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A NOTE ON VARIATION CONTINUITY FOR MULTILINEAR MAXIMAL OPERATORS

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Abstract. This note is devoted to establishing the variation continuity of the one-dimensional discrete uncentered multilinear maximal operator. The above result is based on some refine variation estimates of the above maximal functions on monotone intervals. The main result essentially improves some known ones.

1. Introduction

An active topic of current research is to study the regularity properties of maximal operators. The first work was due to Kinnunen [13] who observed that the centered Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ for all $1 < p \leq \infty$. The same boundedness also holds for the uncentered Hardy–Littlewood maximal operator $\widetilde{\mathcal{M}}$ by a simple modification of Kinnunen's arguments or [12, Theorem 1]. Subsequently, more and more authors are devoted to extending Kinnunen's result to various versions (see, e.g., [8, 10, 14, 15, 23, 24]). Since $\mathcal{M}$ is not necessary sublinear at the derivative level, the continuity of $\mathcal{M} : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $p > 1$ is certainly a nontrivial question. This question was first posed in [12, Question 3], where it was attributed to Iwaniec and was addressed by Luiro in [20]. Some extensions of the above continuity result can be found in [8, 21, 25, 28–31]. Due to the unboundedness of $\mathcal{M} : L^1(\mathbb{R}) \to L^1(\mathbb{R})$, understanding the endpoint Sobolev regularity seems to be a deeper issue. A crucial question was posed by Hajłasz and Onninen in [12]:

Question 1.1 ([12]). *Is the map $f \mapsto |\nabla \mathcal{M} f|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$?*

In 2002, Tanaka [26] firstly established that $\widetilde{\mathcal{M}}f$ is weakly differentiable and

$$\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} \leq 2\|f'\|_{L^1(\mathbb{R})}, \tag{1.1}$$

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if $f \in W^{1,1}(\mathbb{R})$. Later on, Aldaz and Pérez Lázaro [1] improved the above result by showing that $\widetilde{\mathcal{M}}f$ is absolutely continuous and

$$\operatorname{Var}\big(\widetilde{\mathcal{M}}f\big) \le \operatorname{Var}(f), \tag{1.2}$$

under the condition that $f \in \mathrm{BV}(\mathbb{R})$. Here $\operatorname{Var}(f)$ denotes here the total variation of f and $\mathrm{BV}(\mathbb{R})$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ with $\|f\|_{\mathrm{BV}(\mathbb{R})} := \operatorname{Var}(f) < \infty$. As a direct application of (1.2), the bound (1.1) can be improved by

$$\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} \le \|f'\|_{L^1(\mathbb{R})} \tag{1.3}$$

if $f \in W^{1,1}(\mathbb{R})$ (see also [17]). It should be pointed out that both the bounds (1.2) and (1.3) are sharp. Recently, Kurka [16] established the bounds (1.2) and (1.3) for $\mathcal{M}$ (with $C = 240,004$). Other extensions related to the above results are [2,6,9,18]. Recently, the endpoint Sobolev and variation continuity of maximal operators has also been studied by many authors. This topic was first investigated by Carneiro, Madrid and Pierce [7] who showed that, among other things, the map $f \mapsto (\widetilde{\mathcal{M}}f)'$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. Note that $\mathrm{BV}(\mathbb{R})$ is not a Banach space. In order to study the variation continuity of maximal operators, the authors of [7] introduced the space $\widetilde{\mathrm{BV}}(\mathbb{R})$ of functions $f : \mathbb{R} \to \mathbb{R}$ of bounded total variation, which is a Banach space with the norm

$$\|f\|_{\widetilde{\mathrm{BV}}(\mathbb{R})} = |f(-\infty)| + \operatorname{Var}(f),$$

where $f(-\infty) := \lim_{x\to-\infty} f(x)$. In [7], the authors also pointed out that the map $\widetilde{\mathcal{M}} : \widetilde{\mathrm{BV}}(\mathbb{R}) \to \widetilde{\mathrm{BV}}(\mathbb{R})$ is bounded. It is natural to ask whether the map $\widetilde{\mathcal{M}} : \widetilde{\mathrm{BV}}(\mathbb{R}) \to \widetilde{\mathrm{BV}}(\mathbb{R})$ is continuous. This was asked by Carneiro, Madrid and Pierce [7] and was answered by González-Riquelme and Kosz [11].

On the other hand, the endpoint regularity properties of discrete maximal operators have also gotten a mount of attention (see [3, 5, 6, 19, 22, 27, 32] et al.). Before presenting some backgrounds, let us introduce some notation and definitions. Let $1 \le p \le \infty$. For a discrete function $f : \mathbb{Z} \to \mathbb{R}$ we define its ℓ^p-norm as usual:

$$\|f\|_{\ell^p(\mathbb{Z})} = \begin{cases} \Big(\sum_{n\in\mathbb{Z}} |f(n)|^p\Big)^{1/p}, & \text{if } 1 \le p < \infty; \\ \sup_{n\in\mathbb{Z}} |f(n)|, & \text{if } p = +\infty. \end{cases}$$

We also define the first derivative of f by

$$f'(n) = f(n+1) - f(n), \quad n \in \mathbb{Z}.$$

The total variation of $f : \mathbb{Z} \to \mathbb{R}$ is defined as

$$\operatorname{Var}(f) := \sum_{n\in\mathbb{Z}} |f(n+1) - f(n)| = \|f'\|_{\ell^1(\mathbb{Z})}.$$

We denote by $\mathrm{BV}(\mathbb{Z})$ the set of all $f : \mathbb{Z} \to \mathbb{R}$ with $\operatorname{Var}(f) < \infty$.

Let $\mathbb{N} = \{0, 1, \dots, \}$ and $m \in \mathbb{N} \setminus \{0\}$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j : \mathbb{Z} \to \mathbb{R}$ being a discrete function. We consider the discrete uncentered m-sublinear maximal operator

$$\widetilde{\mathfrak{M}}(\vec{f})(n) = \sup_{r,s\in\mathbb{N}} \frac{1}{(r+s+1)^m} \prod_{j=1}^{m} \sum_{k=-r}^{s} |f_j(n+k)|, \quad n \in \mathbb{Z}.$$

Clearly, when $m = 1$, the operator $\widetilde{\mathfrak{M}}$ reduces to the one-dimensional discrete uncentered Hardy-Littlewood maximal operator $\widetilde{M}$.

The regularity properties of maximal operators in the discrete setting were first studied by Bober et al. [3] who proved that

$$\mathrm{Var}(\widetilde{M}f) \leq \mathrm{Var}(f),$$

if $f \in \mathrm{BV}(\mathbb{Z})$. Later on, the above result was extended to a centered version in [27], to a fractional version in [6] and to a multilinear version in [32]. In order to establish the variation continuity of discrete maximal operator, the authors of [7] introduced the space of discrete functions of bounded total variation, which is denoted by $\widetilde{\mathrm{BV}}(\mathbb{Z})$ and is a Banach space with the norm

$$\|f\|_{\widetilde{\mathrm{BV}}(\mathbb{Z})} = |f(-\infty)| + \mathrm{Var}(f),$$

where $f(-\infty) := \lim_{n\to-\infty} f(n)$. It is clear that

$$\|f\|_{\ell^\infty(\mathbb{Z})} \leq \|f\|_{\widetilde{\mathrm{BV}}(\mathbb{Z})} \leq 3\|f\|_{\ell^1(\mathbb{Z})}, \tag{1.4}$$

$$\ell^1(\mathbb{Z}) \subset \widetilde{\mathrm{BV}}(\mathbb{Z}) \subset \mathrm{BV}(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}), \tag{1.5}$$

which are proper inclusions. In [7], the authors showed that the map $\widetilde{M} : \widetilde{\mathrm{BV}}(\mathbb{Z}) \to \widetilde{\mathrm{BV}}(\mathbb{Z})$ is bounded and continuous.

Very recently, Zhang [32] established the following result.

Theorem A ([32])**.** (i) *The map* $\widetilde{\mathfrak{M}} : \widetilde{\mathrm{BV}}(\mathbb{Z}) \times \cdots \times \widetilde{\mathrm{BV}}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ *is bounded.*

(ii) *The map* $\widetilde{\mathfrak{M}} : \ell^1(\mathbb{Z}) \times \cdots \times \ell^1(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ *is continuous.*

Based on Theorem A and (1.5), it is natural to ask the following:

Question 1.2. *Is the map* $\widetilde{\mathfrak{M}} : \widetilde{\mathrm{BV}}(\mathbb{Z}) \times \cdots \times \widetilde{\mathrm{BV}}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ *continuous?*

This is the main motivation of this note. A positive answer for Question 1.2 can be shown by the following:

Theorem 1.3. *The map* $\widetilde{\mathfrak{M}} : \widetilde{\mathrm{BV}}(\mathbb{Z}) \times \cdots \times \widetilde{\mathrm{BV}}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ *is continuous.*

Remark 1.4. (i) Theorem 1.3 improves the continuity result in Theorem A because of (1.5).

(ii) It is unknown whether the map $\widetilde{\mathfrak{M}} : \widetilde{\mathrm{BV}}(\mathbb{Z}) \times \cdots \times \widetilde{\mathrm{BV}}(\mathbb{Z}) \to \widetilde{\mathrm{BV}}(\mathbb{Z})$ is bounded and continuous. In fact, by Theorem 1.3, the above continuity question can be reduced to show that $\lim_{j\to\infty} \widetilde{\mathfrak{M}}(\vec{f_j})(-\infty) = \widetilde{\mathfrak{M}}(\vec{f})(-\infty)$ if

$\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in \widetilde{\mathrm{BV}}(\mathbb{Z})$ and $\vec{f_i} = (f_{1,i}, \ldots, f_{m,i})$ with each $f_{j,i} \to f_j$ in $\widetilde{\mathrm{BV}}(\mathbb{Z})$ as $i \to \infty$ for all $1 \le j \le m$.

The rest of the paper is organized as follows. In Section 2 we present the proof of Theorem 1.3. We would like to remark that the main idea in the proof of Theorem 1.3 is motivated by [7, 32]. The main ingredient of the proof of Theorem 1.3 is the variation bounds of the multilinear maximal functions on monotone intervals (see Lemma 2.2). In addition, Theorem 1.3 provides an application of the classical Brezis–Lieb lemma [4]. Throughout this paper, we use the conventions $\prod_{j\in\emptyset} a_j = 1$ and $\sum_{j\in\emptyset} a_j = 0$.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3. For convenience, given an interval $[a, b] \subset \mathbb{Z}$, we denote $\|f\|_{\ell^1([a,b])} = \sum_{n=a}^{b} |f(n)|$ and

$$\mathrm{Var}(f; [a, b]) = \|f'\|_{\ell^1([a,b-1])} = \sum_{n=a}^{b-1} |f(n + 1) - f(n)|$$

for the variation of f on the interval $[a, b]$.

The following is a simple observation, which plays a key role in our proof.

Lemma 2.1. *Let $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in \ell^\infty(\mathbb{Z})$. For $i \ge 1$ let $\vec{f_i} = (f_{1,i}, \ldots, f_{m,i})$, where $\{f_{j,i}\}_{1\le j\le m, i\ge 1} \subset \ell^\infty(\mathbb{Z})$ and $f_{j,i} \to f_j$ in $\ell^\infty(\mathbb{Z})$ as $i \to \infty$. Then $\widetilde{\mathfrak{M}}(\vec{f_i})(n) \to \widetilde{\mathfrak{M}}(\vec{f})(n)$ uniformly for $n \in \mathbb{Z}$.*

Proof. Given $\epsilon \in (0, 1)$ there exists a positive integer N such that

$$\|f_{j,i} - f_j\|_{\ell^\infty(\mathbb{Z})} < \epsilon \quad \text{and} \quad \|f_{j,i}\|_{\ell^\infty(\mathbb{Z})} \le \|f_j\|_{\ell^\infty(\mathbb{Z})} + 1$$

for any $1 \le j \le m$ and $i \ge N$. By the definition of $\widetilde{\mathfrak{M}}(\vec{f})$, we have that for any $n \in \mathbb{Z}$ and $i \ge N$,

$$\begin{aligned}
&|\widetilde{\mathfrak{M}}(\vec{f_i})(n) - \widetilde{\mathfrak{M}}(\vec{f})(n)| \\
&\le \sup_{r,s\in\mathbb{N}} \frac{1}{(r+s+1)^m} \Big| \prod_{j=1}^{m} \sum_{k=-r}^{s} |f_{j,i}(n+k)| - \prod_{j=1}^{m} \sum_{k=-r}^{s} |f_j(n+k)| \Big| \\
&\le \sup_{r,s\in\mathbb{N}} \frac{1}{(r+s+1)^m} \sum_{l=1}^{m} \Big(\sum_{k=-r}^{s} |f_{l,i}(n+k) - f_l(n+k)| \Big) \\
&\quad \times \Big(\prod_{\mu=1}^{l-1} \sum_{k=-r}^{s} |f_\mu(n+k)| \Big) \Big(\prod_{\nu=l+1}^{m} \sum_{k=-r}^{s} |f_{\nu,i}(n+k)| \Big) \\
&\le \sum_{l=1}^{m} \|f_{l,i} - f_l\|_{\ell^\infty(\mathbb{Z})} \prod_{\mu=1}^{l-1} \|f_\mu\|_{\ell^\infty(\mathbb{Z})} \prod_{\nu=l+1}^{m} \|f_{\nu,i}\|_{\ell^\infty(\mathbb{Z})} \\
&\le \sum_{l=1}^{m} \prod_{\substack{1\le\mu\le m,\\ \mu\ne l}} (\|f_\mu\|_{\ell^\infty(\mathbb{Z})} + 1)\epsilon,
\end{aligned}$$

which yields the conclusion of Lemma 2.1. □

The following result is the main ingredient of the proof of Theorem 1.3.

Lemma 2.2. ([32]) *Let* $\vec{f} = (f_1, \ldots, f_m)$ *with each* $f_j \in \widetilde{\mathrm{BV}}(\mathbb{Z})$ *such that* $\widetilde{\mathfrak{M}}(\vec{f})$ *is non-constant* (*in particular,* $\widetilde{\mathfrak{M}}(\vec{f}) \not\equiv \infty$) *and* a, b *be two integers such that* $a < b$.

(i) *If* $\widetilde{\mathfrak{M}}(\vec{f})$ *is non-decreasing in* $[a, b]$ *and* $\widetilde{\mathfrak{M}}(\vec{f})(b-1) < \widetilde{\mathfrak{M}}(\vec{f})(b)$, *then there exists* $s \in \mathbb{N}$ *such that* $s < b - a$ *and*

$$\sum_{n=a}^{b-1} |\widetilde{\mathfrak{M}}(\vec{f})(n+1) - \widetilde{\mathfrak{M}}(\vec{f})(n)| \leq 2\Big(\sum_{l=1}^{m} \prod_{\substack{1 \leq j \leq m \\ j \neq l}} \|f_j\|_{\ell^\infty(\mathbb{Z})} \mathrm{Var}(f_l; [a, b+s])\Big).$$

(ii) *If* $\widetilde{\mathfrak{M}}(\vec{f})$ *is non-increasing in* $[a, b]$ *and* $\widetilde{\mathfrak{M}}(\vec{f})(a) > \widetilde{\mathfrak{M}}(\vec{f})(a+1)$, *then there exists* $r \in \mathbb{N}$ *such that* $r < b - a$ *and*

$$\sum_{n=a}^{b-1} |\widetilde{\mathfrak{M}}(\vec{f})(n+1) - \widetilde{\mathfrak{M}}(\vec{f})(n)| \leq 2\Big(\sum_{l=1}^{m} \prod_{\substack{1 \leq j \leq m \\ j \neq l}} \|f_j\|_{\ell^\infty(\mathbb{Z})} \mathrm{Var}(f_l; [a-r, b])\Big).$$

In order to prove Theorem 1.3, the following definition is very useful.

Definition. For a discrete function $g : \mathbb{Z} \to \mathbb{R}$ we say that an interval $[n, l]$ is a *string of local maxima* of g if

$$g(n-1) < g(n) = \cdots = g(l) > g(l+1).$$

If $n = -\infty$ or $l = +\infty$ (but not both simultaneously) we modify the definition accordingly, eliminating one of the inequalities. If $n = l$, we set $[n, l] = \{n\}$. The rightmost point l of such a string is a *right local maximum* of g, while the leftmost point n is a *left local maximum* of g. We define *string of local minima*, *right local minimum* and *left local minimum* analogously.

We now present the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in \widetilde{\mathrm{BV}}(\mathbb{Z})$. For $i \geq 1$ let $\vec{f_i} = (f_{1,i}, \ldots, f_{m,i})$, where $\{f_{j,i}\}_{1 \leq j \leq m, i \geq 1} \subset \widetilde{\mathrm{BV}}(\mathbb{Z})$ and $f_{j,i} \to f_j$ in $\widetilde{\mathrm{BV}}(\mathbb{Z})$ as $i \to \infty$. It suffices to show that

$$\|(\widetilde{\mathfrak{M}}(\vec{f_i}) - \widetilde{\mathfrak{M}}(\vec{f}))'\|_{\ell^1(\mathbb{Z})} \to 0. \tag{2.1}$$

By Theorem A we see that $(\widetilde{\mathfrak{M}}(\vec{f}))' \in \ell^1(\mathbb{Z})$ and $\{(\widetilde{\mathfrak{M}}(\vec{f_i}))'\}_{i \geq 1} \subset \ell^1(\mathbb{Z})$. From Lemma 2.1 we know that $\widetilde{\mathfrak{M}}(\vec{f_i})(n) \to \widetilde{\mathfrak{M}}(\vec{f})(n)$ uniformly for $n \in \mathbb{Z}$. It follows that

$$(\widetilde{\mathfrak{M}}(\vec{f_i}))'(n) \to (\widetilde{\mathfrak{M}}(\vec{f}))'(n) \quad \text{as} \quad i \to \infty \tag{2.2}$$

uniformly for $n \in \mathbb{Z}$. By the classical Brezis–Lieb lemma [4], for (2.1) it is enough to show that

$$\lim_{i \to \infty} \|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1(\mathbb{Z})} = \|(\widetilde{\mathfrak{M}}(\vec{f}))'\|_{\ell^1(\mathbb{Z})}. \tag{2.3}$$

We now prove (2.3). By (2.2) and Fatou's lemma, we have

$$\|(\widetilde{\mathfrak{M}}(\vec{f}))'\|_{\ell^1(\mathbb{Z})} \le \liminf_{i\to\infty} \|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1(\mathbb{Z})}.$$

Hence, for (2.3) it suffices to prove

$$\limsup_{i\to\infty} \|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1(\mathbb{Z})} \le \|(\widetilde{\mathfrak{M}}(\vec{f}))'\|_{\ell^1(\mathbb{Z})}. \tag{2.4}$$

Let $\epsilon \in (0,1)$. There exists a lager positive integer $L > 0$ such that

$$\max_{1\le j\le m} \|f_j'\|_{\ell^1((-\infty,-L)\cup[L,\infty))} < \epsilon. \tag{2.5}$$

By (2.2), there exists $N_1 = N_1(\epsilon, \vec{f}) > 0$ such that

$$|\widetilde{\mathfrak{M}}(\vec{f_i})(n) - \widetilde{\mathfrak{M}}(\vec{f})(n)| + |(\widetilde{\mathfrak{M}}(\vec{f_i}) - \widetilde{\mathfrak{M}}(\vec{f}))'(n)| < \frac{\epsilon}{2L+1} \tag{2.6}$$

for any $n \in \mathbb{Z}$ and $i \ge N_1$. Then we get from (2.6) that

$$\|(\widetilde{\mathfrak{M}}(\vec{f_i}) - \widetilde{\mathfrak{M}}(\vec{f}))'\|_{\ell^1([-L,L])} < \epsilon \tag{2.7}$$

for any $i \ge N_1$. We can write

$$\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1(\mathbb{Z})} = \|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1([-L,L])} + \|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1((-\infty,-L]\cup[L,\infty))}.$$

By (2.7), we have

$$\begin{aligned}&\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1([-L,L])}\\ &\le \|(\widetilde{\mathfrak{M}}(\vec{f_i}) - \widetilde{\mathfrak{M}}(\vec{f}))'\|_{\ell^1([-L,L])} + \|(\widetilde{\mathfrak{M}}(\vec{f}))'\|_{\ell^1([-L,L])} \le \epsilon + \|(\widetilde{\mathfrak{M}}(\vec{f}))'\|_{\ell^1(\mathbb{Z})}\end{aligned}$$

for any $i \ge N_1$. Hence, for (2.4) it suffices to show that

$$\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1((-\infty,-L]\cup[L,\infty))} \to 0 \ \text{ as } \ i \to \infty. \tag{2.8}$$

In order to prove (2.8), we only prove that

$$\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1([L,\infty))} \to 0 \ \text{ as } \ i \to \infty \tag{2.9}$$

since

$$\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1((-\infty,-L])} \to 0 \ \text{ as } \ i \to \infty$$

can be proved similarly.

We now prove (2.9). By (1.4) and our assumption, there exist a constant $C > 0$ and a positive integer N_2 such that

$$\|f_{j,i}\|_{\ell^\infty(\mathbb{Z})} < C, \quad \forall i \ge 1 \text{ and } \ 1 \le j \le m \tag{2.10}$$

and

$$\mathrm{Var}(f_{j,i} - f_j) < \epsilon, \quad \forall i \ge N_2. \tag{2.11}$$

Let us fix $i \ge \max\{N_1, N_2\}$. Let $\{[a_j^-, a_j^+]\}_{j\in\mathbb{Z}}$ and $\{[b_j^-, b_j^+]\}_{j\in\mathbb{Z}}$ be the ordered strings of local minima and local maxima of $\widetilde{\mathfrak{M}}(\vec{f_i})$ (we allow the possibilities of a_j^- or b_j^- equal to $-\infty$ and a_j^+ or b_j^+ equal to $+\infty$), i.e.,

$$\cdots < a_{-1}^- \le a_{-1}^+ < b_{-1}^- \le b_{-1}^+ < a_0^- \le a_0^+ < b_0^- \le b_0^+ < a_1^- \le a_1^+ < b_1^- \le b_1^+ < \cdots.$$

Without loss of generality we may assume that the sequence doesn't terminate in both sides (since other cases can be obtained similarly). Clearly, $\widetilde{\mathfrak{M}}(\vec{f_i})(b_j^-) > \max\{\widetilde{\mathfrak{M}}(\vec{f_i})(b_j^- -1), \widetilde{\mathfrak{M}}(\vec{f_i})(a_{j+1}^-)\}$, $\widetilde{\mathfrak{M}}(\vec{f_i})(b_j^+) > \max\{\widetilde{\mathfrak{M}}(\vec{f_i})(b_j^+ + 1), \widetilde{\mathfrak{M}}(\vec{f_i})(a_j^+)\}$, $\widetilde{\mathfrak{M}}(\vec{f_i})$ is non-decreasing in $[a_j^+, b_j^-]$ and $\widetilde{\mathfrak{M}}(\vec{f_i})$ is non-increasing in $[b_j^+, a_{j+1}^-]$. For any $j \in \mathbb{Z}$, invoking Lemma 2.2, there exist $r_j, s_j \in \mathbb{N}$ such that $s_j < a_{j+1}^- - b_j^-$, $r_j < b_j^+ - a_j^+$ and

$$\begin{aligned}&\sum_{n=b_j^+}^{a_{j+1}^- -1} |\widetilde{\mathfrak{M}}(\vec{f_i})(n+1) - \widetilde{\mathfrak{M}}(\vec{f_i})(n)| \\ &\leq 2\sum_{l=1}^{m} \prod_{\substack{1\leq \mu \leq m \\ \mu\neq l}} \|f_{\mu,i}\|_{\ell^\infty(\mathbb{Z})} \mathrm{Var}(f_{l,i}; [b_j^+ - r_j, a_{j+1}^-]),\end{aligned} \tag{2.12}$$

$$\begin{aligned}&\sum_{n=a_j^+}^{b_j^- -1} |\widetilde{\mathfrak{M}}(\vec{f_i})(n+1) - \widetilde{\mathfrak{M}}(\vec{f_i})(n)| \\ &\leq 2\sum_{l=1}^{m} \prod_{\substack{1\leq \mu \leq m \\ \mu\neq l}} \|f_{\mu,i}\|_{\ell^\infty(\mathbb{Z})} \mathrm{Var}(f_{l,i}; [a_j^+, b_j^- + s_j]).\end{aligned} \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\begin{aligned}&\sum_{n=a_j^+}^{a_{j+1}^+ -1} |\widetilde{\mathfrak{M}}(\vec{f})(n+1) - \widetilde{\mathfrak{M}}(\vec{f})(n)| \\ &\leq 4\sum_{l=1}^{m} \prod_{\substack{1\leq \mu \leq m \\ \mu\neq l}} \|f_{\mu,i}\|_{\ell^\infty(\mathbb{Z})} \mathrm{Var}(f_{l,i}; [a_j^+, a_{j+1}^+]).\end{aligned} \tag{2.14}$$

Without loss of generality we may assume that $[a_0^-, a_0^+]$ is the first string of local minima of $\widetilde{\mathfrak{M}}(\vec{f})$ on the interval $[L, \infty)$. Then the following two cases are needed to be considered:

Case 1. ($[b_{-1}^-, b_{-1}^+] \subset [L, a_0^-]$ is not true). In this case we have that $\widetilde{\mathfrak{M}}(\vec{f_i})$ is non-increasing in $[L, a_0^-]$. Then we write

$$\begin{aligned}\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1([L,\infty))} &= \mathrm{Var}(\widetilde{\mathfrak{M}}(\vec{f_i}); [L,\infty)) \\ &= \widetilde{\mathfrak{M}}(\vec{f_i})(L) - \widetilde{\mathfrak{M}}(\vec{f_i})(a_0^-) \\ &\quad + \sum_{j=0}^{\infty} \sum_{n=a_j^+}^{a_{j+1}^+ -1} |\widetilde{\mathfrak{M}}(\vec{f_i})(n+1) - \widetilde{\mathfrak{M}}(\vec{f_i})(n)|.\end{aligned} \tag{2.15}$$

By (2.5) and (2.6), we have

$$\begin{aligned}
&\widetilde{\mathfrak{M}}(\vec{f_i})(L)-\widetilde{\mathfrak{M}}(\vec{f_i})(a_0^-)\\
&=\widetilde{\mathfrak{M}}(\vec{f_i})(L)-\widetilde{\mathfrak{M}}(\vec{f})(L)+\widetilde{\mathfrak{M}}(\vec{f})(L)-\widetilde{\mathfrak{M}}(\vec{f})(a_0^-)\\
&\quad+\widetilde{\mathfrak{M}}(\vec{f})(a_0^-)-\widetilde{\mathfrak{M}}(\vec{f_i})(a_0^-)\\
&\leq 2\epsilon+|\widetilde{\mathfrak{M}}(\vec{f})(L)-\widetilde{\mathfrak{M}}(\vec{f})(a_0^-)|\\
&\leq 2\epsilon+\mathrm{Var}(\widetilde{\mathfrak{M}}(\vec{f});[L,\infty])\leq 3\epsilon.
\end{aligned}\tag{2.16}$$

On the other hand, we get from (2.14), (2.5), (2.10) and (2.11) that

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{n=a_j^+}^{a_{j+1}^+-1}|\widetilde{\mathfrak{M}}(\vec{f_i})(n+1)-\widetilde{\mathfrak{M}}(\vec{f_i})(n)|\\
&\leq 4\sum_{j=0}^{\infty}\sum_{l=1}^{m}\prod_{\substack{1\leq\mu\leq m\\ \mu\neq l}}\|f_{\mu,i}\|_{\ell^\infty(\mathbb{Z})}\mathrm{Var}(f_{l,i};[a_j^+,a_{j+1}^+])\\
&\leq 4C^{m-1}\sum_{l=1}^{m}(\mathrm{Var}(f_{l,i}-f_l;[L,\infty))+\mathrm{Var}(f_l;[L,\infty)))\\
&\leq 8C^{m-1}m\epsilon.
\end{aligned}\tag{2.17}$$

Combining (2.17) with (2.16) and (2.15) leads to

$$\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1([L,\infty))}\leq(8C^{m-1}m+3)\epsilon.$$

This gives (2.9) in this case.

Case 2. ($[b_{-1}^-,b_{-1}^+]\subset[L,a_0^-]$). In this case we have that $\widetilde{\mathfrak{M}}(\vec{f_i})$ is non-decreasing in $[L,b_{-1}^-]$. Then we have

$$\begin{aligned}
&\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1([L,\infty))}\\
&=\mathrm{Var}(\widetilde{\mathfrak{M}}(\vec{f_i});[L,\infty))\\
&=\widetilde{\mathfrak{M}}(\vec{f_i})(b_{-1}^-)-\widetilde{\mathfrak{M}}(\vec{f_i})(L)+\widetilde{\mathfrak{M}}(\vec{f_i})(b_{-1}^+)-\widetilde{\mathfrak{M}}(\vec{f_i})(a_0^-)\\
&\quad+\sum_{j=0}^{\infty}\sum_{n=a_j^+}^{a_{j+1}^+-1}|\widetilde{\mathfrak{M}}(\vec{f_i})(n+1)-\widetilde{\mathfrak{M}}(\vec{f_i})(n)|.
\end{aligned}\tag{2.18}$$

An argument similar to (2.16) gives

$$\widetilde{\mathfrak{M}}(\vec{f_i})(b_{-1}^-)-\widetilde{\mathfrak{M}}(\vec{f_i})(L)\leq 3\epsilon;$$

$$\widetilde{\mathfrak{M}}(\vec{f_i})(b_{-1}^+)-\widetilde{\mathfrak{M}}(\vec{f_i})(a_0^-)\leq 3\epsilon.$$

These above estimates together with (2.17) and (2.18) imply that

$$\|(\widetilde{\mathfrak{M}}(\vec{f_i}))'\|_{\ell^1([L,\infty))}\leq(8C^{m-1}m+6)\epsilon.$$

This gives (2.9) in this case and completes the proof of Theorem 1.3. □

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