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WELL-POSEDNESS AND ASYMPTOTIC BEHAVIOR OF PARTLY DISSIPATIVE REACTION DIFFUSION SYSTEMS WITH MEMORY

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Abstract. In this paper, we consider the asymptotic behavior of solutions for the partly dissipative reaction diffusion systems of the FitzHugh-Nagumo type with hereditary memory and a very large class of nonlinearities, which have no restriction on the upper growth of the nonlinearity. We first prove the existence and uniqueness of weak solutions to the initial boundary value problem for the above-mentioned model. Next, we investigate the existence of a uniform attractor of this problem, where the time-dependent forcing term $h \in L^2_b(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ is the only translation bounded instead of translation compact. Finally, we prove the regularity of the uniform attractor $\mathcal{A}$, i.e., $\mathcal{A}$ is a bounded subset of $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2_\mu(\mathbb{R}^+, H^2(\mathbb{R}^N))$. The results in this paper will extend and improve some previously obtained results, which have not been studied before in the case of non-autonomous, exponential growth nonlinearity and contain memory kernels.

1. Introduction

In this paper, we consider the following initial boundary value problem of partly dissipative reaction diffusion system with memory:

$$\begin{cases} u_t - \Delta u + \lambda u - \int_0^\infty \kappa(s)\Delta u(x,t-s)ds & \\ +f(u) + g(x,v) = h(x,t), & x \in \mathbb{R}^N,\ t > \tau, \\ v_t + \sigma(x)v + \varphi(x,u) = 0, & x \in \mathbb{R}^N,\ t > \tau, \\ u(x,t) = u_\tau(x),\ v(x,t) = v_\tau(x), & x \in \mathbb{R}^N,\ t \le \tau, \end{cases} \tag{1.1}$$

where $N \ge 3$, λ is positive, the nonlinearity f, g, φ and the external force h satisfy some specified conditions later. The system (1.1) (with $\kappa \equiv 0$) arose as model that describes the signal transmission across axons and is a model of FitzHugh-Nagumo equations in neurobiology (see, e.g., [4, 15]).

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The long-time behavior of solutions to problem (1.1) in a bounded domain and the autonomous case, that is, the case when h is independent of time t, has been studied by several authors.

In 1989, Marion [14] considered an initial boundary value problem of partly dissipative reaction diffusion system in $\Omega \subset \mathbb{R}^N$. She proved that the system there exists a unique weak solution and the existence of a global attractor in $L^2(\Omega) \times L^2(\Omega)$. A boundedness of the Hausdorff and fractal dimensions of the attractor are also studied by her. In this direction, there are many results related to the dynamics of the partly reaction diffusion systems (see [2, 12–14, 19, 20]). Recently, L. Jihoon and V. M. Toi [10] also considered the system in exponential growth nonlinearity case and showed that there existence of weak solutions, the regularity of the global attractor and the exponential stability of stationary solutions of the systems.

For the case of unbounded domains, the long-time behavior of the solutions of the partly dissipative reaction diffusion system has been studied by some authors (see [16, 18, 24]). However, most existing papers deal with the partly reaction diffusion systems in which $\kappa \equiv 0$, $h \equiv h(x)$, $g(x, v) \equiv \alpha v$, $\varphi(x, u) \equiv \beta u$ and the nonlinearity f is of Sobolev type.

It is well known that non-autonomous equations appear in many applications in the natural sciences, so they are of great importance and interest. In this paper, we will study the long-time behavior of solutions to the partly dissipative reaction diffusion system by allowing the external force h to depend on time t; exponential growth nonlinearity and contain memory kernels.

To study problem (1.1), we assume that the nonlinearities f, g, φ and the external force h satisfy the following conditions:

(**H1**) The nonlinearities f, g, φ and the function σ satisfy

$$f'_u(u) \geq -\ell, \ f(u)u \geq -\delta_1 u^2 \text{ for all } u \in \mathbb{R}, \text{ and } f(0) = 0, \tag{1.2}$$

$$|g'_v(x, v)| \leq \delta_2, \tag{1.3}$$

$$|\varphi'_u(x, u)| \leq \delta_3, \tag{1.4}$$

$$|\varphi'_{x_j}(x, u)| \leq \delta_3(|\phi_1(x)| + |u|), \ 1 \leq j \leq N, \ \phi_1 \in L^2(\mathbb{R}^N);$$

$$\sigma(x) \text{ is bounded on } \mathbb{R}^N, \ |\sigma'(x)| \leq m \text{ and } \sigma(x) \geq \delta_4 > 0, \tag{1.5}$$

where ℓ, δ_i $(i = 1, \dots, 4)$ are positive constants with $\delta_4 > \frac{2(\delta_5^2 + \delta_6^2)}{\lambda - \delta_1} > 0$, and $\lambda > \ell$.

From the above conditions (1.3) and (1.4), we can choose positive constants δ_5 and δ_6 satisfying

$$|g(x, v)| \leq \delta_5(\phi_2(x) + |v|), \ \forall v \in \mathbb{R}, \ \forall x \in \mathbb{R}^N, \tag{1.6}$$

$$|\varphi(x, u)| \leq \delta_6(\phi_3(x) + |u|), \ \forall u \in \mathbb{R}, \ \forall x \in \mathbb{R}^N, \tag{1.7}$$

where $\phi_2, \phi_3 \in L^2(\mathbb{R}^N)$ are nonnegative functions.

(**H2**) The convolution (or memory) kernel κ is a nonnegative summable function having the explicit form

$$\kappa(s) = \int_s^\infty \mu(r)dr,$$

where $\mu \in L^1(\mathbb{R}^+)$ is a decreasing (hence nonnegative) piecewise absolutely continuous in each interval $[\tau, T]$ with $T > 0$. In particular, μ is allowed to exhibit (infinitely many) jumps. Moreover, we require that

$$\kappa(s) \le \theta\mu(s) \tag{1.8}$$

for some $\theta > 0$ and every $s > 0$. As shown in Gatti et al. [5], this is completely equivalent to the requirement that

$$\mu(r+s) \le Me^{-\delta r}\mu(s) \tag{1.9}$$

for some $M \ge 1$, $\delta > 0$, every $r \ge 0$ and almost every $s > 0$.

(**H3**) The function $h \in L^2_b(\mathbb{R}; H^{-1}(\mathbb{R}^N))$, the space of translation bounded functions in $L^2_{\text{loc}}(\mathbb{R}; H^{-1}(\mathbb{R}^N))$, that is,

$$\|h\|^2_{L^2_b} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|h(r)\|^2_{H^{-1}(\mathbb{R}^N)}dr < +\infty.$$

For $h \in L^2_b\left(\mathbb{R}; H^{-1}(\mathbb{R}^N)\right)$, we denote by $\mathcal{H}_w(h)$ the closure of the set $\{h(\cdot + r) \,|\, h \in \mathbb{R}\}$ in $L^2_b(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ with the weak topology. Noting that, as in [3, Chapter 5, Proposition 4.2], we have: for all $\varsigma \in \mathcal{H}_w(h)$, then

$$\|\varsigma\|^2_{L^2_b} \le \|h\|^2_{L^2_b}.$$

Remark 1.1. We can replace the nonlinear function $f(u)$ in equation (1.1) into a function $f(x,u)$, of the form $f(x,u) = f_1(u) + a(x)f_2(u)$, where $f_i(u)$, $i = 1, 2$ satisfy the hypothesis as (1.2) and $a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then we also get the same results as in the case of $f(u)$.

To this aim, as in [7,8], we consider a new variable which reflects the history of (1.1), that is

$$\eta^t(x,s) = \eta(x,t,s) = \int_0^s u(x,t-r)dr, \ s \ge 0.$$

We can check that

$$\partial_t\eta^t(x,s) = u(x,t) - \partial_s\eta^t(x,s), \ s \ge 0.$$

Since $\mu(s) = -\kappa'(s)$, problem (1.1) can be transformed into the following system

$$\begin{cases} u_t - \Delta u + \lambda u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds & \\ +f(u) + g(\cdot, v) = h(t), & t > \tau, \\ v_t + \sigma(\cdot)v + \varphi(\cdot, u) = 0, & t > \tau, \\ \partial_t\eta^t(s) = -\partial_s\eta^t(s) + u(t), & t > \tau, \ s \ge 0, \\ u(t) = u_\tau, \ v(t) = v_\tau, \ \eta^t(s) = \eta^\tau(s) = \eta_\tau(s) & t \le \tau, \ s \in \mathbb{R}^+. \end{cases} \tag{1.10}$$

Note that in the above system, we use the same notations f, g, h, σ and φ as Nemytskii operators induced from (1.1).

Now, denote

$$z(t) = (u(t), v(t), \eta^t) \text{ and } z_\tau = (u_\tau, v_\tau, \eta_\tau).$$

Let $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ the norm and scalar product in $L^2(\mathbb{R}^N)$, respectively.

In view of (1.9), as in [8], let $L^2_\mu(\mathbb{R}^+, L^2(\mathbb{R}^N))$ be the Hilbert space of functions $\xi\colon \mathbb{R}^+ \to L^2(\mathbb{R}^N)$ endowed with the inner product

$$\langle \xi_1, \xi_2 \rangle_\mu = \int_0^\infty \mu(s) \langle \xi_1(s), \xi_2(s) \rangle \, ds,$$

and let $\|\xi\|_\mu$ denote the corresponding norm. In a similar manner, we introduce the inner products $\langle \cdot, \cdot \rangle_{i,\mu}$ on $L^2_\mu(\mathbb{R}^+, H^i(\mathbb{R}^N))$, $i = 1, 2$ by

$$\langle \cdot, \cdot \rangle_{1,\mu} = \langle \cdot, \cdot \rangle_\mu + \langle \nabla \cdot, \nabla \cdot \rangle_\mu \,, \quad \langle \cdot, \cdot \rangle_{2,\mu} = \langle \cdot, \cdot \rangle_\mu + \langle \nabla \cdot, \nabla \cdot \rangle_\mu + \langle \Delta \cdot, \Delta \cdot \rangle_\mu \,,$$

and the corresponding norms are denoted by $\| \cdot \|_{1,\mu}$, $\| \cdot \|_{2,\mu}$.

We now introduce the following Hilbert spaces

$$\begin{aligned}
\mathcal{H} &= L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N)), \\
\mathcal{H}_1 &= H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2_\mu(\mathbb{R}^+, H^2(\mathbb{R}^N)), \\
\mathcal{H}_2 &= H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2_\mu(\mathbb{R}^+, H^2(\mathbb{R}^N)),
\end{aligned}$$

which are, respectively, endowed with the norms induced on $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are

$$\begin{aligned}
\|(u, v, \xi)\|^2_{\mathcal{H}} &= \|u\|^2 + \|v\|^2 + \int_0^\infty \mu(s) \|\xi(s)\|^2_{H^1(\mathbb{R}^N)} ds, \\
\|(u, v, \xi)\|^2_{\mathcal{H}_1} &= \|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \int_0^\infty \mu(s) \|\xi(s)\|^2_{H^2(\mathbb{R}^N)} ds, \\
\|(u, v, \xi)\|^2_{\mathcal{H}_2} &= \|u\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2 + \|v\|^2 + \|\nabla v\|^2 \\
&\quad + \int_0^\infty \mu(s) \|\xi(s)\|^2_{H^2(\mathbb{R}^N)} ds.
\end{aligned}$$

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of weak solutions of the system (1.10) by using the Faedo-Galerkin method and the technique involving the weak convergence in Orlicz space. In Section 3, we prove the existence of an uniform attractor $\mathcal{A}$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N))$. In the last section, we prove that the uniform attractor $\mathcal{A}$ is bounded in $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2_\mu(\mathbb{R}^+, H^2(\mathbb{R}^N))$.

2. Existence of weak solutions

Firstly, we define a weak solution of problem (1.10) as follows:

Definition 2.1. A function $z = (u, v, \eta^t)$ is called a weak solution of problem (1.10) on the interval $[\tau, T]$ with the initial datum $z(\tau) = z_\tau \in \mathcal{H}$ if

$$
\begin{aligned}
&u \in C([\tau,T]; L^2(\mathbb{R}^N)) \cap L^2(\tau, T; H^1(\mathbb{R}^N)),\\
&v \in C([\tau,T]; L^2(\mathbb{R}^N)),\\
&\eta^t_t + \eta^t_s \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+, L^2(\mathbb{R}^N))) \cap L^2(\tau, T; L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N)))
\end{aligned}
$$

for any $T > \tau$, and the first and second equations of (1.10) hold in $L^1(\tau, T; H^{-1}(\mathbb{R}^N) + L^1(\mathbb{R}^N) + L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^N)))$ and $L^2(\tau, T; L^2(\mathbb{R}^N))$, respectively, and the initial conditions in (1.10) are satisfied.

Using the Galerkin type approximations, we can prove the following existence and uniqueness result.

Theorem 2.1. *Assume that hypotheses* **(H1)-(H3)** *hold. Then, for any $z_\tau = (u_\tau, v_\tau, \eta_\tau) \in \mathcal{H}$, any $\varsigma \in \mathcal{H}_w(h)$ and $T > \tau$, $\tau \in \mathbb{R}$ given, problem* (1.10) (*with ς in place of h*) *has a unique weak solution $z = (u, v, \eta^t)$ on the interval $[\tau, T]$ satisfying*

$$z \in C([\tau,T]; \mathcal{H}).$$

Moreover, the weak solutions depend continuously on the initial data.

Proof. Step 1. Existence. For each integer $n \geq 1$, we denote by P_n and Q_n the projections on the subspaces

$$\mathrm{span}(\psi_1, \ldots, \psi_n) \subset L^2(\mathbb{R}^N) \quad \text{and} \quad \mathrm{span}(\xi_1, \ldots, \xi_n) \subset L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N)),$$

respectively. Consider the approximate solution $z_n(t) = (u_n(t), v_n(t), \eta^t_n)$ in the form

$$u_n(t) = \sum_{j=1}^n u_{nj}(t)\psi_j,\ v_n(t) = \sum_{j=1}^n v_{nj}(t)\psi_j \quad \text{and} \quad \eta^t_n(s) = \sum_{j=1}^n \eta_{nj}(t)\xi_j(s)$$

satisfying

$$
\begin{aligned}
(2.1)\qquad &\langle(\partial_t u_n, \partial_t \eta^t_n), (\psi_k, \xi_j)\rangle_{L^2(\mathbb{R}^N)\times L^2_\mu(\mathbb{R}^+,H^1(\mathbb{R}^N))}\\
&= \langle(\Delta u_n - \lambda u_n + \int_0^\infty \mu(s)\Delta\eta^t_n(s)ds)\rangle_{L^2(\mathbb{R}^N)\times L^2_\mu(\mathbb{R}^+,H^1(\mathbb{R}^N))}\\
&\quad - \langle(P_n f(u_n) - P_n g(\cdot, v_n) + P_n\varsigma, u_n - \partial_s\eta^t_n),\\
&\qquad (\psi_k, \xi_j)\rangle_{L^2(\mathbb{R}^N)\times L^2_\mu(\mathbb{R}^+,H^1(\mathbb{R}^N))},\\
&(\partial_t v_n, \psi_k) = (-\varsigma(\cdot)v_n - P_n\varphi(\cdot, u_n), \psi_k),\\
&u_n(\tau) = P_n u_\tau \to u_\tau = \sum_{j=1}^\infty \alpha_j\psi_j \ \text{ in } L^2(\mathbb{R}^N) \text{ as } n \to \infty,\\
&v_n(\tau) = P_n v_\tau \to v_\tau = \sum_{j=1}^\infty \gamma_j\psi_j \ \text{ in } L^2(\mathbb{R}^N) \text{ as } n \to \infty,
\end{aligned}
$$

$$\eta_n^t(\tau) = Q_n\eta_\tau \to \eta_\tau = \sum_{j=1}^{\infty} \beta_j \xi_j(s) \text{ in } L^2_{\mu_t}(\mathbb{R}^+, H^1(\mathbb{R}^N)) \text{ as } n \to \infty,$$

for a.e. $t \le T$, for every $k, j = 0, \dots, n$, where ψ_0 and ξ_0 are the zero vectors in the respective spaces. Taking (ψ_k, ξ_0), (ψ_0, ξ_k) and ψ_k in the first equation and the second equation in (2.1), respectively, and applying the divergence theorem to the term

$$\left\langle -\int_0^\infty \mu(s)\Delta\eta_n^t(s)ds, \psi_k \right\rangle$$

we get a system of ODE in the variable $a_k(t)$, $c_k(t)$ and $b_k(t)$ of the form

$$\begin{cases} \dfrac{d}{dt}a_k = & -\nu_k a_k - \lambda a_k - \displaystyle\sum_{j=1}^{n} b_j \left\langle \xi_j, \psi_k \right\rangle_{1,\mu} \\ & - \langle f(u_n), \psi_k\rangle - \langle g(\cdot, v_n), \psi_k\rangle + \langle \varsigma, \psi_k\rangle, \\ \dfrac{d}{dt}c_k = & -\varsigma c_k - \langle \varphi(\cdot, u_n), \psi_k\rangle, \\ \dfrac{d}{dt}b_k = & \displaystyle\sum_{j=1}^{n} a_j \left\langle \psi_j, \xi_k \right\rangle_{1,\mu} - \sum_{j=1}^{n} b_j \left\langle \xi_j', \xi_k \right\rangle_{1,\mu}, \end{cases} \tag{2.2}$$

subject to the initial conditions

$$\begin{cases} a_k(\tau) = \langle u_\tau, \psi_k\rangle, \\ c_k(\tau) = \langle v_\tau, \psi_k\rangle, \\ b_k(\tau) = \langle \eta_\tau, \xi_k\rangle_{1,\mu}. \end{cases} \tag{2.3}$$

According to the standard theory of ODEs, we obtain the existence and uniqueness of local solutions of (2.2)-(2.3).

A priori estimate for (u_n, v_n, η_n^t) *in* $L^2(\Omega) \times L^2(\Omega) \times L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N))$. Multiplying the first equation and the second equation of (2.2) by (a_k, c_k) and the third equation by b_k, then summing over k, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2 + \int_0^\infty \mu(s)\|\nabla\eta_n^t(s)\|^2 ds \right) \\ &\quad + \|\nabla u_n\|^2 + \lambda\|u_n\|^2 + \left\langle \partial_s\eta_n^t, \eta_n^t \right\rangle_{1,\mu} \\ &\quad + \langle f(u_n), u_n\rangle + \langle g(\cdot, v_n), u_n\rangle \\ &= \langle \varsigma, u_n\rangle, \end{aligned} \tag{2.4}$$

$$\frac{1}{2}\frac{d}{dt}\|v_n\|^2 + \int_{\mathbb{R}^N} \sigma(x)|v_n|^2 dx + \langle \varphi(\cdot, u_n), v_n\rangle = 0, \tag{2.5}$$

where $\int_0^\infty \mu(s)\langle \nabla\eta_n^t(s), \nabla u_n\rangle ds = \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)\|\nabla\eta_n^t(s)\|^2 ds + \left\langle \partial_s\eta_n^t, \eta_n^t\right\rangle_{1,\mu}$. Integrating by parts and then using condition **(H3)**, we have

$$\left\langle \partial_s\eta_n^t, \eta_n^t \right\rangle_{1,\mu} = -\int_0^\infty \mu'(s)\|\nabla\eta_n^t(s)\|^2 ds \ge 0. \tag{2.6}$$

Thus, the term $\langle \partial_s \eta_n^t, \eta_n^t \rangle_{1,\mu}$ in (2.5) can be neglected. Applying (1.6), (1.7) and using the Cauchy inequality, we obtain

$$\langle f(u_n), u_n \rangle \geq -\delta_1 \|u_n\|^2, \tag{2.7}$$

$$\begin{aligned} |\langle g(\cdot, v_n), u_n \rangle| &\leq \int_{\mathbb{R}^N} \delta_5(\phi_2 + |v_n|)|u_n| dx \\ &\leq \delta_5(\|\phi_2\| + \|v_n\|)\|u_n\| \\ &\leq \frac{2\delta_5^2}{\lambda - \delta_1}\|\phi_2\|^2 + \frac{\lambda - \delta_1}{4}\|u_n\|^2 + \frac{2\delta_5^2}{\lambda - \delta_1}\|v_n\|^2, \end{aligned} \tag{2.8}$$

$$\begin{aligned} |\langle \varphi(\cdot, u_n), v_n \rangle| &\leq \int_{\mathbb{R}^N} \delta_6(\phi_3 + |u_n|)|v_n| dx \\ &\leq \delta_6(\|\phi_3\| + \|u_n\|)\|v_n\| \\ &\leq \frac{\lambda - \delta_1}{4}\|\phi_3\|^2 + \frac{2\delta_6^2}{\lambda - \delta_1}\|v_n\|^2 + \frac{\lambda - \delta_1}{4}\|u_n\|^2, \end{aligned} \tag{2.9}$$

$$\int_{\mathbb{R}^N} \sigma(x)|v_n|^2 dx \geq \delta_4 \|v_n\|^2, \tag{2.10}$$

and

$$\langle \varsigma, u_n \rangle \leq C(\varepsilon_0)\|\varsigma\|^2_{H^{-1}(\mathbb{R}^N)} + \varepsilon_0\|u_n\|^2 + \varepsilon_0\|\nabla u_n\|^2. \tag{2.11}$$

Summing the two inequalities in (2.4) and (2.5). Then using (2.6)-(2.11), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2 + \|v_n\|^2 + \int_0^\infty \mu(s)\|\nabla \eta_n^t(s)\|^2 ds\right) \\ &\quad + (1-\varepsilon_0)\|\nabla u_n\|^2 + \left(\frac{\lambda-\delta_1}{2} - \varepsilon_0\right)\|u_n\|^2 + \left(\delta_4 - \frac{2(\delta_5^2+\delta_6^2)}{\lambda-\delta_1}\right)\|v_n\|^2 \\ &\leq \frac{2\delta_5^2}{\lambda - \delta_1}\|\phi_2\|^2 + \frac{\lambda-\delta_1}{4}\|\phi_3\|^2 + C\|\varsigma\|^2_{H^{-1}(\mathbb{R}^N)}. \end{aligned} \tag{2.12}$$

Integrating (2.12) from τ to $t, t \in [\tau, T]$, we have

$$\begin{aligned} &\|u_n(t)\|^2 + \|v_n(t)\|^2 + \int_0^\infty \mu(s)\|\nabla \eta_n^t(s)\|^2 ds \\ &\quad + \int_\tau^t \left(\frac{\lambda - \delta_1}{4}\|u_n(r)\|^2 + \|\nabla u_n(r)\|^2\right) dr \\ &\quad + 2\left(\delta_4 - \frac{2(\delta_5^2+\delta_6^2)}{\lambda - \delta_1}\right)\int_\tau^t \|v_n(r)\|^2 dr \\ &\leq \|z_n(\tau)\|^2_{\mathcal{H}_1} + \frac{2}{\lambda - \delta_1}\int_\tau^T \|\varsigma(t)\|^2_{H^{-1}(\mathbb{R}^N)} dt \\ &\quad + 2\left(\frac{2\delta_5^2}{\lambda - \delta_1}\|\phi_2\|^2 + \frac{\lambda - \delta_1}{4}\|\phi_3\|^2\right)(T - \tau). \end{aligned} \tag{2.13}$$

This inequality implies that

$$\begin{cases} \{u_n\} \text{ is bounded in } L^\infty(\tau, T; L^2(\mathbb{R}^N)), \\ \{u_n\} \text{ is bounded in } L^2(\tau, T; H^1(\mathbb{R}^N)), \\ \{v_n\} \text{ is bounded in } L^\infty(\tau, T; L^2(\mathbb{R}^N)). \end{cases} \tag{2.14}$$

On the other hand, multiplying the third equation of (1.10) by η_n^t in $L^2_\mu(\mathbb{R}^+, L^2(\mathbb{R}^N))$, we get

$$\frac{d}{dt}\int_0^\infty \mu(s)\|\eta_n^t\|^2 ds - 2\int_0^\infty \mu'(s)\|\eta_n^t\|^2 ds = 2\int_0^\infty \mu(s)\langle \eta_n^t(s), u_n\rangle ds.$$

Therefore,

$$\frac{d}{dt}\int_0^\infty \mu(s)\|\eta_n^t\|^2 ds \le \frac{\kappa(0)}{\lambda}\int_0^\infty \mu(s)\|\eta_n^t\|^2 ds + \lambda\|u_n\|^2. \tag{2.15}$$

Applying Gronwall lemma and using (2.14), we deduce that

$$\begin{aligned} &\int_0^\infty \mu(s)\|\eta_n^t\|^2 ds \\ &\le e^{C(T-\tau)}\|\eta_n^\tau\|^2_{1,\mu} + \lambda\int_\tau^t \|u_n(r)\|^2 e^{C(t-r)} dr \\ &\le e^{C(T-\tau)}\|\eta_n^\tau\|^2_{1,\mu} + \lambda\|u_n\|^2_{L^\infty(\tau,T;L^2(\mathbb{R}^N))}\int_\tau^t e^{C(t-r)} dr \\ &\le e^{C(T-\tau)}\|\eta_n^\tau\|^2_{1,\mu} + \lambda\|u_n\|^2_{L^\infty(\tau,T;L^2(\mathbb{R}^N))} C e^{C(T-\tau)}. \end{aligned} \tag{2.16}$$

Combining (2.13) and (2.16), we get

$$\{\eta_n^t\} \text{ is bounded in } L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N))). \tag{2.17}$$

Therefore, using (2.14) and (2.17), there exists a subsequence of $\{u_n\}$ and $\{\eta_n^t\}$ (still denoted by $\{u_n\}$ and $\{\eta_n^t\}$) such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{weakly in } L^2(\tau, T; H^1(\mathbb{R}^N)), \\ \eta_n^t \rightharpoonup \eta^t \quad \text{weakly-star in } L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N))), \end{cases} \tag{2.18}$$

and

$$\begin{aligned} &\Delta u_n \rightharpoonup \Delta u \text{ weakly in } L^2(\tau, T; H^{-1}(\mathbb{R}^N)), \\ &\Delta \eta_n^t \rightharpoonup \Delta \eta^t \text{ weakly in } L^2(\tau, T; L^2_\mu(\mathbb{R}^+, H^{-1}(\mathbb{R}^N))), \end{aligned}$$

up to a subsequence.

Boundedness and weak convergence of $f(u_n)$ *in* $L^1(\tau, T; L^1(\mathbb{R}^N) + L^2(\tau, T; L^2(\mathbb{R}^N))$.

From (2.4), using the assumption (1.6) and the Cauchy inequality, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2 + \int_0^\infty \mu(s)\|\nabla\eta_n^t\|^2 ds\right) \tag{2.19}$$

$$
\begin{aligned}
&+\|\nabla u_n\|^2+\lambda\|u_n\|^2+\int_{\mathbb{R}^N} f(u_n)u_n dx\\
\le\ & \int_{\mathbb{R}^N} \delta_5(|\phi_2|+|v_n|)|u_n|dx+\|\varsigma(t)\|_{H^{-1}(\mathbb{R}^N)}\|u_n\|_{H^1(\mathbb{R}^N)}\\
\le\ & \varepsilon_0(\|u_n\|^2+\|\nabla u_n\|^2)+\frac{\delta_5^2}{\lambda}(\|\phi_2\|^2+\|v_n\|^2)+C\|\varsigma(t)\|^2_{H^{-1}(\mathbb{R}^N)}.
\end{aligned}
$$

Integrating (2.19) from τ to T and using the bound (2.14), we get

$$
\int_\tau^T \int_{\mathbb{R}^N} f(u_n)u_n dx \le C. \tag{2.20}
$$

We now prove that $\{f(u_n)\}$ is bounded in $L^1(\tau,T;L^1(\mathbb{R}^N))+L^2(\tau,T;L^2(\mathbb{R}^N))$. Let ϑ_{Ω_1} and ϑ_{Ω_2} be the characteristic functions of the sets

$$
\begin{aligned}
\Omega_1&=\{(x,t)\in\mathbb{R}^N\times(\tau,T):|u(x,t)|\le 1\},\\
\Omega_2&=\{(x,t)\in\mathbb{R}^N\times(\tau,T):|u(x,t)|> 1\},
\end{aligned}
$$

and $\chi(r)=f(r)-f(0)+\gamma r$, where $\gamma>\ell$. Note that $\chi'(r)\ge\gamma-\ell>0$ and $\chi(r)r=(f(r)-f(0))r+\gamma r^2=f'(w)r^2+\gamma r^2\ge(\gamma-\ell)r^2\ge 0$ for all $r\in\mathbb{R}$. Since

$$
\begin{aligned}
\int_\tau^T\int_{\mathbb{R}^N}|\chi(u_n(x,t))\vartheta_{\Omega_1}(x,t)|^2dxdt &= \iint_{\Omega_1}|\chi(u_n)|^2\,dxdt\\
&=\iint_{\Omega_1}|\chi(u_n)-\chi(0)|^2\,dxdt,
\end{aligned}
$$

by the Mean Value theorem we have

$$
\begin{aligned}
\int_\tau^T\int_{\mathbb{R}^N}|\chi(u_n(x,t))\vartheta_{\Omega_1}(x,t)|^2dxdt &\le C\iint_{\Omega_1}|u_n(x,t)|^2\,dxdt\\
&\le C\int_\tau^T\int_{\mathbb{R}^N}|u_n(x,t)|^2\,dxdt.
\end{aligned}
$$

Taking into account that $u_n\in L^\infty(\tau,T;L^2(\mathbb{R}^N))$, we obtain

$$
\chi(u_n(x,t))\vartheta_{\Omega_1}\in L^2(\tau,T;L^2(\mathbb{R}^N)).
$$

Besides, since

$$
\begin{aligned}
&\int_\tau^T\int_{\mathbb{R}^N}|\chi(u_n(x,t))\vartheta_{\Omega_2}(x,t)|dxdt\\
=\ &\iint_{\Omega_2}|\chi(u_n)|\,dxdt\\
\le\ &\iint_{\Omega_2}\chi(u_n)u_n dxdt\\
\le\ &\int_\tau^T\int_{\mathbb{R}^N}\chi(u_n)u_n dxdt
\end{aligned}
$$

$$\leq \int_\tau^T \int_{\mathbb{R}^N} f(u_n)u_n dxdt + \gamma\|u_n\|^2_{L^2(\tau,T;L^2(\mathbb{R}^N))}$$
$$\leq C,$$

where we have used (2.14) and (2.20), we get $\chi(u_n)\vartheta_{\Omega_2} \in L^1(\tau,T;L^1(\mathbb{R}^N)$. Since

$$\chi(u_n(x,t)) = \chi(u_n(x,t))\vartheta_{\Omega_1}(x,t) + \chi(u_n(x,t))\vartheta_{\Omega_2}(x,t),$$

we get $\chi(u) \in L^1(\tau,T;L^1(\mathbb{R}^N)) + L^2(\tau,T;L^2(\mathbb{R}^N))$. Therefore $\{f(u_n)\}$ is bounded in $L^1(\tau,T;L^1(\mathbb{R}^N)) + L^2(\tau,T;L^2(\mathbb{R}^N))$.

Besides, since

$$u_{nt} = \Delta u_n - \lambda u_n + \int_0^\infty \mu(s)\Delta Q_n\eta_n^t(s)ds - P_nf(u_n) - P_ng(\cdot,v_n) + P_n\varsigma,$$

we see that $\{u_{nt}\}$ is bounded in $L^2(\tau,T;H^{-1}(\mathbb{R}^N))+L^1(\tau,T;L^1(\mathbb{R}^N))+L^2(\tau,T;L^2_\mu(\mathbb{R}^+,H^1(\mathbb{R}^N)))$ and then in $L^1(\tau,T;H^{-1}(\mathbb{R}^N))+L^1(\mathbb{R}^N)+L^2_\mu(\mathbb{R}^+,H^1(\mathbb{R}^N))$ by (1.6), (2.14), and (2.17).

In addition, for each $m \geq 1$, we denote $B_m = \{x \in \mathbb{R}^N : |x| \leq m\}$. Let $\phi \in C^1([0,+\infty))$ be a function such that $0 \leq \phi \leq 1$, $\phi|_{[0,1]} = 1$ and $\phi(r) = 0$ for all $r \geq 2$. For each n and m we define

$$\bar{u}_{n,m}(x,t) = \phi\left(\frac{|x|^2}{m^2}\right)u_n(x,t).$$

From (2.14) that, for all $m \geq 1$, the sequence $\{\bar{u}_{n,m}\}_{n\geq 1}$ is bounded $L^2(\tau,T;H^1_0(B_{2m}))$. Since B_{2m} is a bounded set, then $H^1_0(B_{2m}) \hookrightarrow L^2(B_{2m})$ compactly. Then, by Theorem 13.3 and Remark 13.1 in [22] we can deduce that

$$\{\bar{u}_{n,m}\} \text{ is precompact in } L^2(\tau,T;L^2(B_{2m})),$$

and thus

$$\{u_n|_{B_m}\} \text{ is precompact in } L^2\left(\tau,T;L^2(B_m)\right).$$

By a diagonal procedure, using (2.18), we deduce that there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$u_n \to u \text{ a.e. in } B_m \times (\tau,T) \text{ as } n \to +\infty,\ \forall m \geq 1,$$

and thus, taking into account that $\bigcup_{m=1}^\infty B_m = \mathbb{R}^N$, we obtain

$$u_n \to u \text{ a.e. in } \mathbb{R}^N \times (\tau,T).$$

Besides, using the definition of $\chi(s)$ and (2.14), (2.20), we have

$$\int_\tau^T \int_{\mathbb{R}^N} \chi(u_n)u_n dxdt \leq C.$$

Thus, arguing as in the proof of Theorem 3.1 in [6], we obtain that $\chi(u) \in L^1(\tau,T;L^1(\mathbb{R}^N)) + L^2(\tau,T;L^2(\mathbb{R}^N))$ and for all test functions $\beta \in L^\infty((\tau,T)\times\mathbb{R}^N)$,

$$\int_\tau^T \int_{\mathbb{R}^N} \chi(u_n(x,t))\beta(x,t)dxdt \to \int_0^T \int_{\mathbb{R}^N} \chi(u(x,t))\beta(x,t)dxdt.$$

Then, $f(u) \in L^1(\tau, T; L^1(\mathbb{R}^N)) + L^2(\tau, T; L^2(\mathbb{R}^N))$ and

$$\int_\tau^T \int_{\mathbb{R}^N} f(u_n(x,t))\beta(x,t)dxdt \to \int_\tau^T \int_{\mathbb{R}^N} f(u(x,t))\beta(x,t)dxdt$$

for all $\beta \in L^\infty((\tau, T) \times \mathbb{R}^N)$.

Next, from the second equation of (1.10), we get that $v_{nt} = -\sigma(\cdot)v_n - \varphi(\cdot, u_n)$ is uniformly bounded in $L^2(\tau, T; L^2(\mathbb{R}^N))$. Therefore, we can extract a subsequence of v_n (still label v_n) such that

$$v_n \rightharpoonup v \text{ weakly in } L^2(\tau, T; L^2(\mathbb{R}^N)),$$
$$v_{nt} \rightharpoonup v_t \text{ weakly in } L^2(\tau, T; L^2(\mathbb{R}^N)).$$

Furthermore, using assumption (1.4), we conclude that

$$\varphi(\cdot, u_n) \to \varphi(\cdot, u) \text{ in } L^2(\tau, T; L^2(\mathbb{R}^N)).$$

Hence if we pass the limit in the second equation of (1.10), then we obtain

$$v_t + \sigma(\cdot)v + \varphi(\cdot, u) = 0 \text{ in } L^2(\tau, T; L^2(\mathbb{R}^N)).$$

By the formula of variation of parameters, for each $n \in \mathbb{N}$, we have the solution v_n of

$$v_{nt} + \sigma(\cdot)v_n + \varphi(\cdot, u_n) = 0 \tag{2.21}$$

given by

$$v_n(\cdot, t) = e^{-\sigma(\cdot)t}v_0 + \int_0^t e^{-\sigma(\cdot)(t-s)}\varphi(\cdot, u_n)ds.$$

Since v also satisfies the equation (2.21), we have

$$v(\cdot, t) = e^{-\sigma(\cdot)t}v_0 + \int_0^t e^{-\sigma(\cdot)(t-s)}\varphi(\cdot, u)ds.$$

Hence we deduce that $v_n \to v$ in $L^2(\tau, T; L^2(\mathbb{R}^N))$. By using (1.3), we derive that $g(\cdot, v_n) \to g(\cdot, v)$ in $L^2(\tau, T; L^2(\mathbb{R}^N))$. Thus, as $n \to \infty$, we see that $z(t) = (u, v, \eta^t)$ satisfies the system (1.10) for a.e. $t \in [\tau, T]$.

As in [6], since $u \in L^\infty(\tau, T; L^2(\mathbb{R}^N)) \cap L^2(\tau, T; H^1(\mathbb{R}^N))$ and $u_t \in L^2(\tau, T; H^{-1}(\mathbb{R}^N)) + L^1(\tau, T; L^1(\mathbb{R}^N))$, we conclude that $u \in C([\tau, T]; L^2(\mathbb{R}^N))$ (see also [11, Lemma 8.1], [21]). Obviously, we know $v \in C([\tau, T]; L^2(\mathbb{R}^N))$ since $v \in L^\infty(\tau, T; L^2(\mathbb{R}^N))$ and $v_t \in L^2(\tau, T; L^2(\mathbb{R}^N))$ (see [17, Proposition 7.1]).

Finally, by standard arguments, we can check that z satisfies the initial condition $z(\tau) = z_\tau$ and this implies that z is a weak solution of problem (1.10).

Step 2. Uniqueness and continuous dependence. We assume that $z_1 = (u_1, v_1, \eta_1^t)$ and $z_2 = (u_2, v_2, \eta_2^t)$ are two solutions subject to initial data $z_1(\tau)$

and $z_2(\tau)$, respectively. Denote $\tilde{z} = (\tilde{u}, \tilde{v}, \tilde{\eta}^t) = (u_1 - u_2, v_1 - v_2, \eta_1^t - \eta_2^t)$, we have

$$
\begin{cases}
\tilde{u}_t - \Delta\tilde{u} - \int_0^\infty \mu(s)\Delta\tilde{\eta}^t(s)ds + \chi(u_1) - \chi(u_2) - \ell\tilde{u} \\
+g(\cdot, v_1) - g(\cdot, v_2) = 0, \\
\tilde{v}_t + \sigma(\cdot)\tilde{v} + \varphi(\cdot, u_1) - \varphi(\cdot, u_2) = 0, \\
\partial_t\tilde{\eta}^t + \partial_s\tilde{\eta}^t = \tilde{u}, \\
\tilde{u}(\tau) = u_{1\tau} - u_{2\tau},\ \tilde{v}(\tau) = v_{1\tau} - v_{2\tau},\ \tilde{\eta}^\tau = \eta_{1\tau} - \eta_{2\tau},
\end{cases}
\tag{2.22}
$$

where $\chi(u_i) = f(u_i) + \ell u_i$, $i = 1, 2$.

Here because $\tilde{u}(t)$ does not belong to $W = H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we cannot choose $\tilde{u}(t)$ as a test function and cannot take the scalar product the first equation of (2.22) by $\tilde{u}(t)$. Consequently, we use an idea in [6] to overcome this difficulty as follows. For $k > 0$, we define

$$
B_k(r) = \begin{cases} k & \text{if } \ r > k, \\ r & \text{if } \ |r| \le k, \\ -k & \text{if } \ r < -k. \end{cases}
$$

Consider the corresponding Nemytskii mapping $\hat{B}_k : W \to W$ defined as follows:

$$
\hat{B}_k(\tilde{u})(x) = B_k(\tilde{u}(x)) \ \text{ for all } x \in \mathbb{R}^N.
$$

By Theorem 4.7 in [9] (see also Lemma 2.3 in [6]), we have that $\|B_k(\tilde{u}) - \tilde{u}\|_W \to 0$ as $k \to \infty$. Now multiplying (2.22) by $B_k(\tilde{u})$, then integrating over $\mathbb{R}^N$, we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^N} \tilde{u}B_k(\tilde{u}))dx - \frac{1}{2}\|B_k(\tilde{u})\|^2\right) + \int_{\mathbb{R}^N} \nabla\tilde{u}\nabla\hat{B}_k(\tilde{u})dx \\
&\quad + \int_0^\infty \mu(s)\int_{\mathbb{R}^N} \nabla\tilde{\eta}^t\nabla\hat{B}_k(\tilde{u})dxds + \int_{\mathbb{R}^N} (\chi(u_1) - \chi(u_2))B_k(\tilde{u})dx \\
&\quad + (\lambda - \ell)\int_{\mathbb{R}^N} \tilde{u}B_k(\tilde{u})dx + \int_{\mathbb{R}^N} (g(x, v_1) - g(x, v_2))B_k(\tilde{u})dx \\
&= 0,
\end{aligned}
$$

where $\tilde{u}\frac{d}{dt}B_k(\tilde{u}) = \frac{1}{2}\frac{d}{dt}(B_k(\tilde{u}))^2$.

Thus,

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^N} \tilde{u}B_k(\tilde{u}))dx - \frac{1}{2}\|B_k(\tilde{u})\|^2 + \frac{1}{2}\int_0^\infty \mu(s)\int_{\{x\in\mathbb{R}^N : |\tilde{u}(x,t)| \le k\}} |\nabla\tilde{\eta}^t|^2 dxds\right) \\
&\quad + \frac{1}{2}\int_{\{\Omega : |\tilde{u}| \le k\}} (\lambda|\tilde{u}|^2 + |\nabla\tilde{u}|^2)dxdt + \int_{\mathbb{R}^N} \chi'(\xi)\tilde{u}B_k(\tilde{u})dx \\
&\le \ell\int_{\mathbb{R}^N} \tilde{u}B_k(\tilde{u})dx - \int_{\mathbb{R}^N} g'(x, \zeta)\tilde{v}B_k(\tilde{u})dx,
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^\infty \mu(s)\int_{\mathbb{R}^N} \nabla\tilde{\eta}^t\nabla\hat{B}_k(\tilde{u})dxds \\
\geq\ & \int_0^\infty \mu(s)\int_{\{x\in\mathbb{R}^N:|\tilde{u}(x,t)|\leq k\}} \nabla\tilde{\eta}^t\nabla\tilde{u}dxds \\
=\ & \int_0^\infty \mu(s)\int_{\{x\in\mathbb{R}^N:|\tilde{u}(x,t)|\leq k\}} \nabla\tilde{\eta}^t\nabla\partial_t\tilde{\eta}^t dxds \\
&+\int_0^\infty \mu(s)\int_{\{x\in\mathbb{R}^N:|\tilde{u}(x,t)|\leq k\}} \nabla\tilde{\eta}^t\nabla\partial_s\tilde{\eta}^t dxds \\
=\ & \int_0^\infty \mu(s)\int_{\{x\in\mathbb{R}^N:|\tilde{u}(x,t)|\leq k\}} \nabla\tilde{\eta}^t\nabla\partial_t\tilde{\eta}^t dxds \\
&-\frac{1}{2}\int_0^\infty \mu'(s)\int_{\{x\in\mathbb{R}^N:|\tilde{u}(x,t)|\leq k\}} |\nabla\tilde{\eta}^t|^2 dxds \\
\geq\ & \frac{1}{2}\int_0^\infty \mu(s)\int_{\{x\in\mathbb{R}^N:|\tilde{u}(x,t)|\leq k\}} \frac{d}{dt}|\nabla\tilde{\eta}^t|^2 dxds.
\end{aligned}
$$

Noting that $\chi'(s)\geq 0$ and $sB_k(s)\geq 0$ for all $s\in\mathbb{R}$, from the above inequality we deduce that

$$
\begin{aligned}
(2.23)\quad &\frac{d}{dt}\left(\int_{\mathbb{R}^N}\tilde{u}B_k(\tilde{u}))dx-\frac{1}{2}\|B_k(\tilde{u})\|^2+\frac{1}{2}\int_0^\infty\mu(s)\int_{\{x\in\mathbb{R}^N:|\tilde{u}(x,t)|\leq k\}}|\nabla\tilde{\eta}^t|^2dxds\right)\\
&\leq \ell\int_{\mathbb{R}^N}\tilde{u}B_k(\tilde{u})dx+\frac{\delta_2}{2}\|\tilde{v}(s)\|^2+\frac{\delta_2}{2}\|B_k(\tilde{u})\|^2,
\end{aligned}
$$

where $\int_{\mathbb{R}^N}|g'(x,\zeta)||\tilde{v}||B_k(\tilde{u})|dx\leq\frac{\delta_2}{2}\|\tilde{v}(s)\|^2+\frac{\delta_2}{2}\|B_k(\tilde{u})\|^2$.

On the other hand, similar to the proof of (2.15), we obtain

$$
(2.24)\qquad \frac{d}{dt}\int_0^\infty\mu(s)\|\tilde{\eta}^t\|^2ds\leq\frac{\lambda}{2}\|\tilde{u}\|^2+\frac{\kappa(0)}{2\lambda}\int_0^\infty\mu(s)\|\tilde{\eta}^t\|^2ds.
$$

Summation of (2.23) and (2.24), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^N}\tilde{u}B_k(\tilde{u}))dx-\frac{1}{2}\|B_k(\tilde{u})\|^2+\frac{1}{2}\int_0^\infty\mu(s)\int_{\{x\in\mathbb{R}^N:|\tilde{u}(x,t)|\leq k\}}(|\tilde{\eta}^t|^2+|\nabla\tilde{\eta}^t|^2)dxds\right)\\
&\leq \ell\int_{\mathbb{R}^N}\tilde{u}B_k(\tilde{u})dx+\frac{\delta_2}{2}\|\tilde{v}(s)\|^2+\frac{\delta_2}{2}\|B_k(\tilde{u})\|^2+\frac{\lambda}{2}\|\tilde{u}\|^2+\frac{\kappa(0)}{2\lambda}\int_0^\infty\mu(s)\|\tilde{\eta}^t\|^2ds.
\end{aligned}
$$

Integrating from τ to t, where $t\in(\tau,T)$, then letting $k\to\infty$ in the above inequality, we get

$$
\begin{aligned}
(2.25)\qquad &\|\tilde{u}(t)\|^2+\|\tilde{\eta}^t\|^2_{1,\mu}\\
&\leq\|\tilde{u}(\tau)\|^2+\|\tilde{\eta}^\tau\|^2_{1,\mu}+(2\ell+\delta_2+\lambda)\int_\tau^t\|\tilde{u}(r)\|^2dr
\end{aligned}
$$

$$+ \delta_2 \int_\tau^t \|\tilde{v}(r)\|^2 dr + \frac{\kappa(0)}{\lambda} \int_\tau^t \int_0^\infty \mu(s) \|\tilde{\eta}^r\|^2 ds dr.$$

Next, we take the inner product of the second equation in (2.22) with $\tilde{v}$ and integrate it from τ to t, $t \in [\tau, T]$. Then we have

$$\begin{aligned} \frac{1}{2}\|\tilde{v}(t)\|^2 + \int_{\mathbb{R}^N} \sigma(x)|\tilde{v}|^2 dx dt &= \frac{1}{2}\|\tilde{v}(\tau)\|^2 - \int_{\mathbb{R}^N} (\varphi(x, u_1) - \varphi(x, u_2))\tilde{v} dx dt \\ &= \frac{1}{2}\|\tilde{v}(\tau)\|^2 - \int_{\mathbb{R}^N} \varphi'_u(x, \xi)\tilde{u}\, \tilde{v} dx ds. \end{aligned}$$

Using (1.4) and (1.5), we obtain

$$\|\tilde{v}(t)\|^2 + 2\delta_4 \int_\tau^t \|\tilde{v}(s)\|^2 ds \le \|\tilde{v}(\tau)\|^2 + 2\delta_3 \int_\tau^t \int_{\mathbb{R}^N} |\tilde{u}||\tilde{v}| dx dr.$$

Using the Cauchy inequality, we get

$$\|\tilde{v}(t)\|^2 \le \|\tilde{v}(\tau)\|^2 + \delta_3 \int_\tau^t \left(\|\tilde{u}(s)\|^2 + \|\tilde{v}(s)\|^2 \right) ds. \tag{2.26}$$

Adding (2.25) and (2.26), we can deduce that

$$\begin{aligned} \tilde{z}(t) \le \tilde{z}(\tau) + (2\ell + \delta_2 + \delta_3 + \lambda) \int_\tau^t \|\tilde{u}(r)\|^2 dr + (\delta_2 + \delta_3) \int_\tau^t \|\tilde{v}(r)\|^2 dr \\ + \frac{\kappa(0)}{\lambda} \int_\tau^t \int_0^\infty \mu(s) \|\tilde{\eta}^r\|^2 ds dr. \end{aligned}$$

Thus

$$\tilde{z}(t) \le \tilde{z}(\tau) + \left(\ell + \delta_2 + \delta_3 + \lambda + \frac{\kappa(0)}{\lambda} \right) \int_\tau^t \tilde{z}(r) dr.$$

Applying the Gronwall inequality of integral form, we have

$$\tilde{z}(t) \le \frac{\tilde{z}(\tau)}{\ell + \delta_2 + \delta_3 + \lambda + \frac{\kappa(0)}{\lambda}} e^{\left(\ell + \delta_2 + \delta_3 + \lambda + \frac{\kappa(0)}{\lambda}\right)(t - \tau)}, \quad \forall t \in [\tau, T].$$

This proves the uniqueness (when $\tilde{z}(\tau) = 0$) and the continuous dependence on the initial data of the weak solutions. This completes the proof. $\square$

3. Existence of a uniform attractor

Theorem 2.1 allows us to define a family of processes $\{U_\varsigma(t, \tau)\}_{\varsigma \in \mathcal{H}_w(h)}$ as follows:

$$U_\varsigma(t, \tau) : \mathcal{H} \to \mathcal{H},$$

where $U_\varsigma(t, \tau) z_\tau$ is the unique weak solution of (1.10) (with ς in place of h) at the time t with the initial datum z_τ at τ.

We are going to prove the family of processes $\{U_\varsigma(t, \tau)\}_{\varsigma \in \mathcal{H}_w(h)}$ generated by (1.10) possesses a uniform attractor $\mathcal{A}_{\varsigma \in \mathcal{H}_w(h)}$.

3.1. Existence of a uniform absorbing set

Firstly, we will provide an auxiliary lemma to serve later sections.

Lemma 3.1. *Assume that hypothesis* **(H2)** *hold. Then, for any* $u \in H^1(\mathbb{R}^N)$ *and* $\eta^t \in L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N))$, *the following inequality hold*

$$\begin{aligned}(3.1)\qquad &\int_0^\infty \kappa(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds\\ &\le \theta\|\eta^t\|^2_{1,\mu} \le \theta(\|u\|^2 + \|v\|^2 + \|\eta^t\|^2_{1,\mu}),\ j = 0,1;\end{aligned}$$

$$\begin{aligned}(3.2)\quad &\frac{d}{dt}\left(\int_0^\infty \kappa(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds\right)\\ \le\ & -\frac{1}{2}\int_0^\infty \mu(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds + 2\theta^2\kappa(0)\left(j\|u\|^2 + \|\nabla u\|^2\right).\end{aligned}$$

Proof. By hypotheses (1.8), we immediately obtain (3.1). Besides, using the third equation of (1.10) and exploiting again (1.8), we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^\infty \kappa(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds\right)\\ =\ & -2\int_0^\infty \kappa(s)\int_{\mathbb{R}^N}\left(j\eta^t_s\eta^t + \nabla\eta^t_s\nabla\eta^t\right)dxds\\ & + 2\int_0^\infty \kappa(s)\left(j\langle\eta^t(s), u\rangle + \langle\nabla\eta^t(s), \nabla u\rangle\right)ds\\ \le\ & -\int_0^\infty \kappa(s)\frac{d}{ds}\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds\\ & + 2\theta\int_0^\infty \mu(s)\int_{\mathbb{R}^N}\left(j\eta^t\cdot u + \nabla\eta^t\cdot\nabla u\right)dxds\\ \le\ & -\kappa(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)\Big|_{s=0}^{s=\infty} + \int_0^\infty \kappa'(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds\\ & + 2\theta j\Big(\int_0^\infty \mu(s)\|\eta^t(s)\|^2 ds\Big)^{1/2}\Big(\int_0^\infty \mu(s)\|u\|^2 ds\Big)^{1/2}\\ & + 2\theta\Big(\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2 ds\Big)^{1/2}\Big(\int_0^\infty \mu(s)\|\nabla u\|^2 ds\Big)^{1/2}\\ \le\ & -\frac{1}{2}\int_0^\infty \mu(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds + 2\theta^2\left(j\|u\|^2 + \|\nabla u\|^2\right)\int_0^\infty \kappa'(s)ds\\ =\ & -\frac{1}{2}\int_0^\infty \mu(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds + 2\theta^2\kappa(0)\left(j\|u\|^2 + \|\nabla u\|^2\right). \qquad \square\end{aligned}$$

Now, we prove the existence of an $(\mathcal{H},\mathcal{H})$-uniform absorbing set for the family of processes $\{U_\varsigma(t,\tau)\}_{\varsigma\in\mathcal{H}_w(h)}$.

Lemma 3.2. *Assume that hypotheses* **(H1)-(H3)** *hold. Then the family of processes* $\{U_\varsigma(t,\tau)\}_{\varsigma\in\mathcal{H}_w(h)}$ *associated to problem* (1.10) *has an* $(\mathcal{H},\mathcal{H})$*-uniform absorbing set.*

Proof. Multiplying the first equation and the second equation of (1.10) by $u(t)$ and $v(t)$ in $L^2(\mathbb{R}^N)$, respectively, then arguing as in Theorem 2.1, we get inequality as in (2.12) as follows:

$$\begin{aligned}(3.3)\qquad &\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\|v\|^2+\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2ds\right)\\ &+(1-\varepsilon_0)\|\nabla u\|^2+\left(\frac{\lambda-\delta_1}{2}-\varepsilon_0\right)\|u\|^2\\ &+\left(\delta_4-\frac{2(\delta_5^2+\delta_6^2)}{\lambda-\delta_1}\right)\|v\|^2\\ \le{}& \frac{2\delta_5^2}{\lambda-\delta_1}\|\phi_2\|^2+\frac{\lambda-\delta_1}{4}\|\phi_3\|^2+C\|\varsigma\|^2_{H^{-1}(\mathbb{R}^N)}.\end{aligned}$$

On the other hand, multiplying the third equation of (1.10) by $j\eta^t$ in $L^2_\mu(\mathbb{R}^+, L^2(\mathbb{R}^N))$, we get

$$\frac{d}{dt}j\int_0^\infty \mu(s)\|\eta^t\|^2ds-2j\int_0^\infty \mu'(s)\|\eta^t\|^2ds=2j\int_0^\infty \mu(s)\langle\eta^t(s),u\rangle ds.$$

Since the term $-2\int_0^\infty \mu'(s)j\|\eta^t\|^2ds>0$ can be neglected and by the Young inequality, we obtain

$$(3.4)\qquad \frac{d}{dt}j\int_0^\infty \mu(s)\|\eta^t\|^2ds\le\frac{2j\kappa(0)}{\gamma}\|u\|^2+\frac{j\gamma}{2}\int_0^\infty \mu(s)\|\eta^t\|^2ds.$$

Summation of (3.3), (3.4), we get

$$\begin{aligned}&\frac{d}{dt}E_j+2\|\nabla u\|^2+\frac{\lambda-\delta_1}{2}\|u\|^2+2\left(\delta_4-\frac{2(\delta_5^2+\delta_6^2)}{\lambda-\delta_1}\right)\|v\|^2\\ \le{}&\Phi+\frac{2}{\lambda-\delta_1}\|\varsigma\|^2+\frac{2j\kappa(0)}{\gamma}\|u\|^2+\frac{j\gamma}{2}\int_0^\infty \mu(s)\|\eta^t\|^2ds,\end{aligned}$$

where

$$\Phi=\frac{4\delta_5^2}{\lambda-\delta_1}\|\phi_2\|^2+\frac{\lambda-\delta_1}{2}\|\phi_3\|^2 \text{ and}$$

$$E_j=\|u\|^2+\|v\|^2+\int_0^\infty \mu(s)\left(j\|\eta^t\|^2+\|\nabla\eta^t\|^2\right)ds,\ j=0,1.$$

Now, for $\gamma>0$ to be fixed, we define the functional

$$\Lambda_j(t)=E_j+8\gamma\int_0^\infty \kappa(s)\left(j\|\eta^t(s)\|^2+\|\nabla\eta^t(s)\|^2\right)ds,\ j=0,1.$$

Then, using Lemma 3.1, we can see that Λ_j satisfies the differential inequality

$$\frac{d}{dt}\Lambda_j+2(1-8\gamma\theta^2\kappa(0)-\varepsilon_0)\|\nabla u\|^2+\frac{\lambda-\delta_1-32j\gamma\theta^2\kappa(0)-\varepsilon_0}{2}\|u\|^2$$

$$+ 2\left(\delta_4 - \frac{2(\delta_5^2+\delta_6^2)}{\lambda-\delta_1}\right)\|v\|^2 + 4\gamma\int_0^\infty \mu(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds$$
$$\le \Phi + C\|\varsigma\|^2_{H^{-1}(\mathbb{R}^N)} + \frac{2j\kappa(0)}{\gamma}\|u\|^2 + \frac{j\gamma}{2}\int_0^\infty \mu(s)\|\eta^t\|^2 ds,$$

Choosing γ small enough such that $0<\gamma<\min\{\frac{\lambda-\delta_1}{4+32\theta^2\kappa(0)}, \frac{1}{1+8\theta^2\kappa(0)}\}$, we have

$$\frac{d}{dt}\Lambda_j + 2\gamma E_j + 2\gamma\|\nabla u\|^2$$
$$\le \Phi + C\|\varsigma\|^2_{H^{-1}(\mathbb{R}^N)} + \frac{2j\kappa(0)}{\gamma}\|u\|^2 + \frac{j\gamma}{2}\int_0^\infty \mu(s)\|\eta^t\|^2 ds.$$

Up to further reducing γ, we also have

$$E_j \le \Lambda_j \le 2E_j.$$

Thus,

$$\frac{d}{dt}\Lambda_j + \gamma\Lambda_j + 2\gamma\|\nabla u\|^2$$
$$\le \frac{2j\kappa(0)}{\gamma}\Lambda_0 + \Phi + C\|\varsigma\|^2_{H^{-1}(\mathbb{R}^N)} + \frac{j\gamma}{2}\int_0^\infty \mu(s)\|\eta^t\|^2 ds. \tag{3.5}$$

From (3.5), let $j=0$, and then applying Gronwall inequality, we get

$$\Lambda_0(t) \le \Lambda_0(\tau)e^{-\gamma(t-\tau)} + \Phi + C\int_\tau^t e^{-\gamma(t-r)}\|\varsigma(r)\|^2_{H^{-1}(\mathbb{R}^N)}dr. \tag{3.6}$$

Besides, we have

$$\int_\tau^t e^{-\gamma(t-r)}\|\varsigma(r)\|^2_{H^{-1}(\mathbb{R}^N)}dr \tag{3.7}$$
$$\le \left(\int_{t-1}^t e^{-\gamma(t-r)}\|\varsigma(r)\|^2_{H^{-1}(\mathbb{R}^N)}dr + \int_{t-2}^{t-1} e^{-\gamma(t-r)}\|\varsigma(r)\|^2_{H^{-1}(\mathbb{R}^N)}dr + \cdots\right)$$
$$\le \left(1+e^{-\gamma}+e^{-2\gamma}+\cdots\right)\|\varsigma\|_b^2 \le \frac{1}{1-e^{-\gamma}}\|h\|_b^2,$$

where we have used the fact that $\|\varsigma\|_b^2 \le \|h\|_b^2$ for all $\varsigma\in\mathcal{H}_w(g)$. Combining (3.6) and (3.7)

$$\Lambda_0(t) \le \Lambda_0(\tau)e^{-\gamma(t-\tau)} + \Phi + \frac{C}{1-e^{-\delta}}\|h\|_b^2$$
$$\le \rho_0. \tag{3.8}$$

On the other hand, we consider (3.5) for $j=1$, using (3.8) and Gronwall inequality, we obtain

$$\Lambda_1(t) \le \Lambda_1(\tau)e^{-\gamma(t-\tau)} + \left(\frac{2\kappa(0)}{\gamma}+\frac{\gamma}{2}\right)\rho_0 + C\int_\tau^t e^{-\gamma(t-r)}\|\varsigma(r)\|^2_{H^{-1}(\mathbb{R}^N)}dr$$
$$\le \Lambda_1(\tau)e^{-\gamma(t-\tau)} + \left(\frac{2\kappa(0)}{\gamma}+\frac{\gamma}{2}\right)\rho_0 + \Phi + \frac{C}{1-e^{-\delta}}\|h\|_b^2.$$

Thus,

$$E_1(t) \le 2E_1(\tau)e^{-\gamma(t-\tau)} + \left(\frac{2\kappa(0)}{\gamma} + \frac{\gamma}{2}\right)\rho_0 + \Phi + \frac{C}{1-e^{-\delta}}\|h\|_b^2.$$

Therefore, there exists $\rho_1 > 0$ such that

$$E_1(t) \le \rho_1 \text{ or } \|z(t)\|_{\mathcal{H}}^2 \le \rho_1 \tag{3.9}$$

for all $z_\tau \in B_{\mathcal{H}}$, $\varsigma \in \mathcal{H}_w(h)$ and for all $t \ge T_B$, where $B_{\mathcal{H}}$ is an arbitrary bounded subset of $\mathcal{H}$. Therefore, for all $t \ge T_B$, we get

$$U(t,\tau)B_{\mathcal{H}}(R) \subset \mathbb{B}_0,$$

where $\mathbb{B}_0 = B_{\mathcal{H}}(\rho_1)$ is an absorbing set for processes $U(t,\tau)$ on $\mathcal{H}$.

Besides, integrating (3.5) from t to $t+1$, we get

$$\int_t^{t+1} \|\nabla u(r)\|^2 dr \le 2\rho_1. \tag{3.10}$$

This completes the proof. □

3.2. Asymptotic compactness

The main difficulty of the problem is, of course, that the embeddings are no longer compact and the external force h is only in $L_b^2(\mathbb{R}; H^{-1}(\mathbb{R}^N))$. Furthermore, the memory term has no smoothing effect. Therefore, we do not directly estimate the regularization for u as in [10], but must use the decomposition method as follows.

3.2.1. *Decomposition of the equation.* Notice that we only assume the external force $h \in L_b^2(\mathbb{R}; H^{-1}(\mathbb{R}^N))$. Since $L^2(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$ is dense, for every $h(\cdot,t) \in H^{-1}(\mathbb{R}^N)$ and any $\varepsilon > 0$, there exists an $h^\varepsilon(\cdot,t) \in L^2(\mathbb{R}^N)$, which depends on h and ε, such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|h - h^\varepsilon\|_{H^{-1}(\mathbb{R}^N)}^2 dr < \frac{\varepsilon}{2}. \tag{3.11}$$

For any $r > 0$ introduce two smooth positive functions $\phi_r^i : \mathbb{R}^N \to \mathbb{R}^+$, $i = 1,2$, such that

$$\phi_r^1(x) + \phi_r^2(x) = 1 \quad \forall x \in \mathbb{R}^N,$$

and

$$\begin{aligned} \phi_r^1(x) &= 0 \ \text{ if } |x| \le r, \\ \phi_r^2(x) &= 0 \ \text{ if } |x| \ge r+1. \end{aligned}$$

Putting $\varsigma_i(x,t) = \varsigma(x, t\cdot\phi_r^i(x))$, $i = 1,2$. The dependence on r of ς_i is omitted for simplicity of notation. Therefore, we can check that

$$\begin{cases} \lim\limits_{r\to\infty} \|\varsigma_1\|_{H^{-1}(\mathbb{R}^N)} = 0, \\ \varsigma_2(x,t) = 0 \ \text{ as } |x| \ge r+1. \end{cases} \tag{3.12}$$

Now, to make the asymptotic regular estimates, we decompose the solution $U_\varsigma(t,\tau)z_\tau = z(t) = (u(t), v(t), \eta^t)$, $z_\tau = (u_\tau, v_\tau, \eta^\tau)$, of problem (1.10) into the sum

$$U_\varsigma(t,\tau)z_\tau = D(t,\tau)z_\tau + K_\varsigma(t,\tau)z_\tau,$$

where $D(t,\tau)z_\tau = z_1(t)$ and $K_\varsigma(t,\tau)z_\tau = z_2(t)$, that is, $z = (u, v, \eta^t) = z_1 + z_2$, the decomposition is as follows:

$$u = u_1 + u_2,\ v = v_1 + v_2,\ \eta^t = \zeta^t + \xi^t,$$
$$z_1 = (u_1, v_1, \zeta^t),\ z_2 = (u_2, v_2, \xi^t),$$

where $z_1(t)$ solves the following equation

$$\begin{cases} \partial_t u_1 - \Delta u_1 - \displaystyle\int_0^\infty \mu(s)\Delta\zeta^t(s)ds + \lambda u_1 + f(u) - f(u_2) \\ = \varsigma_1 + \varsigma_2 - \varsigma_2^\varepsilon, \\ \partial_t v_1 + \sigma(x)v_1 = 0, \\ \partial_t \zeta^t = -\partial_s \zeta^t + u_1, \\ u_1(x,t)|_{t\le\tau} = u_\tau(x), v_1(x,t)|_{t\le\tau} = v_\tau(x), \\ \zeta^t(x,s)|_{t\le\tau} = \eta_\tau(x,s), \end{cases} \tag{3.13}$$

and $z_2(t)$ is the unique solution of the following problem

$$\begin{cases} \partial_t u_2 - \Delta u_2 - \displaystyle\int_0^\infty \mu(s)\Delta\xi^t(s)ds + \lambda u_2 + f(u_2) + g(x,v) = \varsigma_2^\varepsilon, \\ \partial_t v_2 + \sigma(x)v_2 + \varphi(x,u) = 0, \\ \partial_t \xi^t = -\partial_s \xi^t + u_2, \\ u_2(x,t)|_{t\le\tau} = 0, v_2(x,t)|_{t\le\tau} = 0, \\ \xi^t(x,s)|_{t\le\tau} = \xi_\tau(x,s) = 0. \end{cases} \tag{3.14}$$

By using similar arguments as in the proof of Theorem 2.1, one can prove the existence and uniqueness of solutions to problems (3.13) and (3.14). Besides, for problem (3.14), because the initial data are zero (so belong to $\mathcal{H}_1 := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2_\mu(\mathbb{R}^+, H^2(\mathbb{R}^N)))$, we can show that the solution (u_2, v_2, ξ^t) is in fact a strong solution.

3.2.2. *The first a priori estimate.* We begin with the decay estimate for solutions of (3.13).

Lemma 3.3. *Assume that hypotheses* **(H1)-(H3)** *hold. Then the solutions of equation* (3.13) *satisfy the following estimate: there is a constant* $\gamma_1 > 0$ *and there exists* $T > \tau$ *large enough, which depends on* $\|\varsigma\|_{L^2_b}, \|z_\tau\|_{\mathcal{H}}$, *and* $\mathcal{Q}(\cdot)$ *independent of* z, *such that*

$$\|D(t,\tau)z_\tau\|^2_{\mathcal{H}} \le \mathcal{Q}(\|z_\tau\|_{\mathcal{H}})e^{-\gamma_1(t-\tau)} + \varepsilon \quad \text{for all } t \ge T,$$

where $\mathcal{Q}$ *is an increasing function on* $[0,\infty)$.

Proof. Multiplying the first equation and the second of (3.13) by u_1 and v_1, respectively, and then adding the results, we obtain

$$
\begin{aligned}
(3.15)\qquad &\frac{d}{dt}\left(\|u_1\|^2+\|v_1\|^2+\int_{\mathbb{R}^N}\mu(s)\|\nabla\zeta^t(s)\|^2ds\right)\\
&+2\lambda\|u_1\|^2+2\|\nabla u_1\|^2+2\delta_4\|v_1\|^2\\
&-2\int_{\mathbb{R}^N}\mu'(s)\|\nabla\zeta^t(s)\|^2ds+2\langle f(u)-f(u_2),u_1\rangle\\
\le\ &2\langle\varsigma_3,u_1\rangle_{H^{-1},H^1},
\end{aligned}
$$

where $\varsigma_3=\varsigma_1+\varsigma_2-\varsigma_2^{\varepsilon}\in L^2_b(\mathbb{R};H^{-1}(\mathbb{R}^N))$. Similarly to the proof of (3.4), (3.2), we get

$$
\begin{aligned}
(3.16)\qquad &\frac{d}{dt}j\int_0^\infty\mu(s)\|\zeta^t\|^2ds-2j\int_0^\infty\mu'(s)\|\zeta^t\|^2ds\\
\le\ &\frac{2j\kappa(0)}{\gamma_1}\|u_1\|^2+\frac{j\gamma_1}{2}\int_0^\infty\mu(s)\|\zeta^t\|^2ds;
\end{aligned}
$$

and

$$
\begin{aligned}
(3.17)\qquad &\frac{d}{dt}\left(8\gamma_1\int_0^\infty\kappa(s)\left(j\|\zeta^t(s)\|^2+\|\nabla\zeta^t(s)\|^2\right)ds\right)\\
\le\ &-4\gamma_1\int_0^\infty\mu(s)\left(j\|\zeta^t(s)\|^2+\|\nabla\zeta^t(s)\|^2\right)ds\\
&+16\gamma_1\theta^2\kappa(0)\left(j\|u_1\|^2+\|\nabla u_1\|^2\right).
\end{aligned}
$$

Applying Young inequality and using assumption (1.2), we have

$$2\langle\varsigma_3,u_1\rangle_{H^{-1},H^1}\le\varepsilon_0\|u_1\|^2_{H^1(\mathbb{R}^N)}+C(\varepsilon_0)\|\varsigma_3\|^2_{H^{-1}(\mathbb{R}^N)},$$

and

$$2\langle f(u)-f(u_2),v_1\rangle\ge-2\ell\|u_1\|^2;$$

Summing up (3.15), (3.16) and (3.17), and plugging all the above inequalities into (3.15), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\Phi_j+2\left(\lambda-8j\gamma_1\theta^2\kappa(0)-\ell-\varepsilon_0\right)\|u_1\|^2+2(1-8\gamma_1\theta^2\kappa(0)-\varepsilon_0)\|\nabla u_1\|^2\\
&+2\delta_4\|v_1\|^2+4\gamma_1\int_0^\infty\mu(s)\left(j\|\zeta^t(s)\|^2+\|\nabla\zeta^t(s)\|^2\right)ds\\
\le\ &\frac{2j\kappa(0)}{\gamma}\|u_1\|^2+C(\varepsilon_0)\|\varsigma_3\|^2_{H^{-1}(\mathbb{R}^N)},
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_j=\ &\|u_1\|^2+\|v_1\|^2+\int_0^\infty\mu(s)\left(j\|\zeta^t\|^2+\|\nabla\zeta^t\|^2\right)ds\\
&+8\gamma_1\int_0^\infty\kappa(s)\left(j\|\zeta^t(s)\|^2+\|\nabla\zeta^t(s)\|^2\right)ds,\ j=0,1.
\end{aligned}
$$

Choosing ε_0 is small enough and $\gamma_1 < \min\{\delta_4, \frac{\lambda}{2+8\theta^2\kappa(0)+\ell}, \frac{1}{1+8\theta^2\kappa(0)}\}$, we obtain

$$\frac{d}{dt}\Phi_j + 2\gamma_1\Phi_j \leq \frac{2j\kappa(0)}{\gamma_1}\|u_1\|^2 + C(\varepsilon_0)\|\varsigma_3\|^2_{H^{-1}(\mathbb{R}^N)}, \tag{3.18}$$

where $C(\varepsilon_0)\|\varsigma_3\|^2_{H^{-1}(\mathbb{R}^N)} \leq C\|\varsigma_1\|^2_{H^{-1}(\mathbb{R}^N)} + C\|\varsigma_2 - \varsigma_2^\varepsilon\|^2_{H^{-1}(\mathbb{R}^N)}$.

Using (3.11) and (3.12), putting $j = 0$ in (3.18) and subsequently substituting the result into (3.18) with $j = 1$, we obtain

$$\|u_1\|^2 + \|v_1\|^2 + \|\zeta^t\|^2_{1,\mu} \leq \Phi_1(t) \leq Q(\|z_\tau\|_{\mathcal{H}_1})e^{-\gamma_1(t-\tau)} + \varepsilon. \tag{3.19}$$

This completes the proof. □

3.2.3. *The second a priori estimate.* About the solution $z_2(t)$ of (3.14), we have:

Lemma 3.4. *Let* **(H1)-(H3)** *hold. Then for any $z_\tau \in \mathcal{H}$, there exist $M > 0$, $T > \tau$ large enough, which depend on $\|z_\tau\|^2_{\mathcal{H}}$, such that*

$$\|K_\varsigma(t,\tau)z_\tau\|^2_{\mathcal{H}_1} \leq M_0 \quad \text{for all } t \geq T.$$

Proof. Combining (3.9), (3.19) and $(u, v, \eta^t) = (u_1, v_1, \zeta^t) + (u_2, v_2, \xi^t)$, we can see that

$$\|u_2\|^2 + \|v_2\|^2 + \|\xi^t\|^2_{1,\mu} \leq \rho_2. \tag{3.20}$$

Multiplying the first equation of (3.14) by $-\Delta u_2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_2\|^2 + \int_0^\infty \mu(s)\|\Delta\xi^t\|^2 ds\right) + \|\Delta u_2\|^2 + \lambda\|\nabla u_2\|^2 \\ + (f(u_2), -\Delta u_2) + (g(x,v), -\Delta u_2) - \int_0^\infty \mu'(s)\|\Delta\xi^t\|^2 ds \\ \leq \langle \varsigma_2^\varepsilon, -\Delta u_2\rangle.\end{aligned} \tag{3.21}$$

Using (3.9) and the assumptions (1.2), (1.6), (1.7), we get

$$\begin{aligned}(f(u_2), -\Delta u_2) &= \int_{\mathbb{R}^N} f'(u_2)|\nabla u_2|^2 dx \\ &\geq -\ell\|\nabla u_2\|^2; \\ (g(x,v), -\Delta u_2) &\leq \delta_5 \int_{\mathbb{R}^N} (\phi_2(x) + |v|)|\Delta u_2| dx \\ &\leq \delta_5(\|\phi_2\| + \|v\|)\|\Delta u_2\| dx \\ &\leq C\rho_1 + \frac{1}{4}\|\Delta u_2\|^2 \quad \forall t \geq T_B; \\ \langle \varsigma_2^\varepsilon, -\Delta u_2\rangle &\leq \|\varsigma_2^\varepsilon\|\|\Delta u_2\| \\ &\leq \|\varsigma_2^\varepsilon\|^2 + \frac{1}{4}\|\Delta u_2\|^2.\end{aligned}$$

Plugging all the above inequalities into (3.21) and notice that

$$-2\int_0^\infty \mu'(s)\|\Delta\xi^t\|^2 ds \geq 0,$$

it follows that

$$\frac{d}{dt}\left(\|\nabla u_2\|^2 + \int_0^\infty \mu(s)\|\Delta\xi^t\|^2 ds\right) + 2(\lambda-\ell)\|\nabla u_2\|^2 + \|\Delta u_2\|^2 \leq \frac{m^2\rho_2}{\delta^4} + C(\rho_1 + \|\varsigma_2^\varepsilon\|^2). \tag{3.22}$$

We learn from (3.2) that

$$\begin{aligned}\frac{d}{dt} 8\gamma_2 \int_0^\infty \kappa(s)\|\Delta\xi^t(s)\|^2 ds \\ \leq \ -4\gamma_2\int_0^\infty \mu(s)\|\Delta\xi^t(s)\|^2 ds + 16\gamma_2\theta^2\kappa(0)\|\Delta u_2\|^2.\end{aligned} \tag{3.23}$$

Summing up (3.22) and (3.23), it follows that

$$\begin{aligned}&\frac{d}{dt}\Lambda + 2(\lambda-\ell)\|\nabla u_2\|^2 + 4\gamma_2\int_0^\infty \mu(s)\|\nabla\xi^t(s)\|^2 ds \\ &+ (1 - 16\gamma_2\theta^2\kappa(0))\|\Delta u_2\|^2 \\ &\leq C(\rho_1 + \|\varsigma_2^\varepsilon\|^2),\end{aligned}$$

where $\Lambda = \|\nabla u_2\|^2 + \int_0^\infty \mu(s)\|\Delta\xi^t\|^2 ds + 8\gamma_2\int_0^\infty \kappa(s)\|\nabla\xi^t(s)\|^2 ds$.

Choosing $\gamma_2 < \min\{\delta_4, \lambda-\ell, \frac{1}{16\theta^2\kappa(0)}\}$, we obtain

$$\frac{d}{dt}\Lambda + 2\gamma_2\Lambda \leq C(\rho_1 + \|\varsigma_2^\varepsilon\|^2).$$

Applying the Gronwall inequality and using (3.10), we can derive

$$\|\nabla u_2\|^2 + \int_0^\infty \mu(s)\|\Delta\xi^t\|^2 ds \leq 2\Lambda \leq 2\rho_3,\ \forall t \geq T_B + 1. \tag{3.24}$$

We now show that $v_2(t)$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Setting $w_j = \partial v_2/\partial x_j$ $(1 \leq j \leq N)$, then from the second equation of (3.14), we get

$$\begin{cases}\dfrac{\partial w_j}{\partial t} + \sigma w_j = -\sigma'_{x_j} v_2 - \varphi'_{x_j}(x,u) - \varphi(x,u)u_{x_j}, \\ w_j(0) = 0.\end{cases} \tag{3.25}$$

Multiplying (3.25) by w_j, then using (1.4) and (1.5), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w_j\|^2 + \delta_4\|w_j\|^2 \\ &\leq \int_{\mathbb{R}^N}\left(|\sigma'_{x_j}||v_2| + |\varphi'_{x_j}(x,u)| + |\varphi'_u(x,u)|\left|u'_{x_j}\right|\right)|w_j| dx \\ &\leq \frac{\delta_4}{2}\|w_j\|^2 + \frac{1}{2\delta_4}\int_{\mathbb{R}^N}\left(m|v_2| + \delta_3(|\phi_1| + |u| + |u'_{x_j}|)\right)^2 dx\end{aligned}$$

$$\le \frac{\delta_4}{2}\|w_j\|^2 + \frac{3(m+\delta_3)^2}{\delta_4}(\|v_2\|^2 + \|\phi_1\|^2 + \|u\|^2) + \frac{\delta_3^2}{\delta_4}\|u_{x_j}\|^2.$$

Thus we obtain

$$\frac{d}{dt}\|w_j\|^2 + \delta_4\|w_j\|^2 \le \frac{6(m+\delta_3)^2}{\delta_4}(\|v_2\|^2 + \|\phi_1\|^2 + \|u\|^2) + \frac{\delta_3^2}{\delta_4}\|u_{x_j}\|^2. \tag{3.26}$$

By summing (3.26) from $j=1$ to $j=N$ and then using (3.9), (3.20), we obtain

$$\frac{d}{dt}\|\nabla v_2\|^2 + \delta_4\|\nabla v_2\|^2 \le c_0 + \frac{2\delta_3^2}{\delta_4}\|\nabla u\|^2,\ \forall t \ge T_B,$$

where $c_0 = \frac{6(m+\delta_3)^2}{\delta_4}(\rho_2 + \|\phi_1\|^2 + \rho_1)$.

Applying the Gronwall inequality, then using (3.10) and the same argument as in (3.7), it follows that

$$\begin{aligned} \|\nabla v_2(t)\|^2 &\le \frac{c_0}{\delta_5} + \frac{2\delta_3^2}{\delta_4}\int_\tau^t e^{-\delta_5(t-r)}\|\nabla u(r)\|^2 dr \\ &\le \frac{c_0}{\delta_5} + \frac{4\rho_1\delta_3^2}{\delta_4(1-e^{-\gamma})}. \end{aligned} \tag{3.27}$$

Now we combine (3.20) with (3.24) and (3.27), we can see that

$$\|(u_2, v_2, \xi^t)\|_{\mathcal{H}_1}^2 \le \rho_4 := \frac{c_0}{\delta_5} + \frac{4\rho_1\delta_3^2}{\delta_4(1-e^{-\gamma})} + \rho_1 + \rho_2 + 2\rho_3,\ \forall t \ge T_B + 1.$$

This completes the proof. □

To overcome the non-compactness of Sobolev embeddings in $\mathbb{R}^N$, we decompose the whole space $\mathbb{R}^N$ into a bounded ball and its complement. Then the uniform asymptotic compactness of $U(\tau,t)$ will follow from the compact Sobolev embeddings in the bounded ball and the estimates in its complement. We consider the following lemma:

Lemma 3.5. *Let B be a bounded subset in $\mathcal{H}$. Then for any $\omega > 0$, there exist $T_\omega > 0$ and $K_\omega > 0$ such that*

$$\int_{|x|\ge K_\omega}(|u_2|^2 + |v_2|^2)dx + \int_0^\infty \mu(s)\int_{|x|\ge K_\omega}\left(|\xi^t(s)|^2 + |\nabla\xi^t(s)|^2\right)dxds$$
$$< \omega,\ \forall t \ge T_\omega,\ \forall z_\tau \in B.$$

Proof. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a smooth function satisfying $\phi(s) = 0$ for $0 \le s \le 1; 0 \le \phi(s) \le 1$ for $s \in \mathbb{R}^+$ and $\phi(s) = 1$ for $s \ge 2$. It is easy to see that $\phi'(s) \le C$ for all $s \in \mathbb{R}^+$ and $\phi'(s) = 0$ for $s \ge 2$.

Multiplying the first equation of (3.14) by $\phi\left(\frac{|x|^2}{k^2}\right)w$ and then integrating the resulting identity, we find

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx + \lambda\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx \tag{3.28}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\nabla u_2|^2dx+\int_{\mathbb{R}^N}\frac{2x}{k^2}\phi'\left(\frac{|x|^2}{k^2}\right)u_2\nabla u_2dx\\
&+\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\nabla u_2\nabla\xi^t(s)dxds\\
&+\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\frac{2x}{k^2}\phi'\left(\frac{|x|^2}{k^2}\right)u_2\nabla\xi^t(s)dxds\\
&+\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)f(u_2)u_2dx+\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)g(x,v)u_2dx\\
=&\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)u_2\varsigma_2^{\varepsilon}(t)dx.
\end{aligned}
$$

Multiplying the third equation of (3.14) by $j\phi\left(\frac{|x|^2}{k^2}\right)\xi^t$ in $L^2_\mu(\mathbb{R}^+,L^2(\mathbb{R}^N))$, we get

$$
\begin{aligned}
(3.29)\quad &\frac{d}{dt}j\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\xi^t(s)|^2dxds\\
&-2j\int_0^\infty\mu'(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\xi^t(s)|^2dxds\\
&=2j\int_0^\infty\mu(s)\langle\phi\left(\frac{|x|^2}{k^2}\right)\xi^t(s),u_2\rangle ds\\
&\le\frac{2j\kappa(0)}{\gamma_3}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx+\frac{j\gamma_3}{2}\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\xi^t(s)|^2dxds.
\end{aligned}
$$

Since $u_2=\xi^t_t+\xi^t_s$, we have

$$
\begin{aligned}
(3.30)\quad &\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\nabla\xi^t\nabla u_2dxds\\
&=\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\nabla\xi^t\nabla\xi^t_tdxds\\
&\quad+\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\nabla\xi^t\nabla\xi^t_sdxds\\
&=\frac{1}{2}\frac{d}{dt}\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\nabla\xi^t|^2dxds\\
&\quad-\int_0^\infty\mu'(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\nabla\xi^t|^2dxds.
\end{aligned}
$$

By (1.2) and (1.7), we have

$$
\begin{aligned}
(3.31)\quad \int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)f(u_2)u_2dx&\ge-\delta_1\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx\\
&=-\delta_1\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx,
\end{aligned}
$$

and

$$\begin{aligned}(3.32)\quad &\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|g(x,v)u_2|dx\\ &\leq \delta_5\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\phi_2(x)||u_2|dx+\delta_5\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|v||u_2|dx\\ &\leq \frac{\lambda-\delta_1}{2}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx+\frac{\delta_5^2}{\lambda-\delta_1}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)(|\phi_2(x)|^2+|v|^2)dx\\ &\leq \frac{\lambda-\delta_1}{2}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx+\frac{2\delta_5^2}{\lambda-\delta_1}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|v_2|^2dx\\ &\quad+\frac{2\delta_5^2}{\lambda-\delta_1}\int_{|x|\geq k}(|\phi_2(x)|^2+|v_1|^2)dx.\end{aligned}$$

Since $\phi'(s)=0$ for all $s>2$, we have

$$\begin{aligned}(3.33)\qquad \left|\int_{\mathbb{R}^N}\frac{2x}{k^2}\phi'\left(\frac{|x|^2}{k^2}\right)u_2\nabla u_2\right|dx &\leq C\int_{|x|\leq\sqrt{2}k}\frac{2|x|}{k^2}|u_2||\nabla u_2|dx\\ &\leq \frac{C}{k}(\|u_2\|^2+\|\nabla u_2\|^2),\end{aligned}$$

and similarly,

$$\begin{aligned}(3.34)\qquad &\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\frac{2x}{k^2}\phi'\left(\frac{|x|^2}{k^2}\right)u_2\nabla\xi^t(s)dxds\\ &\leq \frac{C}{k}\int_0^\infty\mu(s)\|\nabla\xi^t(s)\|^2ds+\frac{C\kappa(0)}{k}\|u_2\|^2.\end{aligned}$$

By the Cauchy inequality and (3.19), we have

$$\begin{aligned}(3.35)\qquad &2\left|\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)u_2\varsigma_2^\varepsilon\right|dx\\ &\leq \frac{\lambda-\delta_1}{8}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx+\frac{8}{\lambda-\delta_1}\int_{|x|\geq\sqrt{2}k}|\varsigma_2^\varepsilon|^2dx.\end{aligned}$$

Summation of (3.28) and (3.29), then using (3.30)-(3.35), we obtain

$$\begin{aligned}(3.36)\quad &\frac{d}{dt}\left(\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx\right.\\ &\qquad\left.+\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\left(j|\xi^t(s)|^2+|\nabla\xi^t(s)|^2\right)dxds\right)\\ &\quad+\frac{3(\lambda-\delta_1)}{4}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx+2\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\nabla u_2|^2dx\\ &\quad-2\int_0^\infty\mu'(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)(j|\xi^t(s)|^2+|\nabla\xi^t|^2)dxds\\ &\leq \frac{2j\kappa(0)}{\gamma_3}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx+\frac{j\gamma_3}{2}\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\xi^t(s)|^2dxds\end{aligned}$$

$$+\frac{4\delta_5^2}{\lambda-\delta_1}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|v_2|^2dx+C\int_{|x|\geq\sqrt{2}k}(\phi_1(x)+|\phi_2(x)|^2+|v_1|^2)dx$$
$$+C\int_{|x|\geq\sqrt{2}k}|\varsigma_2^\varepsilon(t)|^2dx+\frac{C}{k}\left(\|u_2\|^2+\|\nabla u_2\|^2+\int_0^\infty\mu(s)\|\nabla\xi^t(s)\|^2ds\right).$$

Now, multiplying the second equation of (3.14) by $\phi\left(\frac{|x|^2}{k^2}\right)v_2$ and then using (1.7), we find

$$\begin{aligned}(3.37)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|v_2|^2dx+\delta_4\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|v_2|^2dx\\
&=-\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\varphi(x,u)v_2dx\\
&\leq\delta_6\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\phi_3(x)||v_2|dx+\delta_6\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u||v_2|dx\\
&\leq\left(\frac{2\delta_6^2}{\lambda-\delta_1}+\frac{\lambda-\delta_1}{4}\right)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)(|\phi_3(x)|^2+|u_1|^2)dx\\
&\quad+\frac{\lambda-\delta_1}{4}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx+\frac{2\delta_6^2}{\lambda-\delta_1}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|v_2|^2dx.\end{aligned}$$

Summing up (3.36) and (3.37), and the term

$$-2\int_0^\infty\mu'(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)(j|\xi^t(s)|^2+|\nabla\xi^t|^2)dxds\geq0$$

can be neglected, we find

$$\begin{aligned}(3.38)\quad &\frac{d}{dt}A_j+\frac{\lambda-\delta_1}{4}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|u_2|^2dx+2\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\nabla u_2|^2dx\\
&+2\left(\delta_4-\frac{2(\delta_5^2+\delta_6^2)}{\lambda-\delta_1}\right)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|v_2|^2dx\\
\leq&\frac{2j\kappa(0)}{\gamma_3}\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|w|^2dx+\frac{j\gamma_3}{2}\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)|\xi^t(s)|^2dxds\\
&+C\int_{|x|\geq\sqrt{2}k}|\varsigma_2^\varepsilon(t)|^2dx+C\int_{|x|\geq\sqrt{2}k}(|\phi_2(x)|^2+|\phi_3(x)|^2+|u_1|^2+|v_1|^2)dx\\
&+\frac{C}{k}\left(\|u_2\|^2+\|\nabla u_2\|^2+\int_0^\infty\mu(s)\|\nabla\xi^t(s)\|^2ds\right),\end{aligned}$$

where

$$\begin{aligned}A_j=&\left(\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)(|u_2|^2+|v_2|^2)dx\right.\\
&\left.+\int_0^\infty\mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\left(j|\xi^t(s)|^2+|\nabla\xi^t(s)|^2\right)dxds\right).\end{aligned}$$

Similarly to the proof of (3.2), we have

$$
\begin{aligned}
(3.39)\qquad & \frac{d}{dt}\left(8\gamma_3\int_0^\infty \kappa(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\left(j|\xi^t(s)|^2+|\nabla\xi^t(s)|^2\right)dxds\right)\\
\leq & -4\gamma_3\int_0^\infty \mu(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\left(j|\xi^t(s)|^2+|\nabla\xi^t(s)|^2\right)dxds\\
& +16\gamma_3\theta^2\kappa(0)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\left(j|u_2|^2+|\nabla u_2|^2\right)dx.
\end{aligned}
$$

Summation of (3.36) and (3.39), then choosing $\gamma_3 > 0$ small enough, we end up with

$$
\begin{aligned}
(3.40)\quad & \frac{d}{dt}W_j+\gamma_3W_j\\
\leq & \frac{2j\kappa(0)}{\gamma_3}W_0+C\int_{|x|\geq\sqrt{2}k}(|\phi_2(x)|^2+|\phi_3(x)|^2+|u_1|^2+|v_1|^2)dx\\
& +C\int_{|x|\geq\sqrt{2}k}|\varsigma_2^\varepsilon(t)|^2dx+\frac{C}{k}\left(\|u_2\|^2+\|\nabla u_2\|^2+\int_0^\infty\mu(s)\|\nabla\xi^t(s)\|^2ds\right),
\end{aligned}
$$

where

$$
\begin{aligned}
& W_j = A_j+8\gamma_3\int_0^\infty\kappa(s)\int_{\mathbb{R}^N}\phi\left(\frac{|x|^2}{k^2}\right)\left(j|\xi^t(s)|^2+|\nabla\xi^t(s)|^2\right)dxds, \quad \text{and}\\
& A_j\leq W_j\leq 2A_j.
\end{aligned}
$$

Therefore, let $j = 0$ and multiplying (3.40) by e^{γ_3} and integrating from $T \geq T_B + 1$ to t, we find that

$$
\begin{aligned}
(3.41)\quad & W_0\\
\leq & e^{-\gamma_3(t-T)}W_0(T)+Ce^{-\gamma_3t}\int_T^t\int_{|x|\geq\sqrt{2}k}e^{\gamma_3r}|\varsigma_2^\varepsilon(r)|^2dxdr\\
& +Ce^{-\gamma_3t}\int_T^t\int_{|x|\geq\sqrt{2}k}e^{\gamma_3r}(|\phi_2(x)|^2+|\phi_3(x)|^2+|u_1|^2+|v_1|^2)dxdr\\
& +\frac{Ce^{-\gamma_3t}}{k}\int_T^t e^{\gamma_3r}\left(\|u_2\|^2+\|\nabla u_2\|^2+\int_0^\infty\mu(s)\|\nabla\xi^t(s)\|^2ds\right)dr.
\end{aligned}
$$

Using the assumptions of $\phi_i(x)$, $i = 1, 2, 3$ and Lemma 3.3, we have

$$
(3.42)\quad \limsup_{t\to+\infty}\limsup_{k\to+\infty}e^{-\gamma_3t}\int_\tau^t\int_{|x|\geq k}e^{\gamma_3r}(|\phi_2(x)|^2+|\phi_3(x)|^2+|u_1|^2+|v_1|^2)dxdr=0.
$$

Besides, using (3.12) and (3.20), we can see that

$$
\begin{aligned}
(3.43)\qquad & \limsup_{t\to+\infty}\limsup_{k\to+\infty}e^{-\gamma_3t}\int_\tau^t\int_{|x|\geq k}e^{\gamma_3r}|\varsigma_2^\varepsilon(r)|^2dr=0;\quad\text{and}\\
& e^{-\gamma_3(t-T)}W_0(T)\leq e^{-\gamma_3(t-T)}\rho_1^2\to 0 \text{ as } t\to+\infty;
\end{aligned}
$$

and

$$\begin{aligned}(3.44)\qquad & \frac{Ce^{-\gamma_3 t}}{k}\int_T^t e^{\gamma_3 r}\left(\|u_2\|^2+\|\nabla u_2\|^2+\int_0^\infty \mu(s)\|\nabla\xi^t(s)\|^2 ds\right)dr\\ &\leq \frac{Ce^{-\gamma_3 t}}{k}\int_T^t e^{\gamma_3 r}\rho_2 dr\\ &\leq \frac{C(\rho_2)}{k}\to 0 \text{ as } k\to+\infty.\end{aligned}$$

Combining (3.41)-(3.44), we obtain

$$W_0\to 0 \text{ as } k\to+\infty. \tag{3.45}$$

Finally, we consider (3.40) for $j=1$. Reasoning exactly as in the proof of the case $j=0$ and using (3.45), we can take T_ω and $K_\omega>0$ large enough such that

$$\int_{|x|\geq K_\omega}|K_\varsigma(t,\tau)z_\tau|^2dx<\omega$$

for all $t\geq T_\omega$, $z_\tau\in B$. The proof is complete. $\square$

In addition, for any $\xi_\tau\in L^2_\mu(\mathbb{R}^+,H^1(\mathbb{R}^N))$, the Cauchy problem (see e.g. [1])

$$\begin{cases}\partial_t\xi^t=-\partial_s\xi^t+u_2, & t>\tau,\\ \xi^\tau=\xi_\tau,\end{cases}$$

has a unique solution $\xi^t\in C((\tau,+\infty);L^2_\mu(\mathbb{R}^+,H^1(\mathbb{R}^N)))$, and

$$\xi^t(s)=\begin{cases}\int_0^s u_2(t-r)dr, & \tau<s\leq t,\\ \xi_\tau(s-t)-\xi_\tau(\tau)+\int_\tau^t u_2(t-r)dr, & s>t.\end{cases} \tag{3.46}$$

So, for the equation (3.46), thanks to $\xi^\tau(x,s)=0$, we have

$$\xi^t(s)=\begin{cases}\int_0^s u_2(t-r)dr, & \tau<s\leq t,\\ \int_0^t u_2(t-r)dr, & s>t.\end{cases}$$

Let $\mathbb{B}_0$ be the bounded absorbing set obtained in Lemma 3.2. The same argument as in Lemma 3.6 of [23], we get the following lemma:

Lemma 3.6. *Setting*

$$\mathcal{K}_T=PK_\varsigma(T,\tau)B_0$$

for $T>0$ large enough, where $\{K_\varsigma(t,\tau)\}_{t\geq\tau}$ is the solution process of (3.14), $P:L^2(B(K_\omega))\times L^2(B(K_\omega))\times L^2_\mu(\mathbb{R}^+,H^1_0(B(K_\omega)))\to L^2_\mu(\mathbb{R}^+,H^1_0(B(K_\omega)))$ *is the projection operator. Then there is a positive constant $N_1=N_1(\|B_0\|_{\mathcal{H}})$ such that*

(i) *$\mathcal{K}_T$ is bounded in*

$$L^2_\mu(\mathbb{R}^+,H^2(B(K_\omega))\cap H^1_0(B(K_\omega)))\cap H^1_\mu(\mathbb{R}^+;H^1_0(B(K_\omega)),$$

(ii) $\sup_{\xi\in\mathcal{K}_T}\|\xi(s)\|^2_{H^1_0(B(K_\omega)}\leq N_1$.

Moreover, $\mathcal{K}_T$ is relatively compact in $L^2_\mu(\mathbb{R}^+, H^1_0(B(K_\omega))$.

One of our main theorems is the following:

Theorem 3.1. *The family of processes $\{U_\varsigma(t,\tau)\}_{\varsigma\in\mathcal{H}_w(h)}$ associated to* (1.10) *possesses a uniform attractor $\mathcal{A}$ in the space $\mathcal{H}$. Moreover,*

$$\mathcal{A} = \bigcup_{\varsigma\in\mathcal{H}_w(h)} \mathcal{K}_\varsigma(s), \quad \forall s\in\mathbb{R},$$

where $\mathcal{K}_\varsigma(s)$ is the kernel section at time s of the process $U_\varsigma(t,\tau)$.

Proof. By Lemma 3.2, the family of processes $U_\varsigma(t,\tau)$ has a bounded absorbing $\mathbb{B}_0$ in $\mathcal{H}$. Moreover, $U_\varsigma(t,\tau)$ is uniform asymptotically compact in $\mathcal{H}$ due to Lemmas 3.3 and 3.6. Therefore, the family of process $U_\varsigma(t,\tau)$ has the uniform attractor $\mathcal{A}$ in $\mathcal{H}$. □

4. Regularity of uniform attractor

In what follows, we show that the uniform attractor $\mathcal{A}$ is a bounded subset of $\mathcal{H}_2 = H^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)\times L^2_\mu(\mathbb{R}^+, H^2(\mathbb{R}^N))$. To prove the uniform the boundedness of the uniform attractor, we assume that the external force h satisfies a stronger hypothesis:

(**H3Bis**) The functions $h\in L^\infty(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ and $\partial_t h\in L^2_b(\mathbb{R}; H^{-1}(\mathbb{R}^N))$.

Theorem 4.1. *Assume that* (**H1**), (**H2**) *and* (**H3Bis**) *hold. Then the uniform attractor $\mathcal{A}$ is bounded in $\mathcal{H}_2$.*

Proof. Recall that in this paper we only assume the external force $h(\cdot,t)\in H^{-1}(\mathbb{R}^N)$. Thus, to prove the boundedness of the uniform attractor, we cannot multiply $-\Delta u$ immediately into the first equation. To overcome this difficulty, for fix $\tau\in\mathbb{R}$ and each initial data $z_\tau\in\mathcal{A}_\tau$, using the method of the semigroup decomposition given in Subsection 3.2.1 and according to Lemmas 3.3 and 3.4, we get

$$\|(u_1(t), v_1(t), \zeta^t)\|^2_{\mathcal{H}} = \|D(t,\tau)z_{1\tau}\|^2_{\mathcal{H}} \le \mathcal{Q}(\|z_\tau\|_{\mathcal{H}})e^{-\gamma_1(t-\tau)} + \varepsilon, \tag{4.1}$$

and

$$\|u_2\|^2_{H^1(\mathbb{R}^N)} + \|v_2\|^2_{H^1(\mathbb{R}^N)} + \|\xi^t\|^2_{2,\mu} \le \rho_4. \tag{4.2}$$

Now we will prove $\|(u_2, v_2, \xi^t)\|^2_{\mathcal{H}_2} \le \rho_7$ for some $\rho_7 > 0$. To prove this inequality, we need to take some steps:

Step 1: *The estimate for $\partial_t u_2$ in $L^2(\mathbb{R}^N)$*. We now differentiate the first equation and the third equation in (3.14) with respect to t to get

$$\begin{cases} \partial_t U - \Delta U - \int_0^\infty \mu(s)\Delta Z^t(s)ds + \lambda U + f'(u_2)U + g_v'(x,v)v_t = \partial_t \varsigma_2^\varepsilon, \\ \partial_t v_2 + \sigma(x)v_2 + \varphi(x,u) = 0, \\ \partial_t Z^t = -\partial_s Z^t + U, \\ U(x,t)|_{t\le\tau} = 0, v_2(x,t)|_{t\le\tau} = 0, \\ Z^\tau(x,s) = Z_\tau(x,s) = 0, \end{cases} \tag{4.3}$$

where $U = \partial_t u_2$, $Z^t = \partial_t \xi^t$.

Multiplying the first equation of (4.3) by U, then using the conditions (1.2) and (1.3) and the Young inequality, we have

$$\begin{aligned} &\frac{d}{dt}\left(\|U\|^2 + \int_0^\infty \mu(s)\|\nabla Z^t(s)\|^2 ds\right) + 2(\lambda - \ell - \varepsilon_0)\|U\|^2 \\ &\quad + 2\|\nabla U\|^2 - 2\int_0^\infty \mu'(s)\|\nabla Z^t(s)\|^2 ds \\ &\le C(\varepsilon_0)\delta_2^2\|v_t\|^2 + C(\varepsilon_0)\|\partial_t\varsigma_2^\varepsilon\|^2, \end{aligned}$$

where

$$\begin{aligned} &(f'(u_2)U, U) \ge -\ell\|U\|^2; \\ &|(g_v'(x,v)v_t, U)| \le \delta_2\|v_t\|\|U\| \le C(\varepsilon_0)\delta_2^2\|v_t\|^2 + \varepsilon_0\|U\|^2; \\ &(\partial_t\varsigma_2^\varepsilon, U) \le \varepsilon_0\|U\|^2 + C(\varepsilon_0)\|\partial_t\varsigma_2^\varepsilon\|^2. \end{aligned}$$

Now, for $\gamma_4 > 0$ to be fixed, we define the functional

$$\Lambda(t) = \|U\|^2 + \int_0^\infty \mu(s)\|\nabla Z^t(s)\|^2 ds + 8\gamma_4\int_0^\infty \kappa(s)\|\nabla Z^t(s)\|^2 ds.$$

Up to further reducing γ_4, we also have

$$\|U\|^2 + \int_0^\infty \mu(s)\|\nabla Z^t(s)\|^2 ds \le \Lambda \le 2\left(\|U\|^2 + \int_0^\infty \mu(s)\|\nabla Z^t(s)\|^2 ds\right).$$

Then, using Lemma 3.1, we can see that Λ satisfies the differential inequality

$$\begin{aligned} &\frac{d}{dt}\Lambda(t) + 2(\lambda - \ell - \varepsilon_0)\|U\|^2 + 2(1 - 4\gamma_4\theta^2\kappa(0))\|\nabla U\|^2 \\ &\quad + 4\gamma_4\int_0^\infty \mu(s)\|\nabla Z^t(s)\|^2 ds \\ &\le C(\varepsilon_0)\delta_2^2\|v_t\|^2 + C(\varepsilon_0)\|\partial_t\varsigma_2^\varepsilon\|^2, \end{aligned} \tag{4.4}$$

where $-2\int_0^\infty \mu'(s)\|\nabla Z^t(s)\|^2 ds \ge 0$.

Multiplying the second equation in (1.10) by v_t and using (1.7), (3.9), we obtain

$$\begin{aligned} \|v_t\|^2 &\leq \int_{\mathbb{R}^N} |\sigma(x)|^2|v|^2dx + \int_{\mathbb{R}^N} |\varphi(x,u)|^2dx \\ &\leq 2\rho_4 \sup_{x\in\mathbb{R}^N} |\sigma(x)|^2 + 4\delta_6^2(\|\phi_3\|^2+\rho_1) \\ &\leq \rho_5,\ \forall t \geq T_B+1. \end{aligned} \tag{4.5}$$

Combining (4.4), (4.5) and (3.1), then choosing γ_4 and ε_0 are small enough, it follows that

$$\frac{d}{dt}\Lambda(t) + \gamma_4\Lambda(t) \leq C(\delta_2^2\rho_5 + \|\partial_t\varsigma_2^\varepsilon\|^2),\ \forall t \geq T_B+1.$$

Applying the Gronwall lemma and using (**H3Bis**), we deduce

$$\|U\|^2 + \int_0^\infty \mu(s)\|\nabla Z^t(s)\|^2ds \leq \rho_6,\ \forall t \geq T_B+1.$$

Thus,

$$\|\partial_t u_2(t)\|^2 + \int_0^\infty \mu(s)\|\nabla\partial_t\xi^t(s)\|^2ds \leq \rho_6,\ \forall t \geq T_B+1. \tag{4.6}$$

Step 2: The estimate $\|u_2\|^2_{H^2(\mathbb{R}^N)} \leq \rho_8$. Now multiplying the first equation in (3.14) with $-\Delta u_2$, if we use the conditions (1.2), (1.3), (1.6), we have

$$\begin{aligned} \|\Delta u_2\|^2 =& -(\partial_t u_2, -\Delta u_2) - (f(u_2), -\Delta u_2) - (g(\cdot,v), -\Delta u_2) \\ &+ \int_0^\infty \mu(s)(\Delta\xi^t(s), -\Delta u_2)ds + (\varsigma_2^\varepsilon, -\Delta u_2) \\ \leq& \frac{1}{2}\|\Delta u_2\|^2 + 2\|\partial_t u_2\|^2 + \ell\|\nabla u_2\|^2 \\ &+ C\int_0^\infty \mu(s)\|\Delta\xi^t(s)\|^2ds + 2\delta_5^2(\|\phi_2\|^2+\|v\|^2) + 2\|\varsigma_2^\varepsilon\|^2. \end{aligned}$$

By applying (4.2) and (4.6) and (**H2Bis**), we can take a positive constant ρ_7 satisfying

$$\|\Delta u_2(t)\|^2 \leq \rho_7,\ \forall t \geq T_B+1,$$

and therefore

$$\|(u_2, v_2, \xi^t)\|^2_{\mathcal{H}_2} \leq \rho_8,\ \forall t \geq T_B+1. \tag{4.7}$$

Collecting (4.1) and (4.7), and setting $\mathbb{B}_1 = B_{\mathcal{H}_2}(\rho_8)$, we get

$$dist_{\mathcal{H}}(\mathcal{A}, \mathbb{B}_1) = dist_{\mathcal{H}}(U(t,\tau)\mathcal{A}, \mathbb{B}_1) \leq Ce^{-\gamma_1(t-\tau)} + \varepsilon.$$

Hence, $\mathcal{A} \subset \mathbb{B}_1$, proving that $\mathcal{A}$ is bounded in $\mathcal{H}_2$. □

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