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DEPTH AND STANLEY DEPTH OF TWO SPECIAL CLASSES OF MONOMIAL IDEALS

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Abstract. In this paper, we define two new classes of monomial ideals $I_{l,d}$ and $J_{k,d}$. When $d \geq 2k+1$ and $l \leq d-k-1$, we give the exact formulas to compute the depth and Stanley depth of quotient rings $S/I_{l,d}^t$ for all $t \geq 1$. When $d = 2k = 2l$, we compute the depth and Stanley depth of quotient ring $S/I_{l,d}$. When $d \geq 2k$, we also compute the depth and Stanley depth of quotient ring $S/J_{k,d}$.

1. Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over K in n variables. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. A Stanley decomposition $\mathcal{D}$ of M is a finite direct sum of K-vector spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} u_i K[Z_i],$$

where $u_i \in M$ is homogeneous and $Z_i \subseteq \{x_1, \ldots, x_n\}$, $i = 1, \ldots, r$, and its Stanley depth, sdepth$(\mathcal{D})$, is defined as $\min\{|Z_i| : i = 1, \ldots, r\}$. The number

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called the Stanley depth of M. For a friendly introduction to Stanley depth, we refer the reader to [6, 14].

Stanley conjectured in [16] that sdepth$(M) \geq$ depth(M) for any $\mathbb{Z}^n$-graded S-module M. There are many researches on this conjecture, especially when M has the form S/I or I with I a monomial ideal of S, see [1, 4, 7, 15]. In [5], Duval et al. constructed an explicit counterexample to disprove the Stanley conjecture for S/I, where I is a monomial ideal of S. But it is still important to find new classes of $\mathbb{Z}^n$-graded modules which satisfy the Stanley inequality.

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For the monomial ideal $I \subset S$ it is clear that $\operatorname{depth}(I) = \operatorname{depth}(S/I) + 1$, whereas for Stanley depth this is not the case. In [6], Herzog conjectured:

Conjecture 1.1. *Let $I \subset S$ be a monomial ideal. Then* $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I)$.

The above conjecture has been proved in some special cases by Popescu and Qureshi in [12] and Rauf in [13]. For recent works on the above conjecture, we refer the reader to [8–10].

Let Δ be a simplicial complex on the vertex set $V = \{x_i : 1 \leq i \leq n\}$. Each element of Δ is called a face of Δ, and a face F is called a facet if F is a maximal face under inclusion. Let $\mathcal{F}(\Delta)$ denote the collection of all its facets. For each subset $F \subset V$, we set $x_F = \prod_{x_j \in F} x_j$. By identifying the vertex x_i with the variable x_i in the polynomial ring S, one can associate Δ with a squarefree monomial ideal $I(\Delta) = (x_F : F \in \mathcal{F}(\Delta))$, which is called the facet ideal of Δ. In [19], Zhu computed the depth and Stanley depth of the edge ideals (which are in fact the facet ideals of graphs) of some m-line graphs and m-cyclic graphs with a common vertex. Wei and Gu [18] defined two classes of simplicial complexes $\Delta_{n,d}$ and $\Delta'_{n,d}$, where

$$\begin{aligned}\mathcal{F}(\Delta_{n,d}) = \{&\{x_1, x_2, \ldots, x_d\}, \{x_{d-k+1}, x_{d-k+2}, \ldots, x_{2d-k}\}, \ldots,\\ &\{x_{n-d+1}, x_{n-d+2}, \ldots, x_n\}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(\Delta'_{n,d}) = \{&\{x_1, x_2, \ldots, x_d\}, \{x_{d-k+1}, x_{d-k+2}, \ldots, x_{2d-k}\}, \ldots,\\ &\{x_{n-2d+2k+1}, x_{n-2d+2k+2}, \ldots, x_{n-d+2k}\},\\ &\{x_{n-d+k+1}, \ldots, x_n, x_1, \ldots, x_k\}\}.\end{aligned}$$

They computed the depth and Stanley depth of the facet ideals of these simplicial complexes.

In this paper, we define two new classes of squarefree monomial ideals $I_{l,d}$ and $J_{k,d}$, where $I_{l,d}$ (resp. $J_{k,d}$) is in fact the facet ideal associated to the simplicial complex consisting of the union of $\Delta_{n_1,d}, \ldots, \Delta_{n_s,d}$ (resp. $\Delta'_{n_1,d}, \ldots, \Delta'_{n_s,d}$) with common vertices $x_1, \ldots, x_l$ (resp. $x_1, \ldots, x_k$). These two ideals generalize the constructions of those monomial ideals introduced in [18] and [19]. In this article, we study the depth and Stanley depth of quotient rings of $I_{l,d}$ and $J_{k,d}$, and prove Conjecture 1.1 for these two ideals in some cases.

Our paper is organized as follows: In Section 2, we give the definitions of $I_{l,d}$ and $J_{k,d}$, and review some terminologies, notations and results. In Section 3, we first give the exact formulas for depth and Stanley depth of quotient rings $S/I_{l,d}^t$ for all $t \geq 1$, when $d \geq 2k+1$ and $l \leq d-k-1$. We also compute the depth and Stanley depth of quotient ring $S/I_{l,d}$, when $d = 2k = 2l$. In Section 4, we compute the depth and Stanley depth of quotient ring $S/J_{k,d}$ in two cases: $d \geq 2k+1$ and $d = 2k$.

2. Preliminaries

In this section, we first give the definitions of $I_{l,d}$ and $J_{k,d}$, and review some standard terminologies and notations from algebra. For more details, see [17]. Let $s \geq 1$ be an integer throughout the paper.

Definition 2.1. Let l, k, d and n_i be positive integers with $i \in [s] := \{1, 2, \ldots, s\}$. We define the squarefree monomial ideal

$$I_{l,k,d,(n_i)_{1\leq i\leq s}} := \sum_{i=1}^{s}(x_1 \cdots x_l x_{l+1,i} \cdots x_{d,i},\ x_{d-k+1,i}x_{d-k+2,i} \cdots x_{2d-k,i},\ \ldots,\ x_{n_i-d+1,i}x_{n_i-d+2,i} \cdots x_{n_i,i}),$$

where $1 \leq l \leq d-k$ and $n_i \geq d > k \geq 1$ for $1 \leq i \leq s$. Note that $d-k \mid n_i - k$ for all $i \in [s]$.

Remark 2.2. (1) For simplicity, we denote $I_{l,d} := I_{l,k,d,(n_i)_{1\leq i\leq s}}$ in this paper.

(2) $|G(I_{l,d})| = \sum_{i=1}^{s} \frac{n_i-k}{d-k}$, where $G(I_{l,d})$ denotes the set of minimal monomial generators of $I_{l,d}$.

Example 2.3. Set $s = 3$, $l = k = 1$, $d = 3$, $n_1 = 3$, $n_2 = 5$ and $n_3 = 7$ in Definition 2.1. Then we have $I_{1,3} = (x_1x_{2,1}x_{3,1}, x_1x_{2,2}x_{3,2}, x_{3,2}x_{4,2}x_{5,2}, x_1x_{2,3}x_{3,3}, x_{3,3}x_{4,3}x_{5,3}, x_{5,3}x_{6,3}x_{7,3})$.

Definition 2.4. Let k, d and n_i be positive integers with $i \in [s]$. We define the squarefree monomial ideal

$$J_{k,d,(n_i)_{1\leq i\leq s}} := \sum_{i=1}^{s}(x_1 \cdots x_k x_{k+1,i} \cdots x_{d,i},\ x_{d-k+1,i}x_{d-k+2,i} \cdots x_{2d-k,i},\ \ldots,\ x_{n_i-2d+2k+1,i}x_{n_i-2d+2k+2,i} \cdots x_{n_i-d+2k,i},\ x_{n_i-d+k+1,i} \cdots x_{n_i,i}x_1 \cdots x_k),$$

where $d \geq 2k \geq 2$ and $n_i \geq 3d - 3k$ for $1 \leq i \leq s$. Note that $d-k \mid n_i$ for all $i \in [s]$.

Remark 2.5. (1) For convenience, we denote $J_{k,d} := J_{k,d,(n_i)_{1\leq i\leq s}}$ in this paper.

(2) It is easy to see that $|G(J_{k,d})| = \sum_{i=1}^{s} \frac{n_i}{d-k}$.

Example 2.6. Set $s = 3$, $d = 2k = 2$, $n_1 = 3$, $n_2 = 4$ and $n_3 = 5$ in Definition 2.4, then we get $J_{1,2} = (x_1x_{2,1}, x_{2,1}x_{3,1}, x_{3,1}x_1, x_1x_{2,2}, x_{2,2}x_{3,2}, x_{3,2}x_{4,2}, x_{4,2}x_1, x_1x_{2,3}, x_{2,3}x_{3,3}, x_{3,3}x_{4,3}, x_{4,3}x_{5,3}, x_{5,3}x_1)$.

Let $I \subset S$ be a monomial ideal. The big height of I, denoted by $\text{bight}(I)$, is the maximum height of the minimal prime ideals of I. The arithmetical rank of I, denoted by $\text{ara}(I)$, is the minimum number r of elements of S such that the ideal $(u_1, u_2, \ldots, u_r)$ has the same radical as I. If I is a squarefree monomial ideal, it is well-known that

$$\text{ht}(I) \leq \text{bight}(I) \leq \text{pd}(S/I) \leq \text{ara}(I) \leq |G(I)|,$$

where $\mathrm{pd}(S/I)$ denotes the projective dimension of S/I.

A prime ideal P is associated to I if $P = (I : c)$ for some monomial $c \in S$. The set of prime ideals associated to I will be denoted by $\mathrm{Ass}(S/I)$. The associated prime ideals of a monomial ideal are monomial prime ideals. The set $\mathrm{Min}(S/I)$ consists of all prime ideals that are minimal over I with respect to inclusion. It is known that $\mathrm{Min}(S/I) \subset \mathrm{Ass}(S/I)$. When I is squarefree, $\mathrm{Ass}(S/I) = \mathrm{Min}(S/I)$.

Now we recall some known results that are heavily used in this paper.

Lemma 2.7 (Depth Lemma). *Let S be a local ring or a Noetherian graded ring with S_0 local. If*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of finitely generated S-modules, where the maps are all homogeneous, then ([17, Lemma 1.3.9]):

a) *If* $\mathrm{depth}(B) < \mathrm{depth}(C)$*, then* $\mathrm{depth}(A) = \mathrm{depth}(B)$.

b) *If* $\mathrm{depth}(B) = \mathrm{depth}(C)$*, then* $\mathrm{depth}(A) \geq \mathrm{depth}(B)$.

c) *If* $\mathrm{depth}(B) > \mathrm{depth}(C)$*, then* $\mathrm{depth}(A) = \mathrm{depth}(C) + 1$.

Also (*see* [2, Proposition 1.2.9]):

d) $\mathrm{depth}(A) \geq \min\{\mathrm{depth}(B), \mathrm{depth}(C) + 1\}$.

e) $\mathrm{depth}(B) \geq \min\{\mathrm{depth}(A), \mathrm{depth}(C)\}$.

f) $\mathrm{depth}(C) \geq \min\{\mathrm{depth}(A) - 1, \mathrm{depth}(B)\}$.

In [13], Rauf proved the analog of Lemma 2.7(e) for Stanley depth:

Lemma 2.8. *Let $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of finitely generated $\mathbb{Z}^n$-graded S-modules. Then*

$$\mathrm{sdepth}(M) \geq \min\{\mathrm{sdepth}(U), \mathrm{sdepth}(N)\}.$$

We also need the following lemma, see [13, Theorem 3.1].

Lemma 2.9. *Let $I \subset S_1 = K[x_1, \ldots, x_n]$, $J \subset S_2 = K[y_1, \ldots, y_m]$ be monomial ideals and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Then*

$$\mathrm{sdepth}(S/(IS, JS)) \geq \mathrm{sdepth}(S_1/I) + \mathrm{sdepth}(S_2/J).$$

3. Depth and Stanley depth of the monomial ideal $I_{l,d}$

Throughout this section, we set $S := K[x_1, \ldots, x_l, x_{l+1,1}, \ldots, x_{n_1,1}, \ldots, x_{l+1,s}, \ldots, x_{n_s,s}]$ be the polynomial ring over a field K in n variables, where $n := \sum_{i=1}^{s} n_i - (s-1)l$. Next, we will discuss our main results in two cases.

3.1. The case $d \geq 2k + 1$ and $l \leq d - k - 1$

In this section, we will give some formulas for depth and Stanley depth of quotient rings $S/I_{l,d}^t$ for all $t \geq 1$. Our proofs of the main results make heavy use of the following lemma.

Lemma 3.1. $P = \sum_{i=1}^{s}(x_{d-k,i}, x_{2(d-k),i}, x_{3(d-k),i}, \ldots, x_{n_i-k,i}) \in \mathrm{Min}(S/I_{l,d})$.

Proof. Let $a_{i,j} = x_{1+(i-1)(d-k),j}x_{2+(i-1)(d-k),j}\cdots x_{d+(i-1)(d-k),j}$ and $b_{i,j} = x_{i(d-k),j}$ for $1 \le i \le \frac{n_j-k}{d-k}$ and $1 \le j \le s$, where $x_{1,j} = x_1, \dots, x_{l,j} = x_l$ for all $j \in [s]$. Then $I_{l,d} = \sum_{j=1}^{s}(a_{1,j}, a_{2,j}, \dots, a_{(n_j-k)/(d-k),j})$ and $P = \sum_{j=1}^{s}(b_{1,j}, b_{2,j}, \dots, b_{(n_j-k)/(d-k),j})$.

According to the definitions of $a_{i,j}$ and $b_{i,j}$, b_{i_1,j_1} appears in a_{i_2,j_2} if and only if $i_1 = i_2$ and $j_1 = j_2$. It follows that b_{i_1,j_1} divides a_{i_2,j_2} if and only if $i_1 = i_2$ and $j_1 = j_2$, so $I_{l,d} \subset P$. We assume that P is not minimal over $I_{l,d}$. Let $P_0 \subsetneq P$ be a minimal prime ideal of $I_{l,d}$. Since $I_{l,d}$ is squarefree, $P_0 \subsetneq P$ is a monomial prime ideal, and there exists $a_{i,j}$ such that none of $G(P_0)$ divides $a_{i,j}$. Hence $I_{l,d} \nsubseteq P_0$, a contradiction. □

Proposition 3.2. $\operatorname{bight}(I_{l,d}) = \operatorname{pd}(S/I_{l,d}) = \operatorname{ara}(I_{l,d}) = |G(I_{l,d})| = \sum_{i=1}^{s}\frac{n_i-k}{d-k}$.

Proof. From Lemma 3.1, $P = \sum_{i=1}^{s}(x_{d-k,i}, x_{2(d-k),i}, x_{3(d-k),i}, \dots, x_{n_i-k,i}) \in \operatorname{Min}(S/I_{l,d})$ and $\operatorname{ht}(P) = \sum_{i=1}^{s}\frac{n_i-k}{d-k}$. It follows that $\sum_{i=1}^{s}\frac{n_i-k}{d-k} \le \operatorname{bight}(I_{l,d}) \le \operatorname{pd}(S/I_{l,d}) \le \operatorname{ara}(I_{l,d}) \le |G(I_{l,d})| = \sum_{i=1}^{s}\frac{n_i-k}{d-k}$. Now the result is clear. □

Now, we give the exact formulas for $\operatorname{sdepth}(S/I_{l,d})$ and $\operatorname{depth}(S/I_{l,d})$.

Theorem 3.3. $\operatorname{sdepth}(S/I_{l,d}) = \operatorname{depth}(S/I_{l,d}) = n - \sum_{i=1}^{s}\frac{n_i-k}{d-k}$.

Proof. Since $|G(I_{l,d})| = \sum_{i=1}^{s}\frac{n_i-k}{d-k}$, we have $\operatorname{sdepth}(S/I_{l,d}) \ge n - \sum_{i=1}^{s}\frac{n_i-k}{d-k}$ by [3, Proposition 1.2]. On the other hand, there exists a prime ideal $P \in \operatorname{Ass}(S/I_{l,d})$ such that $\operatorname{ht}(P) = \sum_{i=1}^{s}\frac{n_i-k}{d-k}$ by Lemma 3.1. It follows that $\operatorname{sdepth}(S/I_{l,d}) \le n - \sum_{i=1}^{s}\frac{n_i-k}{d-k}$ by [7, Proposition 1.3]. By the Auslander-Buchsbaum formula and Proposition 3.2, $\operatorname{depth}(S/I_{l,d}) = n - \operatorname{pd}(S/I_{l,d}) = n - \sum_{i=1}^{s}\frac{n_i-k}{d-k}$. □

The following corollary states that the Stanley inequality and Conjecture 1.1 hold for $I_{l,d}$.

Corollary 3.4. $\operatorname{sdepth}(I_{l,d}) \ge \operatorname{sdepth}(S/I_{l,d}) + 1 = \operatorname{depth}(I_{l,d})$.

Proof. Since $|G(I_{l,d})| = \sum_{i=1}^{s}\frac{n_i-k}{d-k}$, $\operatorname{sdepth}(I_{l,d}) \ge \max\{1, n - \lfloor\frac{1}{2}|G(I_{l,d})|\rfloor\} = n - \lfloor\sum_{i=1}^{s}\frac{n_i-k}{2(d-k)}\rfloor \ge n - \sum_{i=1}^{s}\frac{n_i-k}{d-k} + 1 = \operatorname{sdepth}(S/I_{l,d}) + 1 = \operatorname{depth}(S/I_{l,d}) + 1 = \operatorname{depth}(I_{l,d})$ by [11, Theorem 2.3] and Theorem 3.3. □

Let $I_{l,k,d,n_i} := (x_1\cdots x_l x_{l+1,i}\cdots x_{d,i},\ x_{d-k+1,i}\cdots x_{2d-k,i},\ \dots,\ x_{n_i-d+1,i}\cdots x_{n_i,i})$, $i = 1, \dots, s$. For simplicity, we denote $I_{n_i,i} := I_{l,k,d,n_i}$, hence $I_{l,d} = \sum_{i=1}^{s} I_{n_i,i}$. Next, we present a main result of this section.

Theorem 3.5. *For all* $t \ge 1$, $\operatorname{sdepth}(S/I_{l,d}^t) = \operatorname{depth}(S/I_{l,d}^t) = n - \sum_{i=1}^{s}\frac{n_i-k}{d-k}$.

Proof. We use induction on n and t. If $n_i = d$ for $1 \le i \le s$ (i.e., $n = sd - (s-1)l$), then $I_{l,d}^t = (x_1\cdots x_l x_{l+1,1}\cdots x_{d,1}, \dots, x_1\cdots x_l x_{l+1,s}\cdots x_{d,s})^t$. We consider the short exact sequence

$$0 \longrightarrow \frac{S}{(I_{l,d}^t : x_1^t\cdots x_l^t)} \longrightarrow \frac{S}{I_{l,d}^t} \longrightarrow \frac{S}{(I_{l,d}^t, x_1^t\cdots x_l^t)} \longrightarrow 0.$$

Since $x_1^t\cdots x_l^t$ divides any element of $G(I_{l,d}^t)$, it follows that $(I_{l,d}^t : x_1^t\cdots x_l^t) = (x_{l+1,1}\cdots x_{d,1},\dots,x_{l+1,s}\cdots x_{d,s})^t := J^t$ and $(I_{l,d}^t, x_1^t\cdots x_l^t) = (x_1^t\cdots x_l^t)$. It is easy to see that $J\subset S$ is a complete intersection. Then by [4, Theorem 2.15(1)], we obtain

$$\operatorname{sdepth}(S/(I_{l,d}^t : x_1^t\cdots x_l^t)) = \operatorname{sdepth}(S/J^t) = \dim(S/J) = n-s.$$

Similarly, $\operatorname{depth}(S/(I_{l,d}^t : x_1^t\cdots x_l^t)) = n-s$. Note that $(x_1^t\cdots x_l^t)$ is principal, then $\operatorname{sdepth}(S/(I_{l,d}^t, x_1^t\cdots x_l^t)) = \operatorname{depth}(S/(I_{l,d}^t, x_1^t\cdots x_l^t)) = n-1\geq n-s$. It follows that $\operatorname{sdepth}(S/I_{l,d}^t)\geq \operatorname{depth}(S/I_{l,d}^t) = n-s$ by Lemmas 2.7 and 2.8. From Lemma 3.1, $P_0 = (x_{d-k,1}, x_{d-k,2},\dots,x_{d-k,s})\in \operatorname{Min}(S/I_{l,d}) = \operatorname{Min}(S/I_{l,d}^t)\subset \operatorname{Ass}(S/I_{l,d}^t)$ for all $t\geq 1$, and $\operatorname{ht}(P_0)=s$. Then $\operatorname{sdepth}(S/I_{l,d}^t)\leq \dim(S/P_0) = n-s$ by [7, Proposition 1.3], so $\operatorname{sdepth}(S/I_{l,d}^t) = \operatorname{depth}(S/I_{l,d}^t) = n-s$ for all $t\geq 1$. Assume that $n > sd-(s-1)l$ in the following.

If $t=1$, the result holds for all n by Theorem 3.3. Now assume that $t\geq 2$ and $n_s\geq 2d-k$. We denote $u := x_{n_s-d+1,s}\cdots x_{n_s-d+k,s}$, $v := x_{n_s-d+k+1,s}\cdots x_{n_s,s}$ and consider the short exact sequence

$$0\longrightarrow \frac{S}{(I_{l,d}^t : uv)}\longrightarrow \frac{S}{(I_{l,d}^t : v)}\longrightarrow \frac{S}{((I_{l,d}^t : v), u)}\longrightarrow 0.$$

Let $G(I_{l,d}) = \bigcup_{j=1}^s\{a_{1,j}, a_{2,j},\dots,a_{(n_j-k)/(d-k),j}\}$, the same as in the proof of Lemma 3.1, and $w\in G(I_{l,d}^t)$. If $a_{(n_s-k)/(d-k),s}\mid w$, then $\frac{w}{a_{(n_s-k)/(d-k),s}}\in G(I_{l,d}^t : uv)\cap I_{l,d}^{t-1}$. If $a_{(n_s-k)/(d-k),s}\nmid w$ and $a_{(n_s-d)/(d-k),s}\mid w$, then we get $\frac{w}{u}\in G(I_{l,d}^t : uv)$ and $\frac{w}{a_{(n_s-d)/(d-k),s}}\mid \frac{w}{u}$, where $\frac{w}{a_{(n_s-d)/(d-k),s}}\in I_{l,d}^{t-1}$. If $a_{(n_s-k)/(d-k),s}\nmid w$ and $a_{(n_s-d)/(d-k),s}\nmid w$, then $\frac{w}{1}\in G(I_{l,d}^t : uv)$ and w must be divisible by some element of $I_{l,d}^{t-1}$. Thus $(I_{l,d}^t : uv)\subseteq I_{l,d}^{t-1}$. It follows that $(I_{l,d}^t : uv) = I_{l,d}^{t-1}$.

By induction on t, we get $\operatorname{depth}(S/(I_{l,d}^t : uv)) = \operatorname{depth}(S/I_{l,d}^{t-1}) = n-\sum_{i=1}^s\frac{n_i-k}{d-k}$. Similarly, $\operatorname{sdepth}(S/(I_{l,d}^t : uv)) = n-\sum_{i=1}^s\frac{n_i-k}{d-k}$.

Since u divides any element of $G(I_{l,d}^t)$ which is divisible by $a_{(n_s-k)/(d-k),s}$ or $a_{(n_s-d)/(d-k),s}$, it follows that $((I_{l,d}^t : v), u) = (I'S, u)$, where $I' := (I_{n_1,1},\dots, I_{n_{s-1},s-1}, I_{n_s-2d+2k,s})^t\subset S_1 := K[x_1,\dots,x_l, x_{l+1,1},\dots,x_{n_1,1},\dots,x_{l+1,s-1},\dots, x_{n_{s-1},s-1}, x_{l+1,s},\dots,x_{n_s-2d+2k,s}]$. Notice that u is regular on $S/I'S$, hence the induction on n and [7, Lemma 3.6] imply that

$$\begin{aligned}\operatorname{depth}_S(S/((I_{l,d}^t : v), u)) &= \operatorname{depth}_{S_1}(S_1/I') + (2d-2k) - 1\\ &= n-\sum_{i=1}^{s-1}\frac{n_i-k}{d-k} - \frac{(n_s-2d+2k)-k}{d-k} - 1\\ &= n-\sum_{i=1}^{s}\frac{n_i-k}{d-k} + 1.\end{aligned}$$

Similarly, $\operatorname{sdepth}(S/((I_{l,d}^t : v), u)) = n - \sum_{i=1}^s \frac{n_i - k}{d-k} + 1$. Then we have $\operatorname{sdepth}(S/(I_{l,d}^t : v)) \geq \operatorname{depth}(S/(I_{l,d}^t : v)) = n - \sum_{i=1}^s \frac{n_i-k}{d-k}$ by Lemmas 2.7 and 2.8.

Since v divides any element of $G(I_{l,d}^t)$ which is divisible by $a_{(n_s-k)/(d-k),s}$, we get $(I_{l,d}^t, v) = (I''S, v)$, where $I'' := (I_{n_1,1}, \ldots, I_{n_{s-1},s-1}, I_{n_s-d+k,s})^t \subset S_2 := K[x_1, \ldots, x_l, x_{l+1,1}, \ldots, x_{n_1,1}, \ldots, x_{l+1,s-1}, \ldots, x_{n_{s-1},s-1}, x_{l+1,s}, \ldots, x_{n_s-d+k,s}]$. Note that v is regular on $S/I''S$, by induction on n and [7, Lemma 3.6], we deduce that

$$\begin{aligned}\operatorname{depth}_S(S/(I_{l,d}^t, v)) &= \operatorname{depth}_{S_2}(S_2/I'') + (d-k) - 1\\ &= n - \sum_{i=1}^{s-1} \frac{n_i - k}{d-k} - \frac{(n_s - d + k) - k}{d-k} - 1\\ &= n - \sum_{i=1}^{s} \frac{n_i - k}{d-k}.\end{aligned}$$

Similarly, $\operatorname{sdepth}(S/(I_{l,d}^t, v)) = n - \sum_{i=1}^s \frac{n_i-k}{d-k}$. By applying Lemmas 2.7 and 2.8 to the short exact sequence

$$0 \longrightarrow \frac{S}{(I_{l,d}^t : v)} \longrightarrow \frac{S}{I_{l,d}^t} \longrightarrow \frac{S}{(I_{l,d}^t, v)} \longrightarrow 0,$$

we obtain $\operatorname{sdepth}(S/I_{l,d}^t) \geq \operatorname{depth}(S/I_{l,d}^t) = n - \sum_{i=1}^s \frac{n_i-k}{d-k}$.

From Lemma 3.1, $P_1 = \sum_{i=1}^s (x_{d-k,i}, x_{2(d-k),i}, \ldots, x_{n_i-k,i}) \in \operatorname{Ass}(S/I_{l,d}^t)$ for all $t \geq 1$, and $\operatorname{ht}(P_1) = \sum_{i=1}^s \frac{n_i-k}{d-k}$. Then $\operatorname{sdepth}(S/I_{l,d}^t) \leq \dim(S/P_1) = n - \sum_{i=1}^s \frac{n_i-k}{d-k}$ by [7, Proposition 1.3]. This completes the proof. $\square$

Remark 3.6. Set $s = 1$ in Theorem 3.5, then $\operatorname{sdepth}(S/I_{l,d}^t) = \operatorname{depth}(S/I_{l,d}^t) = n_1 - \frac{n_1-k}{d-k}$ for all $t \geq 1$. Thus our results generalize [18, Theorem 2.7].

3.2. The case $d = 2k = 2l$

In this section, we will give some formulas for depth and Stanley depth of quotient ring $S/I_{l,d}$. We adopt the following notations:

$$\begin{aligned}\alpha &:= \textstyle\sum_{i=1}^s \left(\frac{(d-2)n_i}{d} + \lceil \frac{2n_i - 2d}{3d} \rceil\right) + s + k - sk,\\ \beta &:= \textstyle\sum_{i=1}^s \left(\frac{(d-2)n_i}{d} + \lceil \frac{2n_i}{3d} \rceil\right) + k - sk,\\ \gamma &:= \textstyle\sum_{i=1}^s \left(\frac{(d-2)n_i}{d} + \lceil \frac{2n_i - d}{3d} \rceil\right) + s + k - sk - 1.\end{aligned}$$

Now, we prove the main results of this section.

Theorem 3.7. *With the notations introduced one has*

$$\operatorname{sdepth}(S/I_{l,d}) \geq \operatorname{depth}(S/I_{l,d}) = \begin{cases} \alpha, & \text{if } \frac{n_i}{k} \equiv 2 \pmod 3 \text{ for some } i \in [s],\\ \beta, & \text{if } \frac{n_i}{k} \equiv 1 \pmod 3 \text{ for all } i \in [s],\\ \gamma, & \text{otherwise.}\end{cases}$$

Proof. It is easy to see that $(I_{l,d} : x_1 \cdots x_k) = \sum_{i=1}^{s}(x_{k+1,i} \cdots x_{d,i}, L_{1,i})$ and $(I_{l,d}, x_1 \cdots x_k) = \sum_{i=1}^{s} L_{2,i} + (x_1 \cdots x_k)$, where $L_{1,i} := (x_{d+1,i} \cdots x_{2d,i}, \ldots, x_{n_i-d+1,i} \cdots x_{n_i,i})$ and $L_{2,i} := (x_{k+1,i} \cdots x_{3k,i}, \ldots, x_{n_i-d+1,i} \cdots x_{n_i,i})$ for $1 \leq i \leq s$. We denote $a_i := x_{k+1,i} \cdots x_{d,i}$, $X_i := \{x_{d+1,i}, \ldots, x_{n_i,i}\}$ and $Y_i := \{x_{k+1,i}, \ldots, x_{n_i,i}\}$ for $1 \leq i \leq s$. Thus we obtain

$$\frac{S}{(I_{l,d} : x_1 \cdots x_k)} \cong \frac{K[Y_1 \setminus X_1]}{(a_1)} \otimes_K \cdots \otimes_K \frac{K[Y_s \setminus X_s]}{(a_s)} \otimes_K \frac{K[X_1]}{L_{1,1}} \otimes_K \cdots \otimes_K \frac{K[X_s]}{L_{1,s}} \otimes_K K[x_1, \ldots, x_k],$$

and

$$\frac{S}{(I_{l,d}, x_1 \cdots x_k)} \cong \frac{K[Y_1]}{L_{2,1}} \otimes_K \cdots \otimes_K \frac{K[Y_s]}{L_{2,s}} \otimes_K \frac{K[x_1, \ldots, x_k]}{(x_1 \cdots x_k)}.$$

For $i \in [s]$, (a_i) and $(x_1 \cdots x_k)$ are complete intersections. Therefore, by Lemma 2.9, [17, Proposition 2.2.20, Theorem 2.2.21] and [18, Theorem 2.11], we deduce that $\operatorname{sdepth}(S/(I_{l,d} : x_1 \cdots x_k)) \geq \operatorname{depth}(S/(I_{l,d} : x_1 \cdots x_k)) = \sum_{i=1}^{s}\left(\frac{(d-2)(n_i-d)}{d} + \lceil \frac{2(n_i-d)}{3d} \rceil\right) + (sk-s) + k = \alpha$, and $\operatorname{sdepth}(S/(I_{l,d}, x_1 \cdots x_k)) \geq \operatorname{depth}(S/(I_{l,d}, x_1 \cdots x_k)) = \sum_{i=1}^{s}\left(\frac{(d-2)(n_i-k)}{d} + \lceil \frac{2(n_i-k)}{3d} \rceil\right) + (k-1) = \gamma$.

If $\frac{n_i}{k} \equiv 2 \pmod 3$ for some $i \in [s]$, then $\alpha \leq \gamma$. Using Lemmas 2.7 and 2.8 on the short exact sequence

$$0 \longrightarrow S/(I_{l,d} : x_1 \cdots x_k) \longrightarrow S/I_{l,d} \longrightarrow S/(I_{l,d}, x_1 \cdots x_k) \longrightarrow 0, \tag{1}$$

we conclude that $\operatorname{sdepth}(S/I_{l,d}) \geq \operatorname{depth}(S/I_{l,d}) = \alpha$.

Assume that $\frac{n_i}{k} \not\equiv 2 \pmod 3$ for all $i \in [s]$. Set $b_i := x_{d+1,i} \cdots x_{3k,i}$, $L_{3,i} := (x_{3k+1,i} \cdots x_{5k,i}, \ldots, x_{n_i-d+1,i} \cdots x_{n_i,i})$ and $A_i := Y_i \setminus X_i$ for $1 \leq i \leq s$. Then we have

$$\begin{aligned}
\frac{(I_{l,d} : x_1 \cdots x_k)}{I_{l,d}} \cong\ & a_1\Big(\frac{K[x_1, \ldots, x_k]}{(x_1 \cdots x_k)} \otimes_K \frac{K[X_1]}{(b_1, L_{3,1})} \\
& \otimes_K \frac{K[Y_2]}{L_{2,2}} \otimes_K \cdots \otimes_K \frac{K[Y_s]}{L_{2,s}}\Big)[A_1] \\
& \oplus a_2\Big(\frac{K[x_1, \ldots, x_k]}{(x_1 \cdots x_k)} \otimes_K \frac{K[Y_1]}{(a_1, L_{1,1})} \otimes_K \frac{K[X_2]}{(b_2, L_{3,2})} \\
& \otimes_K \frac{K[Y_3]}{L_{2,3}} \otimes_K \cdots \otimes_K \frac{K[Y_s]}{L_{2,s}}\Big)[A_2] \\
& \oplus \cdots \\
& \oplus a_s\Big(\frac{K[x_1, \ldots, x_k]}{(x_1 \cdots x_k)} \otimes_K \frac{K[Y_1]}{(a_1, L_{1,1})} \otimes_K \cdots \\
& \otimes_K \frac{K[Y_{s-1}]}{(a_{s-1}, L_{1,s-1})} \otimes_K \frac{K[X_s]}{(b_s, L_{3,s})}\Big)[A_s].
\end{aligned}$$

Next, we consider the following two cases.

Case 1: $\frac{n_i}{k} \equiv 1 \pmod 3$ for all $i \in [s]$. Using Lemma 2.9, [17, Proposition 2.2.20, Theorem 2.2.21], [18, Theorem 2.11] and the isomorphism, we obtain $\operatorname{sdepth}((I_{l,d} : x_1 \cdots x_k)/I_{l,d}) \geq \operatorname{depth}((I_{l,d} : x_1 \cdots x_k)/I_{l,d}) = (k-1) + \sum_{i=1}^{s-1}\Big(\frac{(d-2)(n_i-d)}{d} + \lceil\frac{2(n_i-d)}{3d}\rceil + (k-1)\Big) + \Big(\frac{(d-2)(n_s-3k)}{d} + \lceil\frac{2(n_s-3k)}{3d}\rceil + (k-1)\Big) + k = \beta = \alpha$. Now, applying Lemmas 2.7 and 2.8 to the short exact sequence

$$0 \longrightarrow (I_{l,d} : x_1 \cdots x_k)/I_{l,d} \longrightarrow S/I_{l,d} \longrightarrow S/(I_{l,d} : x_1 \cdots x_k) \longrightarrow 0, \tag{2}$$

we get $\operatorname{sdepth}(S/I_{l,d}) \geq \operatorname{depth}(S/I_{l,d}) = \beta$.

Case 2: $\frac{n_i}{k} \equiv 0 \pmod 3$ for some $i \in [s]$. We assume that $\frac{n_s}{k} \equiv 0 \pmod 3$. Using Lemma 2.9, [17, Proposition 2.2.20, Theorem 2.2.21], [18, Theorem 2.11] and the isomorphism, it follows that $\operatorname{sdepth}((I_{l,d} : x_1 \cdots x_k)/I_{l,d}) \geq \operatorname{depth}((I_{l,d} : x_1 \cdots x_k)/I_{l,d}) = (k-1) + \sum_{i=1}^{s-1}\Big(\frac{(d-2)(n_i-d)}{d} + \lceil\frac{2(n_i-d)}{3d}\rceil + (k-1)\Big) + \Big(\frac{(d-2)(n_s-3k)}{d} + \lceil\frac{2(n_s-3k)}{3d}\rceil + (k-1)\Big) + k = \gamma = \alpha - 1$. By applying Lemmas 2.7 and 2.8 to the short exact sequence (2), we conclude that $\operatorname{sdepth}(S/I_{l,d}) \geq \operatorname{depth}(S/I_{l,d}) = \gamma$. □

Let $u \in S$ be a monomial and $I \subset S$ a monomial ideal. We set $\operatorname{supp}_1(u) := \{i : x_i \mid u\}$, $\operatorname{supp}_2(u) := \{(i,j) : x_{i,j} \mid u\}$, $\operatorname{supp}_1(I) := \{i : x_i \mid v \text{ for some } v \in G(I)\}$ and $\operatorname{supp}_2(I) := \{(i,j) : x_{i,j} \mid v \text{ for some } v \in G(I)\}$. Let $\mathcal{C}$ denote the set $\bigcup_{i=1}^{s}\{(k+1,i),\dots,(n_i,i)\}$. With these notations and the same arguments as used in the proof of [8, Lemma 3.3], one can prove the following lemma.

Lemma 3.8. *Let $I \subset S$ be a squarefree monomial ideal with $\operatorname{supp}_1(I) = [k]$ and $\operatorname{supp}_2(I) = \mathcal{C}$. Let $v \in S/I$ be a squarefree monomial such that $x_i v \in I$ for all $i \in [k] \setminus \operatorname{supp}_1(v)$ and $x_{i,j} v \in I$ for all $(i,j) \in \mathcal{C} \setminus \operatorname{supp}_2(v)$. Then $\operatorname{sdepth}(S/I) \leq |\operatorname{supp}_1(v)| + |\operatorname{supp}_2(v)|$.*

Next, we give another main result of this section.

Theorem 3.9. *With the notations introduced one has*

$$\operatorname{sdepth}(S/I_{l,d}) \leq \begin{cases} \alpha, & \text{if } \frac{n_i}{k} \not\equiv 0 \pmod 3 \text{ for some } i \in [s], \\ \gamma, & \text{otherwise.} \end{cases}$$

Proof. We prove the result by the following two cases.

Case 1: $\frac{n_i}{k} \not\equiv 0 \pmod 3$ for some $i \in [s]$. For $1 \leq i \leq \frac{n_j-k}{k}$ and $1 \leq j \leq s$, we define the monomial $a_{i,j} \in S$ by

$$a_{i,j} = \begin{cases} x_{ik+1,j} \cdots x_{(i+1)k,j}, & \text{if } i \equiv 0 \pmod 3, \\ x_{ik+1,j} \cdots x_{(i+1)k-1,j}, & \text{otherwise.} \end{cases}$$

If $\frac{n_j}{k} \equiv 1 \pmod 3$ or $\frac{n_j}{k} \equiv 2 \pmod 3$, we set $u_j := a_{1,j}a_{2,j} \cdots a_{(n_j-k)/k,j}$. If $\frac{n_j}{k} \equiv 0 \pmod 3$, we set $u_j := a_{1,j}a_{2,j} \cdots a_{(n_j-k)/k,j}x_{n_j,j}$. Since $v := x_1 \cdots x_k \cdot \prod_{j=1}^{s} u_j \in S/I_{l,d}$, but $x_i v \in I_{l,d}$ for all $i \in [k] \setminus \operatorname{supp}_1(v)$ and $x_{i,j} v \in I_{l,d}$ for all $(i,j) \in \mathcal{C} \setminus \operatorname{supp}_2(v)$, thus we obtain $\operatorname{sdepth}(S/I_{l,d}) \leq |\operatorname{supp}_1(v)| + |\operatorname{supp}_2(v)| = \alpha$ by Lemma 3.8.

Case 2: $\frac{n_i}{k} \equiv 0 \pmod 3$ for all $i \in [s]$. For $1 \le i \le \frac{n_j-k}{k}$ and $1 \le j \le s$, we define the monomial $b_{i,j} \in S$ by

$$b_{i,j} = \begin{cases} x_{ik+1,j} \cdots x_{(i+1)k,j}, & \text{if } i \equiv 1 \pmod 3, \\ x_{ik+1,j} \cdots x_{(i+1)k-1,j}, & \text{otherwise.} \end{cases}$$

We set $u_j := b_{1,j} b_{2,j} \cdots b_{(n_j-k)/k,j}$, $j = 1, \ldots, s$. Since $w := x_1 \cdots x_{k-1} \cdot \prod_{j=1}^{s} u_j \in S/I_{l,d}$, but $x_i w \in I_{l,d}$ for all $i \in [k] \setminus \mathrm{supp}_1(w)$ and $x_{i,j} w \in I_{l,d}$ for all $(i,j) \in \mathcal{C} \setminus \mathrm{supp}_2(w)$, therefore we get $\mathrm{sdepth}(S/I_{l,d}) \le |\mathrm{supp}_1(w)| + |\mathrm{supp}_2(w)| = \gamma$ by Lemma 3.8. □

Remark 3.10. Set $s = 1$ in Theorems 3.7 and 3.9, then we have $\mathrm{sdepth}(S/I_{l,d}) = \mathrm{depth}(S/I_{l,d}) = \frac{(d-2)n_1}{d} + \lceil \frac{2n_1}{3d} \rceil$, which generalizes [18, Theorem 2.11]. Our results also generalize [19, Theorems 3.3 and 3.4], where $d = 2k = 2l = 2$.

As a consequence of Theorems 3.7 and 3.9, we get the following corollary.

Corollary 3.11. $\mathrm{sdepth}(I_{l,d}) \ge \mathrm{sdepth}(S/I_{l,d}) + 1 \ge \mathrm{depth}(I_{l,d})$.

Proof. Since $|G(I_{l,d})| = \sum_{i=1}^{s} \frac{n_i-k}{k}$, $\mathrm{sdepth}(I_{l,d}) \ge \max\{1, n - \lfloor \frac{1}{2}|G(I_{l,d})| \rfloor\} = n - \lfloor \sum_{i=1}^{s} \frac{n_i-k}{2k} \rfloor \ge \mathrm{sdepth}(S/I_{l,d}) + 1 \ge \mathrm{depth}(S/I_{l,d}) + 1 = \mathrm{depth}(I_{l,d})$ by Theorems 3.7, 3.9 and [11, Theorem 2.3]. □

4. Depth and Stanley depth of the monomial ideal $J_{k,d}$

Throughout this section we set $S := K[x_1, \ldots, x_k, x_{k+1,1}, \ldots, x_{n_1,1}, \ldots, x_{k+1,s}, \ldots, x_{n_s,s}]$ be the polynomial ring over a field K in n variables, where $n := \sum_{i=1}^{s} n_i - (s-1)k$. Next, we will discuss our main results in two cases.

4.1. The case $d \ge 2k+1$

In this section, we will give some formulas for depth and Stanley depth of quotient ring $S/J_{k,d}$. The following lemma will be useful in several proofs.

Lemma 4.1. $P = \sum_{i=1}^{s} (x_{d-k,i}, x_{2(d-k),i}, x_{3(d-k),i}, \ldots, x_{n_i,i}) \in \mathrm{Min}(S/J_{k,d})$.

Proof. With the same arguments as used in the proof of Lemma 3.1, one can show that P is a minimal prime ideal of $J_{k,d}$. □

Proposition 4.2. $\mathrm{bight}(J_{k,d}) = \mathrm{pd}(S/J_{k,d}) = \mathrm{ara}(J_{k,d}) = |G(J_{k,d})| = \sum_{i=1}^{s} \frac{n_i}{d-k}$.

Proof. We have $P = \sum_{i=1}^{s} (x_{d-k,i}, x_{2(d-k),i}, x_{3(d-k),i}, \ldots, x_{n_i,i}) \in \mathrm{Min}(S/J_{k,d})$ and $\mathrm{ht}(P) = \sum_{i=1}^{s} \frac{n_i}{d-k}$ by Lemma 4.1. Then $\sum_{i=1}^{s} \frac{n_i}{d-k} \le \mathrm{bight}(J_{k,d}) \le \mathrm{pd}(S/J_{k,d}) \le \mathrm{ara}(J_{k,d}) \le |G(J_{k,d})| = \sum_{i=1}^{s} \frac{n_i}{d-k}$. The proof is completed. □

Now, we give the exact formulas for depth and Stanley depth of $S/J_{k,d}$.

Theorem 4.3. $\mathrm{sdepth}(S/J_{k,d}) = \mathrm{depth}(S/J_{k,d}) = n - \sum_{i=1}^{s} \frac{n_i}{d-k}$.

Proof. Since $|G(J_{k,d})| = \sum_{i=1}^{s} \frac{n_i}{d-k}$, we have $\mathrm{sdepth}(S/J_{k,d}) \geq n - \sum_{i=1}^{s} \frac{n_i}{d-k}$ by [3, Proposition 1.2]. On the other hand, there exists a prime ideal $P \in \mathrm{Ass}(S/J_{k,d})$ such that $\mathrm{ht}(P) = \sum_{i=1}^{s} \frac{n_i}{d-k}$ by Lemma 4.1. It follows that $\mathrm{sdepth}(S/J_{k,d}) \leq n - \sum_{i=1}^{s} \frac{n_i}{d-k}$ by [7, Proposition 1.3]. By the Auslander-Buchsbaum formula and Proposition 4.2, we obtain $\mathrm{depth}(S/J_{k,d}) = n - \mathrm{pd}(S/J_{k,d}) = n - \sum_{i=1}^{s} \frac{n_i}{d-k}$. □

Remark 4.4. Set $s = 1$ in Theorem 4.3, then $\mathrm{sdepth}(S/J_{k,d}) = \mathrm{depth}(S/J_{k,d}) = n_1 - \frac{n_1}{d-k}$, which generalizes [18, Theorem 2.9].

The following corollary implies that the Stanley inequality and Conjecture 1.1 hold for $J_{k,d}$.

Corollary 4.5. $\mathrm{sdepth}(J_{k,d}) > \mathrm{sdepth}(S/J_{k,d}) + 1 = \mathrm{depth}(J_{k,d})$.

Proof. Since $|G(J_{k,d})| = \sum_{i=1}^{s} \frac{n_i}{d-k}$ and $n_i \geq 3d - 3k$ for $1 \leq i \leq s$, we have $\mathrm{sdepth}(J_{k,d}) \geq \max\{1, n - \lfloor \frac{1}{2}|G(J_{k,d})| \rfloor\} = n - \lfloor \sum_{i=1}^{s} \frac{n_i}{2(d-k)} \rfloor > n - \sum_{i=1}^{s} \frac{n_i}{d-k} + 1 = \mathrm{sdepth}(S/J_{k,d}) + 1 = \mathrm{depth}(S/J_{k,d}) + 1 = \mathrm{depth}(J_{k,d})$ by [11, Theorem 2.3] and Theorem 4.3. □

4.2. The case $d = 2k$

In this section, we will give some formulas to compute the depth and Stanley depth of quotient ring of $J_{k,d}$. We adopt the following notations:

$$\alpha := \sum_{i=1}^{s} \Big(\frac{(d-2)n_i}{d} + \lceil \frac{2n_i}{3d} \rceil \Big) + k - sk,$$
$$\beta := \sum_{i=1}^{s} \Big(\frac{(d-2)n_i}{d} + \lceil \frac{2n_i - d}{3d} \rceil \Big) + s + k - sk - 1.$$

Now, we prove the main results of this section.

Theorem 4.6. *With the notations introduced one has*

$$\mathrm{sdepth}(S/J_{k,d}) \geq \mathrm{depth}(S/J_{k,d}) = \begin{cases} \alpha, & \text{if } \frac{n_i}{k} \not\equiv 1 \pmod 3 \text{ for some } i \in [s], \\ \beta, & \text{otherwise.} \end{cases}$$

Proof. We get $(J_{k,d} : x_1 \cdots x_k) = \sum_{i=1}^{s} (x_{k+1,i} \cdots x_{d,i}, x_{n_i-k+1,i} \cdots x_{n_i,i}, L_{1,i})$ and $(J_{k,d}, x_1 \cdots x_k) = \sum_{i=1}^{s} L_{2,i} + (x_1 \cdots x_k)$, where $L_{1,i} := (x_{d+1,i} \cdots x_{2d,i}, \ldots, x_{n_i-3k+1,i} \cdots x_{n_i-k,i})$ and $L_{2,i} := (x_{k+1,i} \cdots x_{3k,i}, \ldots, x_{n_i-d+1,i} \cdots x_{n_i,i})$ for $1 \leq i \leq s$. We denote $a_{1,i} := x_{k+1,i} \cdots x_{d,i}$, $a_{2,i} := x_{n_i-k+1,i} \cdots x_{n_i,i}$, $A_{1,i} := \{x_{d+1,i}, \ldots, x_{n_i-k,i}\}$, $A_{2,i} := \{x_{k+1,i}, \ldots, x_{d,i}\}$, $A_{3,i} := \{x_{n_i-k+1,i}, \ldots, x_{n_i,i}\}$ and $A_{4,i} := A_{1,i} \cup A_{2,i} \cup A_{3,i}$ for all $i \in [s]$. Then we obtain

$$\begin{aligned} \frac{S}{(J_{k,d} : x_1 \cdots x_k)} \cong \; & \frac{K[A_{1,1}]}{L_{1,1}} \otimes_K \cdots \otimes_K \frac{K[A_{1,s}]}{L_{1,s}} \\ & \otimes_K \frac{K[A_{2,1}]}{(a_{1,1})} \otimes_K \cdots \otimes_K \frac{K[A_{2,s}]}{(a_{1,s})} \\ & \otimes_K \frac{K[A_{3,1}]}{(a_{2,1})} \otimes_K \cdots \otimes_K \frac{K[A_{3,s}]}{(a_{2,s})} \otimes_K K[x_1, \ldots, x_k], \end{aligned}$$

and

$$\frac{S}{(J_{k,d}, x_1\cdots x_k)} \cong \frac{K[A_{4,1}]}{L_{2,1}} \otimes_K \cdots \otimes_K \frac{K[A_{4,s}]}{L_{2,s}} \otimes_K \frac{K[x_1,\ldots,x_k]}{(x_1\cdots x_k)}.$$

Note that $(a_{1,i})$, $(a_{2,i})$ and $(x_1\cdots x_k)$ are complete intersections for all $1 \le i \le s$. Thus, by Lemma 2.9, [17, Proposition 2.2.20, Theorem 2.2.21] and [18, Theorem 2.11], we have $\operatorname{sdepth}(S/(J_{k,d} : x_1\cdots x_k)) \ge \operatorname{depth}(S/(J_{k,d} : x_1\cdots x_k)) = \sum_{i=1}^{s}\Big(\frac{(d-2)(n_i-3k)}{d} + \lceil\frac{2(n_i-3k)}{3d}\rceil\Big) + 2(sk-s)+k = \alpha$, and we get $\operatorname{sdepth}(S/(J_{k,d}, x_1\cdots x_k)) \ge \operatorname{depth}(S/(J_{k,d}, x_1\cdots x_k)) = \sum_{i=1}^{s}\Big(\frac{(d-2)(n_i-k)}{d} + \lceil\frac{2(n_i-k)}{3d}\rceil\Big) + (k-1) = \beta$.

If $\frac{n_i}{k} \not\equiv 1 \pmod 3$ for some $i\in[s]$, then $\alpha\le\beta$. Using Lemmas 2.7 and 2.8 on the short exact sequence

$$0 \longrightarrow S/(J_{k,d} : x_1\cdots x_k) \longrightarrow S/J_{k,d} \longrightarrow S/(J_{k,d}, x_1\cdots x_k) \longrightarrow 0,$$

we conclude that $\operatorname{sdepth}(S/J_{k,d}) \ge \operatorname{depth}(S/J_{k,d}) = \alpha$.

Assume that $\frac{n_i}{k} \equiv 1 \pmod 3$ for all $i\in[s]$. For any $1\le i\le s$, we set $X_{1,i} := A_{1,i}\cup A_{3,i}$, $X_{2,i} := A_{4,i}$, $X_{3,i} := A_{1,i}\cup A_{2,i}$, $b_{1,i} := x_{d+1,i}\cdots x_{3k,i}$, $b_{2,i} := x_{n_i-d+1,i}\cdots x_{n_i-k,i}$, $L_{3,i} := (x_{3k+1,i}\cdots x_{5k,i},\ldots,x_{n_i-d+1,i}\cdots x_{n_i,i})$ and $L_{4,i} := (x_{d+1,i}\cdots x_{2d,i},\ldots,x_{n_i-2d+1,i}\cdots x_{n_i-d,i})$. Then it follows that

$$\begin{aligned}
\frac{(J_{k,d} : x_1\cdots x_k)}{J_{k,d}} \cong\ & a_{1,1}\Big(R\otimes_K \frac{K[X_{1,1}]}{(b_{1,1}, L_{3,1})} \otimes_K \frac{K[X_{2,2}]}{L_{2,2}} \otimes_K \cdots \\
&\qquad \otimes_K \frac{K[X_{2,s}]}{L_{2,s}}\Big)[A_{2,1}] \\
\oplus\ & a_{2,1}\Big(R\otimes_K \frac{K[X_{3,1}]}{(a_{1,1}, b_{2,1}, L_{4,1})} \otimes_K \frac{K[X_{2,2}]}{L_{2,2}} \otimes_K \cdots \otimes_K \frac{K[X_{2,s}]}{L_{2,s}}\Big)[A_{3,1}] \\
\oplus\ & a_{1,2}\Big(R\otimes_K \frac{K[X_{2,1}]}{M_1} \otimes_K \frac{K[X_{1,2}]}{(b_{1,2}, L_{3,2})} \otimes_K \frac{K[X_{2,3}]}{L_{2,3}} \otimes_K \cdots \otimes_K \frac{K[X_{2,s}]}{L_{2,s}}\Big)[A_{2,2}] \\
\oplus\ & a_{2,2}\Big(R\otimes_K \frac{K[X_{2,1}]}{M_1} \otimes_K \frac{K[X_{3,2}]}{(a_{1,2}, b_{2,2}, L_{4,2})} \otimes_K \frac{K[X_{2,3}]}{L_{2,3}} \otimes_K \cdots \\
&\qquad \otimes_K \frac{K[X_{2,s}]}{L_{2,s}}\Big)[A_{3,2}] \\
\oplus\ & \cdots \\
\oplus\ & a_{1,s}\Big(R\otimes_K \frac{K[X_{2,1}]}{M_1} \otimes_K \cdots \otimes_K \frac{K[X_{2,s-1}]}{M_{s-1}} \otimes_K \frac{K[X_{1,s}]}{(b_{1,s}, L_{3,s})}\Big)[A_{2,s}] \\
\oplus\ & a_{2,s}\Big(R\otimes_K \frac{K[X_{2,1}]}{M_1} \otimes_K \cdots \otimes_K \frac{K[X_{2,s-1}]}{M_{s-1}} \otimes_K \frac{K[X_{3,s}]}{(a_{1,s}, b_{2,s}, L_{4,s})}\Big)[A_{3,s}],
\end{aligned}$$

where $R := \frac{K[x_1,\ldots,x_k]}{(x_1\cdots x_k)}$ and $M_i := (a_{1,i}, a_{2,i}, L_{1,i})$ for $1\le i\le s-1$. Using Lemma 2.9, [17, Proposition 2.2.20, Theorem 2.2.21], [18, Theorem 2.11] and the isomorphism, it follows that $\operatorname{sdepth}((J_{k,d} : x_1\cdots x_k)/J_{k,d}) \ge \operatorname{depth}((J_{k,d} :$

$x_1 \cdots x_k)/J_{k,d}) = (k-1) + \sum_{i=1}^{s-1}\Big(\frac{(d-2)(n_i-3k)}{d} + \lceil \frac{2(n_i-3k)}{3d} \rceil + (d-2)\Big) + \Big(\frac{(d-2)(n_s-2d)}{d} + \lceil \frac{2(n_s-2d)}{3d} \rceil + (d-2)\Big) + k = \beta < \alpha$. Now, applying Lemmas 2.7 and 2.8 to the short exact sequence

$$0 \longrightarrow (J_{k,d} : x_1 \cdots x_k)/J_{k,d} \longrightarrow S/J_{k,d} \longrightarrow S/(J_{k,d} : x_1 \cdots x_k) \longrightarrow 0,$$

we have $\mathrm{sdepth}(S/J_{k,d}) \geq \mathrm{depth}(S/J_{k,d}) = \beta$, as desired. □

Theorem 4.7. $\mathrm{sdepth}(S/J_{k,d}) \leq \sum_{i=1}^{s}\Big(\frac{(d-2)n_i}{d} + \lceil \frac{2n_i}{3d} \rceil\Big) + k - sk = \alpha.$

Proof. For $1 \leq i \leq \frac{n_j-k}{k}$ and $1 \leq j \leq s$, we define the monomial $a_{i,j} \in S$ by

$$a_{i,j} = \begin{cases} x_{ik+1,j} \cdots x_{(i+1)k,j}, & \text{if } i \equiv 0 \pmod 3, \\ x_{ik+1,j} \cdots x_{(i+1)k-1,j}, & \text{otherwise.} \end{cases}$$

If $\frac{n_j}{k} \equiv 0 \pmod 3$ or $\frac{n_j}{k} \equiv 2 \pmod 3$, we set $u_j := a_{1,j}a_{2,j} \cdots a_{(n_j-k)/k,j}$. If $\frac{n_j}{k} \equiv 1 \pmod 3$, we set $u_j := \frac{a_{1,j} \cdots a_{(n_j-k)/k,j}x_{n_j-k,j}}{x_{n_j,j}}$. Since $v := x_1 \cdots x_k \cdot \prod_{j=1}^{s} u_j \in S/J_{k,d}$, but $x_i v \in J_{k,d}$ for all $i \in [k]\backslash\mathrm{supp}_1(v)$ and $x_{i,j}v \in J_{k,d}$ for all $(i,j) \in \mathcal{C}\backslash\mathrm{supp}_2(v)$, it follows that $\mathrm{sdepth}(S/J_{k,d}) \leq |\mathrm{supp}_1(v)| + |\mathrm{supp}_2(v)| = \alpha$ by Lemma 3.8. □

Remark 4.8. Theorems 4.6 and 4.7 generalize [18, Proposition 2.16, Theorem 2.18], where $s = 1$. Our results also generalize [19, Theorems 4.2 and 4.3], where $d = 2$ and $k = 1$.

As a consequence of Theorems 4.6 and 4.7, we get the following corollary, which says that the Stanley inequality and Conjecture 1.1 hold for $J_{k,d}$.

Corollary 4.9. $\mathrm{sdepth}(J_{k,d}) \geq \mathrm{sdepth}(S/J_{k,d})$ *and* $\mathrm{sdepth}(J_{k,d}) \geq \mathrm{depth}(J_{k,d})$.

Proof. Since $|G(J_{k,d})| = \sum_{i=1}^{s} \frac{n_i}{k}$, $\mathrm{sdepth}(J_{k,d}) \geq \max\{1, n - \lfloor \frac{1}{2}|G(J_{k,d})| \rfloor\} = n - \lfloor \sum_{i=1}^{s} \frac{n_i}{2k} \rfloor \geq \max\{\mathrm{sdepth}(S/J_{k,d}), \mathrm{depth}(J_{k,d})\}$ by Theorems 4.6, 4.7 and [11, Theorem 2.3]. □

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