{n+1}})}{{}^\#\mathcal{G}_n(u)} f(u) = f(u).
\end{aligned}$$

Hence $\{\varphi_\xi : \xi \in \pi_{G_{n+1}}(\Sigma)\}$ is a quasi-basis for E_n, thus E_n is of index-finite type. $\square$

Lemma 3.3. *For $k \geq 0$ and $\xi, \eta \in \pi_{G_{k+1}}(\Sigma)$, let $e^k_{\xi,\eta} = {}^\#\Lambda_k(\xi) \cdot I_\xi e_k I_\eta \in C^*(\mathcal{R}, \mathcal{E})$, where I_ξ and I_η are as in Lemma 3.2. We have*

(i) *$I_\xi = e^k_{\xi,\xi}$;*

(ii) *$e^k_{\xi,\eta} \neq 0$ if and only if $(\xi, \eta) \in \Lambda_k$. Moreover, when $k = 0$, $e^0_{\xi,\eta} = I_\xi I_\eta \neq 0$ if and only if $\xi = \eta$;*

(iii) *For $(\xi, \eta), (\varsigma, \zeta) \in \Lambda_k$, $e^k_{\xi,\eta} e^k_{\varsigma,\zeta} = \delta_{\eta,\varsigma} e^k_{\xi,\zeta}$, where $\delta_{\eta,\varsigma}$ is the Kronecker symbol.*

Proof. When $k = 0$, we have $\Lambda_0 = \{(\xi, \xi) : \xi \in \pi_{G_1}(\Sigma)\}$, so $e^0_{\xi,\eta} = I_\xi I_\eta = \delta_{\xi,\eta} I_\xi$ for $\xi, \eta \in \pi_{G_1}(\Sigma)$. In this case, (i), (ii) and (iii) hold. Hence we assume that $k \geq 1$ in the following proof.

(i) By Lemma 3.2, $\{\varphi_\xi : \xi \in \pi_{G_{k+1}}(\Sigma)\}$ is a quasi-basis for E_k, where $\varphi_\xi = \sqrt{{}^\#\Lambda_k(\xi)} I_\xi$ for $\xi \in \pi_{G_{k+1}}(\Sigma)$. It follows from [2, 6.2(i)] that

$$\sum_{\eta \in \pi_{G_{k+1}}(\Sigma)} \varphi_\eta e_k \varphi_\eta = e_0 = I.$$

Hence for $\xi \in \pi_{G_{k+1}}(\Sigma)$, we have

$$I_\xi = \sum_{\eta \in \pi_{G_{k+1}}(\Sigma)} I_\xi \varphi_\eta e_k \varphi_\eta = \varphi_\xi e_k \varphi_\xi = {}^\#\Lambda_k(\xi) \cdot I_\xi e_k I_\xi = e^k_{\xi,\xi}.$$

(ii) For $\xi, \eta \in \pi_{G_{k+1}}(\Sigma)$ and $u \in \Sigma$, we have

$$I_\xi E_k(I_\eta)(u) = \frac{1}{{}^\#\mathcal{G}_k(u)} I_\xi(u) \sum_{v \in \mathcal{G}_k(u)} I_\eta(v),$$

so that $I_\xi E_k(I_\eta) \neq 0$ if and only if there exists $(u, v) \in \mathcal{G}_k$ such that $u|_{G_{k+1}} = \xi$ and $v|_{G_{k+1}} = \eta$, i.e., $(\xi, \eta) \in \Lambda_k$. Moreover, in the case that $I_\xi E_k(I_\eta) \neq 0$, we have $I_\xi E_k(I_\eta)(u) = \frac{1}{{}^\#\mathcal{G}_k(u)} I_\xi(u) = \frac{1}{{}^\#\Lambda_k(\xi)} I_\xi(u)$ for each $u \in \Sigma$.

Since $e^k_{\xi,\eta}(e^k_{\xi,\eta})^* = {}^\#\Lambda_k(\xi)^2 I_\xi E_k(I_\eta)e_k I_\xi$, we have $e^k_{\xi,\eta} \neq 0$ if and only if $I_\xi E_k(I_\eta) \neq 0$ if and only if $(\xi,\eta) \in \Lambda_k$.

(iii) For $(\xi,\eta),(\varsigma,\zeta) \in \Lambda_k$, we have $e^k_{\xi,\eta}e^k_{\varsigma,\zeta} = {}^\#\Lambda_k(\xi) \cdot {}^\# \Lambda_k(\varsigma)\ I_\xi e_k I_\eta I_\varsigma e_k I_\zeta$. Thus if $\eta \neq \varsigma$, then $I_\eta I_\varsigma = 0$, so that $e^k_{\xi,\eta}e^k_{\varsigma,\zeta} = 0$.

If $\eta = \varsigma$, then $(\xi,\zeta) \in \Lambda_k$. In this case, from the proof in (ii), we have $I_\xi E_k(I_\eta) = \frac{1}{{}^\#\Lambda_k(\xi)} I_\xi$, thus

$$e^k_{\xi,\eta}e^k_{\eta,\zeta} = {}^\#\Lambda_k(\xi)^2 I_\xi E_k(I_\eta)e_k I_\zeta = {}^\#\Lambda_k(\xi) I_\xi e_k I_\zeta = e^k_{\xi,\zeta}. \qquad \square$$

Lemma 3.4. *Given $k \geq 0$, for $(\xi,\eta) \in \Lambda_k$, let $\Phi = \{(\widetilde{\xi},\widetilde{\eta}) \in \Lambda_{k+1} : \widetilde{\xi}|_{G_{k+1}} = \xi,\ \widetilde{\eta}|_{G_{k+1}} = \eta\}$. Then $e^k_{\xi,\eta} = \sum\limits_{(\widetilde{\xi},\widetilde{\eta})\in\Phi} e^{k+1}_{\widetilde{\xi},\widetilde{\eta}}$.*

Proof. Let $k = 0$. For $(\xi,\xi) \in \Lambda_0$, we have $\Phi = \{(\widetilde{\xi},\widetilde{\xi}) \in \Lambda_1 : \widetilde{\xi}|_{G_1} = \xi\}$. One can check that $I_\xi = \sum\limits_{\widetilde{\xi}\in\pi_{G_2}(\Sigma),\, \widetilde{\xi}|_{G_1}=\xi} I_{\widetilde{\xi}}$. Hence the statement holds for $k = 0$. Next we assume that $k \geq 1$.

For $\zeta \in \pi_{G_{k+2}}(\Sigma)$ and $u \in \Sigma$, one checks that $\sum\limits_{v\in\mathcal{G}_k(u)} I_\zeta(v) \leq 1$, and the equality holds if and only if $u|_{G_{k+2}\setminus G_k} = \zeta|_{G_{k+2}\setminus G_k}$, so that $\sum\limits_{v\in\mathcal{G}_k(u)} I_\zeta(v) = I_{\zeta|_{G_{k+2}\setminus G_k}}(u)$, where $I_{\zeta|_{G_{k+2}\setminus G_k}}(u)$ is the characteristic function of the set

$$\Sigma_{\zeta|_{G_{k+2}\setminus G_k}} = \{u \in \Sigma : u|_{G_{k+2}\setminus G_k} = \zeta|_{G_{k+2}\setminus G_k}\}.$$

Hence, given $\xi \in \pi_{G_{k+1}}(\Sigma)$, $\zeta \in \pi_{G_{k+2}}(\Sigma)$ and $u \in \Sigma$, by noting that ${}^\#\mathcal{G}_k(u) = {}^\#\Lambda_k(u|_{G_{k+1}})$, we have

$$\begin{aligned}(I_\xi E_k(\varphi_\zeta))(u) &= \frac{\sqrt{{}^\#\Lambda_{k+1}(\zeta)}}{{}^\#\mathcal{G}_k(u)} I_\xi(u) \sum_{v\in\mathcal{G}_k(u)} I_\zeta(v) \\ &= \frac{\sqrt{{}^\#\Lambda_{k+1}(\zeta)}}{{}^\#\Lambda_k(\xi)} I_\xi(u) I_{\zeta|_{G_{k+2}\setminus G_k}}(u),\end{aligned}$$

so

$$I_\xi E_k(\varphi_\zeta) = E_k(\varphi_\zeta) I_\xi = \frac{\sqrt{{}^\#\Lambda_{k+1}(\zeta)}}{{}^\#\Lambda_k(\xi)} I_\xi I_{\zeta|_{G_{k+2}\setminus G_k}}.$$

From the proof in Lemma 3.3, $\{\varphi_\zeta : \zeta \in \pi_{G_{k+2}}(\Sigma)\}$ is a quasi-basis for E_{k+1}, it follows from [2, 6.1] that $\{E_k(\varphi_\zeta) : \zeta \in \pi_{G_{k+2}}(\Sigma)\}$ is a quasi-basis for the restriction of E_{k+1} to $\mathcal{R}_k$. By [2, 6.2] again, we have

$$\sum_{\zeta\in\pi_{G_{k+2}}(\Sigma)} E_k(\varphi_\zeta)e_{k+1}E_k(\varphi_\zeta) = e_k.$$

For $(\xi,\eta) \in \Lambda_k$ and $(\widetilde{\xi},\widetilde{\eta}) \in \Phi$, we have ${}^\#\Lambda_k(\xi) = {}^\#\Lambda_k(\eta)$, $\xi|_{G_{k+1}\setminus G_k} = \eta|_{G_{k+1}\setminus G_k}$ and $\widetilde{\xi}|_{G_{k+2}\setminus G_{k+1}} = \widetilde{\eta}|_{G_{k+2}\setminus G_{k+1}}$. Hence each $(\widetilde{\xi},\widetilde{\eta})$ in Φ is determined uniquely by $\widetilde{\xi}$. Let $\mathcal{E} = \{\widetilde{\xi} : (\widetilde{\xi},\widetilde{\eta}) \in \Phi\}$ and $\mathcal{E}(\widetilde{\xi}) = \{\zeta \in \pi_{G_{k+2}}(\Sigma) :$

$\zeta|_{G_{k+2}\setminus G_k} = \widetilde{\xi}|_{G_{k+2}\setminus G_k}\}$ for $\widetilde{\xi} \in \mathcal{E}$. Since $e^k_{\xi,\eta} = {}^{\#}\Lambda_k(\xi) I_\xi e_k I_\eta$, it follows from the equality for e_k that we have

$$\begin{aligned}
e^k_{\xi,\eta} &= {}^{\#}\Lambda_k(\xi) \sum_{\zeta\in\pi_{G_{k+2}}(\Sigma)} I_\xi E_k(\varphi_\zeta) e_{k+1} E_k(\varphi_\zeta) I_\eta \\
&= \sum_{\zeta\in\pi_{G_{k+2}}(\Sigma)} \frac{{}^{\#}\Lambda_{k+1}(\zeta)}{{}^{\#}\Lambda_k(\xi)} I_\xi I_{\zeta|_{G_{k+2}\setminus G_k}} e_{k+1} I_{\zeta|_{G_{k+2}\setminus G_k}} I_\eta \\
&= \sum_{\zeta\in\pi_{G_{k+2}}(\Sigma),\zeta|_{G_{k+1}\setminus G_k}=\xi|_{G_{k+1}\setminus G_k}} \frac{{}^{\#}\Lambda_{k+1}(\zeta)}{{}^{\#}\Lambda_k(\xi)} I_\xi I_{\zeta|_{G_{k+2}\setminus G_k}} e_{k+1} I_{\zeta|_{G_{k+2}\setminus G_k}} I_\eta \\
&= \sum_{\widetilde{\xi}\in\mathcal{E}} \sum_{\zeta\in\mathcal{E}(\widetilde{\xi})} \frac{{}^{\#}\Lambda_{k+1}(\zeta)}{{}^{\#}\Lambda_k(\xi)} I_\xi I_{\widetilde{\xi}|_{G_{k+2}\setminus G_{k+1}}} e_{k+1} I_{\widetilde{\xi}|_{G_{k+2}\setminus G_{k+1}}} I_\eta \\
&= \sum_{\widetilde{\xi}\in\mathcal{E}} {}^{\#}\Lambda_{k+1}(\widetilde{\xi}) I_{\widetilde{\xi}} e_{k+1} I_{\widetilde{\eta}} \\
&= \sum_{(\widetilde{\xi},\widetilde{\eta})\in\Phi} e^{k+1}_{\widetilde{\xi},\widetilde{\eta}}.
\end{aligned}$$

□

Proposition 3.5. *Given $k \geq 0$, let T_k be the subalgebra generated by $\{e^k_{\xi,\eta} : (\xi,\eta) \in \Lambda_k\}$ in $C^*(\mathcal{R},\mathcal{E})$. Then*

(i) *T_k is a finite dimensional subalgebra in $C^*(\mathcal{R},\mathcal{E})$;*

(ii) *$T_k \subseteq T_{k+1}$, and $C^*(\mathcal{R},\mathcal{E})$ is the norm-closure of $\cup_{k=0}^\infty T_k$, so it is a unital AF-algebra.*

Proof. Recall that, for each $k \geq 0$, $\Lambda_k \subseteq \pi_{G_{k+1}}(\Sigma)\times\pi_{G_{k+1}}(\Sigma)$ is an equivalence relation on $\pi_{G_{k+1}}(\Sigma)$. Let $\{\Lambda_k(\xi_1), \Lambda_k(\xi_2), \ldots, \Lambda_k(\xi_{n_k})\}$ be the list of all Λ_k-equivalence classes on $\pi_{G_{k+1}}(\Sigma)$ and denoted by m^i_k the cardinal of the set $\Lambda_k(\xi_i)$ for $i = 1, 2, \ldots, n_k$.

By Lemma 3.3, one can check that $\{e^k_{\xi,\eta} : \xi, \eta \in \Lambda_k(\xi_i)\}$ is a complete set of matrix units with $p^i_k = \sum_{\xi\in\Lambda_k(\xi_i)} e^k_{\xi,\xi}$. Moreover, $\sum_{i=1}^{n_k} p^i_k = I$. Hence the subalgebra, T^i_k, generated by $\{e^k_{\xi,\eta} : \xi, \eta \in \Lambda_k(\xi_i)\}$ in $C^*(\mathcal{R},\mathcal{E})$ for $1 \leq i \leq n_k$ is isomorphic to the $m^i_k \times m^i_k$ matrix algebra $M_{m^i_k}(\mathbb{C})$, thus $T_k = T^1_k \oplus T^2_k \oplus \cdots \oplus T^{n_i}_k$, is isomorphic to the direct sum of matrix algebras $M_{m^1_k}(\mathbb{C}) \oplus M_{m^2_k}(\mathbb{C}) \oplus \cdots \oplus M_{m^{n_k}_k}(\mathbb{C})$. By the above lemma, $T_k \subseteq T_{k+1}$.

Let $\mathcal{B}$ be the norm-closure of $\cup_{k=0}^\infty T_k$ in $C^*(\mathcal{R},\mathcal{E})$. Since $\{\Sigma_\xi : \xi \in \pi_{G_k}(\Sigma), k \geq 0\}$ generates the topology on Σ and $I_\xi = e^k_{\xi,\xi}$ for each $\xi \in \pi_{G_k}(\Sigma)$, we have $\{I_\xi : \xi \in \pi_{G_k}(\Sigma), k \geq 0\}$ generates $C(\Sigma)$ as a C^*-algebra, thus $C(\Sigma)$ is contained in $\mathcal{B}$. Also since

$$e_k = (\sum_{i=1}^{n_k} p^i_k) e_k (\sum_{i=1}^{n_k} p^i_k) = \sum_{i,j=1}^{n_k} \sum_{\eta\in\Lambda_k(\xi_i)} \sum_{\zeta\in\Lambda_k(\xi_j)} I_\eta e_k I_\zeta$$

$$= \sum_{i,j=1}^{n_k} \sum_{\eta \in \Lambda_k(\xi_i)} \sum_{\zeta \in \Lambda_k(\xi_j)} \frac{1}{\#\Lambda_k(\eta)} e_{\eta,\zeta}^k,$$

we have $e_k \in \mathcal{B}$. Hence $C^*(\mathcal{R}, \mathcal{E})$ is the norm-closure of $\cup_{k=0}^{\infty} T_k$. □

Recall that $\mathcal{G}$ is the homoclinic equivalence relation on Σ. In the following we will show that $C^*(\mathcal{R}, \mathcal{E})$ is $*$-isomorphic to the reduced groupoid C^*-algebra $C_r^*(\mathcal{G})$. As in the proof of above proposition, for each $k \geq 0$, we let $\{\Lambda_k(\xi_1), \Lambda_k(\xi_2), \ldots, \Lambda_k(\xi_{n_k})\}$ be the list of all Λ_k-equivalence classes on $\pi_{G_{k+1}}(\Sigma)$ and denote by m_k^i the cardinal of the set $\Lambda_k(\xi_i)$ for $i = 1, 2, \ldots, n_k$.

Firstly, we have the following characterizations of $C_r^*(\mathcal{G})$ given in [4].

Lemma 3.6. *Given $k \geq 0$, for $(\xi, \eta) \in \Lambda_k$, let $F(\xi, \eta) = \{(u, v) \in \mathcal{G}_k,\ u|_{G_{k+1}} = \xi, v|_{G_{k+1}} = \eta\}$ and let $f_{(\xi,\eta)}^k$ be the characteristic function of $F(\xi, \eta)$. Then*

(i) *Each $f_{(\xi,\eta)}^k$ is in $C_r^*(\mathcal{G})$ for $(\xi, \eta) \in \Lambda_k$;*
(ii) *For each $i = 1, 2, \ldots, n_k$, $\{f_{\xi,\eta}^k : \xi, \eta \in \Lambda_k(\xi_i)\}$ is a set of matrix units with $p_k^i = \sum_{\xi \in \Lambda_k(\xi_i)} f_{\xi,\xi}^k$ and $\sum_{i=1}^{n_k} p_k^i = I$. So the subalgebra S_k^i generated by $\{f_{\xi,\eta}^k : \xi, \eta \in \Lambda_k(\xi_i)\}$ in $C_r^*(\mathcal{G})$ is isomorphic to $m_k^i \times m_k^i$ matrix algebra $M_{m_k^i}(\mathbb{C})$, and the subalgebra S_k generated by $\{f_{\xi,\eta}^k : (\xi, \eta) \in \Lambda_k\}$ in $C_r^*(\mathcal{G})$ is equal to $S_k^1 \oplus S_k^2 \oplus \cdots \oplus S_k^{n_k}$;*
(iii) *For $(\xi, \eta) \in \Lambda_k$, $f_{\xi,\eta}^k = \sum_{(\widetilde{\xi},\widetilde{\eta}) \in \Phi} f_{\widetilde{\xi},\widetilde{\eta}}^{k+1}$, where $\Phi = \{(\widetilde{\xi}, \widetilde{\eta}) \in \Lambda_{k+1} : \widetilde{\xi}|_{G_{k+1}} = \xi,\ \widetilde{\eta}|_{G_{k+1}} = \eta\}$ is as in Lemma 3.4, so that $S_k \subseteq S_{k+1}$;*
(iv) *$\cup_{k=0}^{\infty} S_k$ is dense in $C_r^*(\mathcal{G})$ under the reduced norm, thus $C_r^*(\mathcal{G})$ is a unital AF C^*-algebra.*

Lemma 3.7. *For $n \geq 0$, let $\check{e}_n(u, v) = \begin{cases} \frac{1}{\#\mathcal{G}_n(u)}, & \text{if } (u, v) \in \mathcal{G}_n \\ 0, & \text{otherwise,} \end{cases}$ for $(u, v) \in \mathcal{G}$. Then*

(i) *$\{\check{e}_n\}_{n \geq 0}$ is a decreasing sequence of projections in $C_r^*(\mathcal{G})$ with $\check{e}_0 = I$, the unit element in $C_r^*(\mathcal{G})$. Moreover, $\check{e}_n * f * \check{e}_n = E_n(f) * \check{e}_n$ for each n;*
(ii) *For $(\xi, \eta) \in \Lambda_n$, we have $I_\xi \check{e}_n I_\eta = \frac{1}{\#\Lambda_n(\xi)} f_{(\xi,\eta)}^n$.*

By the universal property of the Toeplitz algebra, there is a unique $*$-homomorphism $\pi : \mathcal{T}(\mathcal{R}, \mathcal{E}) \to C_r^*(\mathcal{G})$ such that $\pi(f) = f$ and $\pi(\hat{e}_n) = \check{e}_n$ for each n and $f \in C(\Sigma)$.

Theorem 3.8. *$C^*(\mathcal{R}, \mathcal{E})$ is $*$-isomorphic to the reduced groupoid C^*-algebra $C_r^*(\mathcal{G})$.*

Proof. Let $n \geq 0$, $k_i \in \widehat{\mathcal{K}}_i$ for $0 \leq i \leq n$ such that $\sum_{i=0}^{n} k_i x = 0$ for every $x \in \widehat{\mathcal{K}}_n$, write $k = \sum_{i=0}^{n} k_i$. Then $\pi(k) \in C_r^*(G) \subseteq C_0(\mathcal{G})$. Since the support of each $\pi(k_i)$ is on $\mathcal{G}_n$, we have the support of $\pi(k)$ is also supported on $\mathcal{G}_n$.

For each $(u,v) \in \mathcal{G}_n$, choose $f \in C(\Sigma)$ such that $f(v) = 1$ and $f(w) = 0$ for every $w \in \mathcal{G}_n(v) \setminus \{v\}$. Since $f\hat{e}_n \in \widehat{\mathcal{K}}_n$, we have $kf\widehat{e}_n = 0$, so that $\pi(k) * \pi(f) * \pi(e_n) = 0$, thus $\pi(k) * \pi(f) * \pi(e_n)(u,v) = 0$. By calculation, we have $\pi(k)(u,v) = 0$. Then $\pi(k) = 0$. Hence the restriction of π to the redunancy vanishes, so that it induces a $*$-homomorphism, still denoted by π, from $C^*(\mathcal{R},\mathcal{E})$ onto $C_r^*(\mathcal{G})$. Since $\pi(f) = f$, $\pi(e_n) = \check{e}_n$, it follows from Lemma 3.7 that $\pi(e^k_{\xi,\eta}) = f^k_{(\xi,\eta)}$ for each $k \geq 0$ and $(\xi,\eta) \in \Lambda_k$. Hence π is an isomorphism. $\square$

References

[1] T. Ceccherini-Silberstein and M. Coornaert, *The Myhill property for strongly irreducible subshifts over amenable groups*, Monatsh. Math. **165** (2012), no. 2, 155–172. https://doi.org/10.1007/s00605-010-0256-2

[2] R. Exel and A. Lopes, *C^*-algebras, approximately proper equivalence relations and thermodynamic formalism*, Ergodic Theory Dynam. Systems **24** (2004), no. 4, 1051–1082. https://doi.org/10.1017/S0143385704000148

[3] R. Exel and J. Renault, *AF-algebras and the tail-equivalence relation on Bratteli diagrams*, Proc. Amer. Math. Soc. **134** (2006), no. 1, 193–206. https://doi.org/10.1090/S0002-9939-05-08129-3

[4] C. Hou, *On C^*-algebras from homoclinic equivalences on subshifts of finite type*, Sci. China Math. **64** (2021), no. 4, 747–762. https://doi.org/10.1007/s11425-018-9421-2

[5] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995. https://doi.org/10.1017/CBO9780511626302

[6] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, 793, Springer, Berlin, 1980.

[7] D. Ruelle, *Noncommutative algebras for hyperbolic diffeomorphisms*, Invent. Math. **93** (1988), no. 1, 1–13. https://doi.org/10.1007/BF01393685

[8] K. Thomsen, *C^*-algebras of homoclinic and heteroclinic structure in expansive dynamics*, Mem. Amer. Math. Soc. **206** (2010), no. 970, x+122 pp. https://doi.org/10.1090/S0065-9266-10-00581-8

[9] S. Wacker, *The Moore and the Myhill property for strongly irreducible subshifts of finite type over group sets*, Preprint at https://arxiv.org/abs/1706.05827, 2017.

[10] Y. Watatani, *Index for C^*-subalgebras*, Mem. Amer. Math. Soc. **83** (1990), no. 424, vi+117 pp. https://doi.org/10.1090/memo/0424

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