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ON THE SEMIGROUP OF PARTITION-PRESERVING TRANSFORMATIONS WHOSE CHARACTERS ARE BIJECTIVE

Mosarof Sarkar and Shubh N. Singh

Abstract. Let $\mathcal{P} = \{X_i : i \in I\}$ be a partition of a set X. We say that a transformation $f : X \to X$ preserves $\mathcal{P}$ if for every $X_i \in \mathcal{P}$, there exists $X_j \in \mathcal{P}$ such that $X_i f \subseteq X_j$. Consider the semigroup $\mathcal{B}(X, \mathcal{P})$ of all transformations f of X such that f preserves $\mathcal{P}$ and the character (map) $\chi^{(f)} : I \to I$ defined by $i\chi^{(f)} = j$ whenever $X_i f \subseteq X_j$ is bijective. We describe Green's relations on $\mathcal{B}(X, \mathcal{P})$, and prove that $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$ if $\mathcal{P}$ is finite. We give a necessary and sufficient condition for $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$. We characterize unit-regular elements in $\mathcal{B}(X, \mathcal{P})$, and determine when $\mathcal{B}(X, \mathcal{P})$ is a unit-regular semigroup. We alternatively prove that $\mathcal{B}(X, \mathcal{P})$ is a regular semigroup. We end the paper with a conjecture.

1. Introduction

Throughout this paper, let X be a nonempty set, let $\mathcal{P} = \{X_i : i \in I\}$ be a partition of X, and let E be the equivalence relation on X corresponding to the partition $\mathcal{P}$. Denote by $T(X)$ (resp. $\mathrm{Sym}(X)$) the full transformation semigroup (resp. symmetric group) on X. The semigroup $T(X)$ and its subsemigroups play a vital role in semigroup theory, since every semigroup can be embedded in some $T(Z)$ (cf. [13, Theorem 1.1.2]). This famous result is analogous to Cayley's theorem for groups, which states that every group can be embedded in some $\mathrm{Sym}(X)$.

We say that any transformation $f : X \to X$ *preserves* the partition $\mathcal{P}$ if for every $X_i \in \mathcal{P}$, there exists $X_j \in \mathcal{P}$ such that $X_i f \subseteq X_j$. In 1994, Pei [14] first studied the subsemigroup $T(X, \mathcal{P})$ of $T(X)$ consisting of all transformations that preserve the partition $\mathcal{P}$. Using symbols,

$$\begin{aligned} T(X, \mathcal{P}) &= \{f \in T(X) : (\forall X_i \in \mathcal{P})\ (\exists X_j \in \mathcal{P})\ X_i f \subseteq X_j\} \\ &= \{f \in T(X) : \forall x, y \in X, (x, y) \in E \Longrightarrow (xf, yf) \in E\}. \end{aligned}$$

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In that paper, Pei [14, Theorem 2.8] proved that $T(X, \mathcal{P})$ is exactly the semigroup of all continuous selfmaps on X with respect to the topology having $\mathcal{P}$ as a basis. The semigroup $T(X, \mathcal{P})$ and its subsemigroups have been extensively studied by a number of authors (see [1, 2, 10, 11, 15–18, 21, 23] for some references).

In what follows, the letter I denotes the index set of the partition $\mathcal{P}$. Let $f \in T(X, \mathcal{P})$. The *character* (*map*) of f is the selfmap $\chi^{(f)}\colon I \to I$ defined by $i\chi^{(f)} = j$ whenever $X_i f \subseteq X_j$. For finite X, the character $\chi^{(f)}$ with the notation $\bar{f}$ has been studied by Araújo et al. [1], Dolinka and East [8], and Dolinka et al. [9]. For arbitrary X, the character $\chi^{(f)}$ was first considered by Purisang and Rakbud [19, p. 220]. The character $\chi^{(f)}$ of f has also been received attention (see [20–23]).

Using the notion of character $\chi^{(f)}$, Purisang and Rakbud [19] introduced the following subsemigroup of $T(X, \mathcal{P})$:

$$\mathcal{B}(X, \mathcal{P}) = \{f \in T(X, \mathcal{P})\colon \chi^{(f)} \in \mathrm{Sym}(I)\}.$$

In that paper, the authors [19, Theorem 3.5(1)] proved that $\mathcal{B}(X, \mathcal{P})$ is a regular semigroup. The semigroup $\mathcal{B}(X, \mathcal{P})$ generalizes both $T(X)$ and $\mathrm{Sym}(X)$ in the sense that $\mathcal{B}(X, \mathcal{P}) = T(X)$ if $|\mathcal{P}| = 1$, and $\mathcal{B}(X, \mathcal{P}) = \mathrm{Sym}(X)$ if $\mathcal{P}$ consists of singleton sets. When $\mathcal{P}$ is finite, the authors [21, Corollary 3.5] proved that $\mathcal{B}(X, \mathcal{P}) = \Sigma(X, \mathcal{P}) = T_{E^*}(X)$, where

$$\Sigma(X, \mathcal{P}) = \{f \in T(X, \mathcal{P})\colon Xf \cap X_i \neq \varnothing \ \forall X_i \in \mathcal{P}\},$$
$$T_{E^*}(X) = \{f \in T(X)\colon \forall x, y \in X,\ (x, y) \in E \Longrightarrow (xf, yf) \in E\}.$$

For arbitrary X, both the semigroups $\Sigma(X, \mathcal{P})$ and $T_{E^*}(X)$ have been studied (see [1, 2, 21–23] and [5–7, 24], respectively). In particular, the authors [1, 2] considered the semigroup $\Sigma(X, \mathcal{P})$ for finite X to calculate the rank of the finite semigroup $T(X, \mathcal{P})$.

This paper is motivated by various results on $T(X, \mathcal{P})$ and its two subsemigroups $\Sigma(X, \mathcal{P})$ and $T_{E^*}(X)$. The rest of the paper is organized as follows. In the next section, we define concepts, introduce notation, and recall some results needed in this paper. In Section 3, we give an alternative proof of Theorem 3.5(1) in [19], which ascertain that the semigroup $\mathcal{B}(X, \mathcal{P})$ is regular. We next describe unit-regular elements in $\mathcal{B}(X, \mathcal{P})$, and determine when $\mathcal{B}(X, \mathcal{P})$ is a unit-regular semigroup. In Section 4, we describe Green's relations on $\mathcal{B}(X, \mathcal{P})$. We also give a necessary and sufficient condition for $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$.

2. Preliminaries and notation

Let X be a nonempty set. The cardinality of X is denoted by $|X|$, and the identity map on X is denoted by id_X. For any sets A and B, let $A \setminus B$ denote the set $\{x \in A\colon x \notin B\}$. A *partition* of X is a collection of pairwise disjoint nonempty subsets, called *blocks*, whose union is X. A *trivial* partition is a partition that has only singleton blocks or a single block. A partition is *uniform* if all its blocks have the same cardinality. A *transversal* of an equivalence relation ρ on X is a subset of X that contains exactly one element from each ρ-class. The set of all positive integers is denoted by $\mathbb{N}$. For $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, \ldots, n\}$.

We compose mappings from left to right and denote their composition by juxtaposition. Let $f\colon X \to Y$ be a mapping. We denote by xf the image of an element $x \in X$ under f. For any $A \subseteq X$ (resp. $B \subseteq Y$), we denote by Af (resp. Bf^{-1}) the set $\{af : a \in A\}$ (resp. $\{x \in X : xf \in B\}$). Furthermore, if $B = \{b\}$, then we write bf^{-1} instead of $\{b\}f^{-1}$. Let $\mathrm{d}(f) = |Y \setminus Xf|$. The *kernel* of f, denoted by $\ker(f)$, is an equivalence relation on X defined by $\ker(f) = \{(a, b) \in X \times X : af = bf\}$. The symbol $\pi(f)$ denotes the partition of X induced by $\ker(f)$, and the symbol T_f denotes any transversal of $\ker(f)$. Note that $|X \setminus T_f|$ is independent of the choice of transversal of $\ker(f)$ (cf. [12, p. 1356]). Set $\mathrm{c}(f) = |X \setminus T_f|$. Let $g\colon X \to X$ be a mapping. For any nonempty subset A of the domain of g, the *restriction* of g to A is the mapping $g_{\restriction A}\colon A \to X$ defined by $x(g_{\restriction A}) = xg$ for all $x \in A$. Moreover if B is a subset of the codomain of g such that $Ag \subseteq B$, then we reserve the same notation $g_{\restriction A}$ for the mapping from A to B that assigns xg to each $x \in A$.

Let S be a semigroup and $a \in S$. We say that a is *regular* in S if there exists $b \in S$ such that $aba = a$. The set of all regular elements in S is denoted by $\mathrm{reg}(S)$. If $\mathrm{reg}(S) = S$, then S is a *regular semigroup*. In addition, let S contains the identity. Then the set of all unit elements of S is denoted by $U(S)$. We say that a is *unit-regular* in S if there exists $u \in U(S)$ such that $aua = a$. The set of all unit-regular elements in S is denoted by $\mathrm{ureg}(S)$. If $\mathrm{ureg}(S) = S$, then S is a *unit-regular semigroup*. Note that $U(T(X)) = \mathrm{Sym}(X)$. It is evident that $U(T(X, \mathcal{P})) = T(X, \mathcal{P}) \cap \mathrm{Sym}(X)$ and $U(\mathcal{B}(X, \mathcal{P})) = U(T(X, \mathcal{P}))$.

Let S be a semigroup and $a, b \in S$. The Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$, and $\mathcal{J}$ on S defined as follows: $(a, b) \in \mathcal{L}$ if $S^1a = S^1b$, $(a, b) \in \mathcal{R}$ if $aS^1 = bS^1$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, and $(a, b) \in \mathcal{J}$ if $S^1aS^1 = S^1bS^1$, where S^1 is the semigroup S with an identity adjoined (if necessary). If $\mathcal{K}$ is any Green's relation on S, then the equivalence class of a with respect to $\mathcal{K}$ is denoted by K_a. Since $\mathcal{L}, \mathcal{R}$, and $\mathcal{J}$ are defined in terms of ideals, which are partially ordered by inclusion, we have induced partial orders on the sets of the equivalence classes of $\mathcal{L}, \mathcal{R}$, and $\mathcal{J}$ [13, p. 47, (2.1.3)]: $L_a \leq L_b$ if $S^1a \subseteq S^1b$, $R_a \leq R_b$ if $aS^1 \subseteq bS^1$, and $J_a \leq J_b$ if $S^1aS^1 \subseteq S^1bS^1$.

We refer the reader to [13] for any undefined concepts, notation, and results of semigroup theory. We end this section by stating a list of preliminary results.

Lemma 2.1 ([23, Lemma 3.1])**.** *Let $f\colon X \to Y$ and $g\colon Y \to X$ be mappings. If $fgf = f$, then $X(fg)$ is a transversal of the equivalence relation* $\ker(f)$.

Lemma 2.2 ([19, Lemma 2.3])**.** *We have $\chi^{(fg)} = \chi^{(f)}\chi^{(g)}$ for all $f, g \in T(X, \mathcal{P})$.*

Lemma 2.3 ([21, Lemma 5.2])**.** *Let $\mathcal{P} = \{X_i : i \in I\}$ be a partition of X and $f \in T(X)$. Then $f \in T(X, \mathcal{P})$ if and only if there exists a unique family $B(f, I) := \{f_{\restriction X_i} : i \in I\}$ such that the codomain of each $f_{\restriction X_i}$ is a block of $\mathcal{P}$.*

Theorem 2.4 ([21, Theorem 5.8])**.** *Let $\mathcal{P} = \{X_i : i \in I\}$ be a partition of X and $f \in T(X, \mathcal{P})$. Then f is bijective if and only if*

(i) *every mapping of $B(f, I)$ is bijective*;
(ii) $\chi^{(f)} \in \mathrm{Sym}(I)$.

3. Unit-regularity for $\mathcal{B}(X, \mathcal{P})$

In this section, we describe unit-regular elements in $\mathcal{B}(X, \mathcal{P})$ and then give a necessary and sufficient condition for $\mathcal{B}(X, \mathcal{P})$ to be unit-regular. We begin by giving an alternative proof of the following proposition which was first appeared in [19, Theorem 3.5(1)].

Proposition 3.1. *The semigroup $\mathcal{B}(X, \mathcal{P})$ is regular.*

Proof. Let $f \in \mathcal{B}(X, \mathcal{P})$, and write $(\chi^{(f)})^{-1} = \alpha$. Fix $x' \in xf^{-1}$ for each $x \in Xf$, and also fix $x_i \in X_i$ for each $i \in I$. Define $g \in T(X)$ as follows: Given $x \in X$, there exists $i \in I$ such that $x \in X_i$; we let

$$xg = \begin{cases} x' & \text{if } x \in X_i \cap Xf, \\ x_{i\alpha} & \text{if } x \in X_i \setminus Xf. \end{cases}$$

It is easy to see that $g \in T(X, \mathcal{P})$ and $\chi^{(g)} = \alpha$, so $g \in \mathcal{B}(X, \mathcal{P})$. To prove $fgf = f$, let $x \in X$ and write $xf = y$. Then $x \in X_i$ for some $i \in I$. Therefore, since $y' \in yf^{-1}$, we obtain $x(fgf) = (xf)gf = (yg)f = y'f = y = xf$, which yields $fgf = f$. Hence $f \in \mathrm{reg}(\mathcal{B}(X, \mathcal{P}))$ as required. □

The following theorem describes unit-regular elements in $\mathcal{B}(X, \mathcal{P})$.

Theorem 3.2. *Let $f \in \mathcal{B}(X, \mathcal{P})$. Then $f \in \mathrm{ureg}(\mathcal{B}(X, \mathcal{P}))$ if and only if $\mathrm{c}(f_{\restriction X_i}) = \mathrm{d}(f_{\restriction X_i})$ for all $i \in I$.*

Proof. Suppose that $f \in \mathrm{ureg}(\mathcal{B}(X, \mathcal{P}))$. Then there exists $g \in U(\mathcal{B}(X, \mathcal{P}))$ such that $fgf = f$. This gives $\chi^{(f)}\chi^{(g)}\chi^{(f)} = \chi^{(f)}$ by Lemma 2.2. Therefore $\chi^{(g)} = (\chi^{(f)})^{-1}$, since $\chi^{(f)}, \chi^{(g)} \in \mathrm{Sym}(I)$. Write $\chi^{(f)} = \alpha$.

To prove the desired result, let $i \in I$ and write $i\alpha = j$. Then $j\chi^{(g)} = i$. Therefore, since $fgf = f$, it is easy to see that $f_{\restriction X_i} g_{\restriction X_j} f_{\restriction X_i} = f_{\restriction X_i}$, where $f_{\restriction X_i} : X_i \to X_j$ (resp. $g_{\restriction X_j} : X_j \to X_i$) is a map of $B(f, I)$ (resp. $B(g, I)$). It follows from Lemma 2.1 that $T_{f_{\restriction X_i}} := X_i(f_{\restriction X_i} g_{\restriction X_j})$ is a transversal of $\ker(f_{\restriction X_i})$. Since $g \in U(T(X, \mathcal{P}))$, it follows from Theorem 2.4 that every map of $B(g, I)$ is bijective. Therefore

$$(X_j \setminus X_i f_{\restriction X_i}) g_{\restriction X_j} = X_j g_{\restriction X_j} \setminus X_i (f_{\restriction X_i} g_{\restriction X_j}) = X_i \setminus T_{f_{\restriction X_i}},$$

which yields $|X_j \setminus X_i f_{\restriction X_i}| = |X_i \setminus T_{f_{\restriction X_i}}|$. Hence $\mathrm{c}(f_{\restriction X_i}) = |X_i \setminus T_{f_{\restriction X_i}}| = |X_j \setminus X_i f_{\restriction X_i}| = \mathrm{d}(f_{\restriction X_i})$.

Conversely, suppose that $\mathrm{c}(f_{\restriction X_i}) = \mathrm{d}(f_{\restriction X_i})$ for all $i \in I$. To prove $f \in \mathrm{ureg}(\mathcal{B}(X,\mathcal{P}))$, we shall construct $g \in U(\mathcal{B}(X,\mathcal{P}))$ such that $fgf = f$. For this, let $i \in I$ and write $i(\chi^{(f)})^{-1} = j$. Let $T_{f_{\restriction X_j}}$ be a transversal of $\ker(f_{\restriction X_j})$, where $f_{\restriction X_j}\colon X_j \to X_i$ is a map of $B(f,I)$. It is easy to see that the map $\varphi_i\colon X_j f_{\restriction X_j} \to T_{f_{\restriction X_j}}$ defined by $x\varphi_i = x'$ whenever $x(f_{\restriction X_j})^{-1} \cap T_{f_{\restriction X_j}} = \{x'\}$ is bijective. Since $|X_i \setminus X_j f_{\restriction X_j}| = |X_j \setminus T_{f_{\restriction X_j}}|$ by hypothesis, there exists a bijection $\psi_i\colon X_i \setminus X_j f_{\restriction X_j} \to X_j \setminus T_{f_{\restriction X_j}}$. Define $h_i\colon X_i \to X_j$ by

$$xh_i = \begin{cases} x\varphi_i & \text{if } x \in X_j f_{\restriction X_j}, \\ x\psi_i & \text{if } x \in X_i \setminus X_j f_{\restriction X_j}. \end{cases}$$

Clearly, h_i is bijective. We now prove that the map $g \in T(X)$, defined by $xg = xh_i$ whenever $x \in X_i$ for some $i \in I$, is a unit element of $\mathcal{B}(X,\mathcal{P})$. For this, let $i \in I$ and write $i(\chi^{(f)})^{-1} = j$. Then, since $h_i\colon X_i \to X_j$ is bijective, we get $X_i g = X_i h_i = X_j$. Therefore $g \in T(X,\mathcal{P})$ and $\chi^{(g)} = (\chi^{(f)})^{-1}$. Since $\chi^{(g)} \in \mathrm{Sym}(I)$ and every map $g_{\restriction X_i} = h_i$ of $B(g,I)$ is bijective, it follows from Theorem 2.4 that $g \in U(\mathcal{B}(X,\mathcal{P}))$. Finally, we prove that $fgf = f$. For this, let $x \in X$. Then $x \in X_i$ for some $i \in I$. Write $xf_{\restriction X_i} = y$ and $i\chi^{(f)} = j$. Therefore, since $y' \in y(f_{\restriction X_i})^{-1} \cap T_{f_{\restriction X_i}}$, we obtain $x(fgf) = (xf_{\restriction X_i})g_{\restriction X_j} f_{\restriction X_i} = (yg_{\restriction X_j})f_{\restriction X_i} = y'f_{\restriction X_i} = y = xf_{\restriction X_i} = xf$, which yields $fgf = f$. Hence $f \in \mathrm{ureg}(\mathcal{B}(X,\mathcal{P}))$ as required. $\square$

It is well-known that $T(X)$ is unit-regular if and only if X is finite (cf. [4, Proposition 5]). In the following theorem, we describe the unit-regularity for $\mathcal{B}(X,\mathcal{P})$.

Theorem 3.3. *The semigroup $\mathcal{B}(X,\mathcal{P})$ is unit-regular if and only if*

(i) *$\mathcal{P}$ is a uniform partition*;
(ii) *the cardinality of every block of $\mathcal{P}$ is finite.*

Proof. Suppose that $\mathcal{B}(X,\mathcal{P})$ is a unit-regular semigroup.

(i) Suppose to the contrary that there are distinct $j,k \in I$ such that $|X_j| \neq |X_k|$. Consider the following two possible cases:

Case 1: Suppose $|X_j| < |X_k|$. Then there exists a mapping $\varphi\colon X_j \to X_k$ that is injective, but not surjective. Therefore $\mathrm{c}(\varphi) = 0$ and $\mathrm{d}(\varphi) \geq 1$.

Case 2: Suppose $|X_j| > |X_k|$. Then there exists a mapping $\varphi\colon X_j \to X_k$ that is surjective, but not injective. Therefore $\mathrm{c}(\varphi) \geq 1$ and $\mathrm{d}(\varphi) = 0$.

In either case, there is a mapping $\varphi\colon X_j \to X_k$ such that $\mathrm{c}(\varphi) \neq \mathrm{d}(\varphi)$. We now choose $x_i \in X_i$ for each $i \in I \setminus \{k\}$, and consider $\alpha \in \mathrm{Sym}(I)$ such that $j\alpha = k$. Define $f \in T(X)$ by

$$xf = \begin{cases} x\varphi & \text{if } x \in X_j, \\ x_{i\alpha} & \text{if } x \in X_i, \text{ where } i \in I \setminus \{j\}. \end{cases}$$

Clearly, $f \in T(X, \mathcal{P})$ and $\chi^{(f)} = \alpha$, so $f \in \mathcal{B}(X, \mathcal{P})$. However, since $\mathrm{c}(\varphi) \neq \mathrm{d}(\varphi)$, we see that $\mathrm{c}(f_{\restriction X_j}) = \mathrm{c}(\varphi) \neq \mathrm{d}(\varphi) = \mathrm{d}(f_{\restriction X_j})$. Therefore $f \notin \mathrm{ureg}(\mathcal{B}(X, \mathcal{P}))$ by Theorem 3.2, which is a contradiction. Hence $\mathcal{P}$ is uniform.

(ii) Suppose to the contrary that X_i is infinite for some $i \in I$. Then there exists a map $\psi\colon X_i \to X_i$ that is surjective, but not injective. Therefore $\mathrm{c}(\psi) \geq 1$ and $\mathrm{d}(\psi) = 0$, so $\mathrm{c}(\psi) \neq \mathrm{d}(\psi)$. Define $g \in T(X)$ by

$$xg = \begin{cases} x\psi & \text{if } x \in X_i, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $g \in T(X, \mathcal{P})$ and $\chi^{(g)} = \mathrm{id}_I$, so $g \in \mathcal{B}(X, \mathcal{P})$. However, since $\mathrm{c}(\psi) \neq \mathrm{d}(\psi)$, we see that $\mathrm{c}(f_{\restriction X_i}) = \mathrm{c}(\psi) \neq \mathrm{d}(\psi) = \mathrm{d}(f_{\restriction X_i})$. Therefore $f \notin \mathrm{ureg}(\mathcal{B}(X, \mathcal{P}))$ by Theorem 3.2, which is a contradiction. Hence the cardinality of every block of $\mathcal{P}$ is finite.

Conversely, suppose that the given conditions hold. Let $f \in \mathcal{B}(X, \mathcal{P})$ and let $i \in I$. Consider the map $f_{\restriction X_i}\colon X_i \to X_{i\chi^{(f)}}$ of $B(f, I)$. By condition (i), we have $|X_i| = |X_{i\chi^{(f)}}|$. Note from condition (ii) that X_i is finite, so $\mathrm{c}(f_{\restriction X_i}) = \mathrm{d}(f_{\restriction X_i})$. Thus $\mathrm{c}(f_{\restriction X_i}) = \mathrm{d}(f_{\restriction X_i})$ for all $i \in I$, where $f_{\restriction X_i} \in B(f, I)$. Hence $f \in \mathrm{ureg}(\mathcal{B}(X, \mathcal{P}))$ by Theorem 3.2 as required. □

4. Green's relations on $\mathcal{B}(X, \mathcal{P})$

In this section, we give a complete description of Green's relations on $\mathcal{B}(X, \mathcal{P})$. We begin with a lemma that is useful in describing the relation $\mathcal{L}$ on $\mathcal{B}(X, \mathcal{P})$.

Lemma 4.1. *Let $f, g \in \mathcal{B}(X, \mathcal{P})$. Then $L_f \leq L_g$ in $\mathcal{B}(X, \mathcal{P})$ if and only if there exists $\alpha \in \mathrm{Sym}(I)$ such that $X_i f \subseteq X_{i\alpha} g$ for all $i \in I$.*

Proof. Suppose that $L_f \leq L_g$ in $\mathcal{B}(X, \mathcal{P})$. Then there exists $h \in \mathcal{B}(X, \mathcal{P})$ such that $f = hg$. This gives $\chi^{(f)} = \chi^{(h)}\chi^{(g)}$ by Lemma 2.2. Write $\chi^{(h)} = \alpha$. Then, since $h \in \mathcal{B}(X, \mathcal{P})$, we have $\alpha \in \mathrm{Sym}(I)$. To prove the desired result, let $i \in I$. Note that $X_i h \subseteq X_{i\alpha}$, and therefore $X_i f = X_i(hg) = (X_i h)g \subseteq X_{i\alpha} g$ as required.

Conversely, suppose that the given condition holds. To prove $L_f \leq L_g$ in $\mathcal{B}(X, \mathcal{P})$, we shall construct $h \in \mathcal{B}(X, \mathcal{P})$ such that $f = hg$. For this, let $i \in I$. By hypothesis, we have $X_i f \subseteq X_{i\alpha} g$. Therefore for each $x \in X_i$, fix $x' \in X_{i\alpha}$ such that $xf = x'g$. Define $h \in T(X)$ as follows: Given $x \in X$, there exists $i \in I$ such that $x \in X_i$; we let $xh = x'$. Clearly, $X_i h \subseteq X_{i\alpha}$, since $x' \in X_{i\alpha}$. Therefore $f \in T(X, \mathcal{P})$ and $\chi^{(h)} = \alpha$, so $h \in \mathcal{B}(X, \mathcal{P})$. Finally, we prove that $f = hg$. For this, let $x \in X$. Then $x \in X_i$ for some $i \in I$. Therefore, since $xf = x'g$, we obtain $x(hg) = (xh)g = x'g = xf$, which yields $f = hg$. Thus $L_f \leq L_g$ in $\mathcal{B}(X, \mathcal{P})$. □

In the following theorem, we describe the relation $\mathcal{L}$ on $\mathcal{B}(X, \mathcal{P})$.

Theorem 4.2. *Let $f, g \in \mathcal{B}(X, \mathcal{P})$. Then $(f, g) \in \mathcal{L}$ in $\mathcal{B}(X, \mathcal{P})$ if and only if there exists $\alpha \in \mathrm{Sym}(I)$ such that $X_i f = X_{i\alpha} g$ for all $i \in I$.*

Proof. Suppose that $(f, g) \in \mathcal{L}$ in $\mathcal{B}(X, \mathcal{P})$. Then there exist $h, h' \in \mathcal{B}(X, \mathcal{P})$ such that $f = hg$ and $g = h'f$. It follows from Lemma 2.2 that $\chi^{(f)} = \chi^{(h)}\chi^{(g)}$ and $\chi^{(g)} = \chi^{(h')}\chi^{(f)}$. Note that $\chi^{(f)}, \chi^{(g)}, \chi^{(h)}, \chi^{(h')} \in \mathrm{Sym}(I)$, write $\chi^{(h)} = \alpha$ and $\chi^{(h')} = \beta$. Clearly, $\beta = \alpha^{-1}$. To prove the desired result, let $i \in I$. Note that $X_i h \subseteq X_{i\alpha}$, and so $X_i f = X_i(hg) = (X_i h)g \subseteq X_{i\alpha} g$. For the reverse inclusion, note that $\beta = \alpha^{-1}$ and $X_{i\alpha} h' \subseteq X_{(i\alpha)\beta} = X_i$. Therefore, since $g = h'f$, we obtain $X_{i\alpha} g = X_{i\alpha}(h'f) = (X_{i\alpha} h')f \subseteq X_i f$ as required.

Conversely, suppose that there exists $\alpha \in \mathrm{Sym}(I)$ such that $X_i f = X_{i\alpha} g$ for all $i \in I$. Then $L_f \leq L_g$ in $\mathcal{B}(X, \mathcal{P})$ by Lemma 4.1. Next, note that $\alpha^{-1} \in \mathrm{Sym}(I)$ and write $\alpha^{-1} = \beta$. Then by hypothesis, we get $X_i g = X_{(i\beta)\alpha} g = X_{i\beta} f$ for all $i \in I$. Therefore $L_g \leq L_f$ in $\mathcal{B}(X, \mathcal{P})$ by Lemma 4.1. Thus $(f, g) \in \mathcal{L}$ in $\mathcal{B}(X, \mathcal{P})$. □

To describe the relation $\mathcal{R}$ on $\mathcal{B}(X, \mathcal{P})$, we need the following terminology and Lemma 4.3.

Let $f, g \in T(X)$. We say that $\pi(f)$ *refines* $\pi(g)$, denoted by $\pi(f) \preceq \pi(g)$, if $\ker(f) \subseteq \ker(g)$. Equivalently, we have $\pi(f) \preceq \pi(g)$ if for every $P \in \pi(f)$, there exists $Q \in \pi(g)$ such that $P \subseteq Q$. We shall write $\pi(f) = \pi(g)$ if $\pi(f) \preceq \pi(g)$ and $\pi(g) \preceq \pi(f)$.

Lemma 4.3. *Let $f, g \in \mathcal{B}(X, \mathcal{P})$. Then $R_f \leq R_g$ in $\mathcal{B}(X, \mathcal{P})$ if and only if $\pi(g) \preceq \pi(f)$.*

Proof. Suppose that $R_f \leq R_g$ in $\mathcal{B}(X, \mathcal{P})$. Then there exists $h \in \mathcal{B}(X, \mathcal{P})$ such that $f = gh$. It follows from [3, Lemma 2.6] that $\pi(g) \preceq \pi(f)$, since $f, g, h \in T(X)$.

Conversely, suppose that $\pi(g) \preceq \pi(f)$. To prove $R_f \leq R_g$ in $\mathcal{B}(X, \mathcal{P})$, we shall construct $h \in \mathcal{B}(X, \mathcal{P})$ such that $f = gh$. For this, fix $y_i \in X_i$ for each $i \in I$; fix $x' \in xg^{-1}$ for each $x \in Xg$. Note that $\chi^{(f)}, \chi^{(g)} \in \mathrm{Sym}(I)$, and write $(\chi^{(g)})^{-1}\chi^{(f)} = \alpha$. Define $h \in T(X)$ as follows: Given $x \in X$, there exists $i \in I$ such that $x \in X_i$; we let

$$xh = \begin{cases} x'f & \text{if } x \in X_i \cap Xg, \\ y_{i\alpha} & \text{if } x \in X_i \setminus Xg. \end{cases}$$

Clearly, h is well-defined, since $\pi(g) \preceq \pi(f)$. We also observe that $X_i h \subseteq X_{i\alpha}$, since $x' \in X_{i(\chi^{(g)})^{-1}}$ whenever $x \in X_i \cap Xg$, and $(\chi^{(g)})^{-1}\chi^{(f)} = \alpha$. Therefore $h \in T(X, \mathcal{P})$ and $\chi^{(h)} = \alpha$, so $h \in \mathcal{B}(X, \mathcal{P})$. Finally, we prove that $f = gh$. For this, let $x \in X$. Then $x \in X_i$ for some $i \in I$. Therefore, since $x' \in xg^{-1}$ and $xg^{-1} \in \pi(f)$, we obtain $x(gh) = (xg)h = x'f = xf$, which yields $f = gh$. Thus $R_f \leq R_g$ in $\mathcal{B}(X, \mathcal{P})$. □

Theorem 4.4. *Let $f, g \in \mathcal{B}(X, \mathcal{P})$. Then $(f, g) \in \mathcal{R}$ in $\mathcal{B}(X, \mathcal{P})$ if and only if $\pi(f) = \pi(g)$.*

Proof. This proof follows directly from Lemma 4.3. $\square$

To describe the relation $\mathcal{D}$ on $\mathcal{B}(X,\mathcal{P})$, we need the following terminology.

For any $A \subseteq X$ and any $f \in T(X)$, let $\pi_A(f) = \{M \in \pi(f) \colon M \cap A \neq \varnothing\}$. We shall write $\pi(f)$ instead of $\pi_X(f)$ if $A = X$.

Theorem 4.5. *Let $f, g \in \mathcal{B}(X,\mathcal{P})$. Then $(f,g) \in \mathcal{D}$ in $\mathcal{B}(X,\mathcal{P})$ if and only if there exists $\alpha \in \mathrm{Sym}(I)$ such that $|X_i f| = |X_{i\alpha} g|$ for all $i \in I$.*

Proof. Suppose that $(f,g) \in \mathcal{D}$ in $\mathcal{B}(X,\mathcal{P})$. Then there exists $h \in \mathcal{B}(X,\mathcal{P})$ such that $(f,h) \in \mathcal{L}$ in $\mathcal{B}(X,\mathcal{P})$ and $(h,g) \in \mathcal{R}$ in $\mathcal{B}(X,\mathcal{P})$. Since $(f,h) \in \mathcal{L}$, it follows from Theorem 4.2 that there exists $\alpha \in \mathrm{Sym}(I)$ such that $X_k f = X_{k\alpha} h$ for all $k \in I$. To prove the desired result, let $i \in I$ and write $i\alpha = j$. Then $X_i f = X_j h$, so $|X_i f| = |X_j h|$. Now since $(h,g) \in \mathcal{R}$ in $\mathcal{B}(X,\mathcal{P})$, it follows from Theorem 4.4 that $\pi(h) = \pi(g)$. This yields $\pi_{X_j}(h) = \pi_{X_j}(g)$, so $|\pi_{X_j}(h)| = |\pi_{X_j}(g)|$. Notice that $|\pi_{X_j}(h)| = |X_j h|$ and $|\pi_{X_j}(g)| = |X_j g|$, so $|X_j h| = |X_j g|$. Thus $|X_i f| = |X_j g|$ as required.

Conversely, suppose that there exists $\alpha \in \mathrm{Sym}(I)$ such that $|X_i f| = |X_{i\alpha} g|$ for all $i \in I$. To prove $(f,g) \in \mathcal{D}$ in $\mathcal{B}(X,\mathcal{P})$, we shall construct $h \in \mathcal{B}(X,\mathcal{P})$ such that $(f,h) \in \mathcal{L}$ in $\mathcal{B}(X,\mathcal{P})$ and $(h,g) \in \mathcal{R}$ in $\mathcal{B}(X,\mathcal{P})$. For this, let $i \in I$ and write $\beta = \alpha^{-1}$. Then $|X_i g| = |X_{(i\beta)\alpha} g| = |X_{i\beta} f|$ by hypothesis, so there is a bijection $\varphi_i \colon X_i g \to X_{i\beta} f$. Now define $h \in T(X)$ as follows: Given $x \in X$, there exists $i \in I$ such that $x \in X_i$; we let $xh = x(g\varphi_i)$. Clearly, $h \in T(X,\mathcal{P})$ and $\chi^{(h)} = \beta\chi^{(f)}$, since $X_i h = X_i(g\varphi_i) = X_{i\beta} f \subseteq X_{i(\beta\chi^{(f)})}$ for all $i \in I$. Therefore $h \in \mathcal{B}(X,\mathcal{P})$. Recall that $\beta \in \mathrm{Sym}(I)$ and $X_i h = X_{i\beta} f$ for all $i \in I$. Thus $(f,h) \in \mathcal{L}$ in $\mathcal{B}(X,\mathcal{P})$ by Theorem 4.2 and the fact that $\mathcal{L}$ is a symmetric relation. It is easy to see from definition of h that $\pi_{X_i}(h) = \pi_{X_i}(g)$ for all $i \in I$, which yields $\pi(h) = \pi(g)$. Therefore $(h,g) \in \mathcal{R}$ in $\mathcal{B}(X,\mathcal{P})$ by Theorem 4.4. Thus, since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, we conclude that $(f,g) \in \mathcal{D}$ in $\mathcal{B}(X,\mathcal{P})$. $\square$

To describe the relation $\mathcal{J}$ on $\mathcal{B}(X,\mathcal{P})$, we need the following lemma.

Lemma 4.6. *Let $f, g \in \mathcal{B}(X,\mathcal{P})$. Then $J_f \leq J_g$ in $\mathcal{B}(X,\mathcal{P})$ if and only if there exists $\alpha \in \mathrm{Sym}(I)$ such that $|X_i f| \leq |X_{i\alpha} g|$ for all $i \in I$.*

Proof. Suppose that $J_f \leq J_g$ in $\mathcal{B}(X,\mathcal{P})$. Then there exist $h, h_1 \in \mathcal{B}(X,\mathcal{P})$ such that $f = hgh_1$. Set $\chi^{(h)} = \alpha$. Clearly, $\alpha \in \mathrm{Sym}(I)$, since $h \in \mathcal{B}(X,\mathcal{P})$. To prove the desired result, let $i \in I$. Notice that $X_i h \subseteq X_{i\alpha}$ and therefore $|X_i f| = |(X_i h)gh_1| \leq |(X_{i\alpha} g)h_1| \leq |X_{i\alpha} g|$ as required.

Conversely, suppose that there exists $\alpha \in \mathrm{Sym}(I)$ such that $|X_i f| \leq |X_{i\alpha} g|$ for all $i \in I$. Then for every $i \in I$, there exists an injection $\varphi_i \colon X_i f \to X_{i\alpha} g$. Now choose $y' \in (y\varphi_i)g^{-1}$ for each $y \in X_i f$. To prove the desired result, we shall construct $h, h_1 \in \mathcal{B}(X,\mathcal{P})$ such that $f = hgh_1$.

We first define $h \in T(X)$ as follows: Given $x \in X$, there exists $i \in I$ such that $x \in X_i$; write $xf = y$, and we let $xh = y'$. To prove $h \in \mathcal{B}(X,\mathcal{P})$, let $i \in I$ and $x \in X_i$. Set $xf = z$. Then $xh = z'$ by definition of h, where $z' \in (z\varphi_i)g^{-1}$. Since $z \in X_i f$, it follows that $z\varphi_i \in X_{i\alpha} g$ by definition of φ_i. This gives

$(z\varphi_i)g^{-1} \subseteq X_{i\alpha}$, since $\chi^{(g)} \in \mathrm{Sym}(I)$. Therefore, since $z' \in (z\varphi_i)g^{-1}$, we get $xh = z' \in X_{i\alpha}$. Thus $h \in T(X, \mathcal{P})$ and $\chi^{(h)} = \alpha$, so $h \in \mathcal{B}(X, \mathcal{P})$.

To define $h_1 \in \mathcal{B}(X, \mathcal{P})$, we observe for every $i \in I$ that $X_i(f\varphi_i) = X_i(hg)$, since $(xf)' \in ((xf)\varphi_i)g^{-1}$ and $x(hg) = (xh)g = (xf)'g = (xf)\varphi_i$ for all $x \in X_i$. Also, fix $x_i \in X_i$ for each $i \in I$. Let $\beta = (\alpha\chi^{(g)})^{-1}\chi^{(f)}$. Clearly, $\beta \in \mathrm{Sym}(I)$ and $\chi^{(f)} = \alpha\chi^{(g)}\beta$, since $\chi^{(f)}, \chi^{(g)}, \alpha \in \mathrm{Sym}(I)$. Define $h_1 \in T(X)$ as follows: Given $x \in X$, there exists $i \in I$ such that $x \in X_i$; write $i(\alpha\chi^{(g)})^{-1} = j$, and we let

$$xh_1 = \begin{cases} y & \text{if } x \in X_i \cap X(hg) \text{ and } x\varphi_j^{-1} = \{y\}, \\ x_{i\beta} & \text{if } x \in X_i \setminus X(hg). \end{cases}$$

To prove $h_1 \in \mathcal{B}(X, \mathcal{P})$, let $i \in I$ and $x \in X_i$. Consider the following two possible cases:

Case 1: Suppose $x \in X_i \cap X(hg)$. Write $i(\alpha\chi^{(g)})^{-1} = k$. Then, since $\chi^{(h)} = \alpha$, we see that $X_k(hg) \subseteq X_{k\alpha}g \subseteq X_{k(\alpha\chi^{(g)})} = X_i$. As $\alpha, \chi^{(g)} \in \mathrm{Sym}(I)$, it follows that $X_k(hg) = X_i \cap X(hg)$. Therefore $x \in X_k(hg) \subseteq X_{k\alpha}g$. Note that $\varphi_k \colon X_kf \to X_{k\alpha}g$ is injective and $X_k(f\varphi_k) = X_k(hg)$. It follows that $x\varphi_k^{-1} = \{z\}$ for some $z \in X_kf$. Therefore, since $i(\alpha\chi^{(g)})^{-1} = k$ and $\beta = (\alpha\chi^{(g)})^{-1}\chi^{(f)}$, we obtain $k\chi^{(f)} = (i(\alpha\chi^{(g)})^{-1})\chi^{(f)} = i\beta$, which yields $X_kf \subseteq X_{i\beta}$. Thus, since $z \in X_kf$, we get $xh_1 = z \in X_{i\beta}$.

Case 2: Suppose $x \in X_i \setminus X(hg)$. Then $xh_1 = x_{i\beta}$ by definition of h_1, so $xh_1 \in X_{i\beta}$.

In either case, we have $xh_1 \in X_{i\beta}$. Hence $X_ih_1 \subseteq X_{i\beta}$. This gives $h_1 \in T(X, \mathcal{P})$ and $\chi^{(h_1)} = \beta$, so $h_1 \in \mathcal{B}(X, \mathcal{P})$.

Finally, we prove that $f = hgh_1$. For this, let $x \in X$. Then $x \in X_i$ for some $i \in I$. Recall that $\varphi_i \colon X_if \to X_{i\alpha}g$ is an injection. Therefore, since $xh = (xf)' \in ((xf)\varphi_i)g^{-1} \subseteq X_{i\alpha}$, we obtain $x(hgh_1) = (xh)(gh_1) = (xf)'(gh_1) = (x(f\varphi_i))h_1 = xf$, which yields $f = hgh_1$ as required. □

Theorem 4.7. *Let $f, g \in \mathcal{B}(X, \mathcal{P})$. Then $(f, g) \in \mathcal{J}$ in $\mathcal{B}(X, \mathcal{P})$ if and only if there exist $\alpha, \beta \in \mathrm{Sym}(I)$ such that $|X_if| \leq |X_{i\alpha}g|$ and $|X_ig| \leq |X_{i\beta}f|$ for all $i \in I$.*

Proof. This proof follows directly from Lemma 4.6. □

In the following proposition, we prove that the relations $\mathcal{D}$ and $\mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$ coincide when $\mathcal{P}$ is finite.

Proposition 4.8. *If $\mathcal{P}$ is a finite partition of X, then $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$.*

Proof. Let $\mathcal{P} = \{X_i \colon i \in [m]\}$, where $m \in \mathbb{N}$. In general, we have $\mathcal{D} \subseteq \mathcal{J}$. For the reverse inclusion, let $(f, g) \in \mathcal{J}$ in $\mathcal{B}(X, \mathcal{P})$. Then by Theorem 4.7, there exist $\alpha, \beta \in \mathrm{Sym}([m])$ such that $|X_if| \leq |X_{i\alpha}g|$ and $|X_ig| \leq |X_{i\beta}f|$ for all $i \in [m]$. Therefore for every $i \in [m]$, we obtain

$$|X_if| \leq |X_{i\alpha}g| \leq |X_{(i\alpha)\beta}f| = |X_{i(\alpha\beta)}f|. \tag{1}$$

Note that m is finite and $\alpha\beta \in \mathrm{Sym}([m])$. Therefore there exists $k \in \mathbb{N}$ such that $j(\alpha\beta)^k = j$ for all $j \in [m]$.

In view of Theorem 4.5, in order to show that $(f, g) \in \mathcal{D}$ in $\mathcal{B}(X, \mathcal{P})$, it suffices to prove that $|X_i f| = |X_{i\alpha} g|$ for all $i \in [m]$. For this, let $i \in [m]$. Then from inequality (1), we have $|X_i f| \leq |X_{i(\alpha\beta)} f|$. Therefore, since $(\alpha\beta)^k$ is identity map on $[m]$, we obtain by using inequality (1) that

$$|X_i f| \leq |X_{i(\alpha\beta)} f| \leq |X_{i(\alpha\beta)^2} f| \leq \cdots \leq |X_{i(\alpha\beta)^k} f| = |X_i f|$$

whence $|X_i f| = |X_{i(\alpha\beta)} f|$. Note from inequality (1) that $|X_i f| \leq |X_{i\alpha} g| \leq |X_{i(\alpha\beta)} f|$, so $|X_i f| = |X_{i\alpha} g|$. Hence, since $\alpha \in \mathrm{Sym}([m])$, we conclude from Theorem 4.5 that $(f, g) \in \mathcal{D}$ in $\mathcal{B}(X, \mathcal{P})$ as required. □

We now introduce the following terminology.

For any $f \in \mathcal{B}(X, \mathcal{P})$, let $\mathcal{P}_f = \{X_i f : i \in I\}$. It is clear that $\mathcal{P}_f$ is a partition of the range set Xf of f, since $\chi^{(f)} \in \mathrm{Sym}(I)$.

Note that if $\mathcal{P}$ is an infinite partition consisting of only singleton sets, then $\mathcal{B}(X, \mathcal{P}) = \mathrm{Sym}(X)$, so $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$. However, for an infinite partition $\mathcal{P}$, it is not always true that $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$, as shown in the following example.

Example 4.9. Let $X = \mathbb{N}$. Consider the partition $\mathcal{P} = \{X_i : i \in I\}$ of X, where $I = \{1, 3, 5k, 5k+1 : k \in \mathbb{N}\}$; $X_1 = \{1, 2\}$, $X_3 = \{3, 4\}$, and for each $k \in \mathbb{N}$, $X_{5k} = \{5k\}$, $X_{5k+1} = \{5k+1, 5k+2, 5k+3, 5k+4\}$. Take mappings $f, g : X \to X$ defined by $xg = x$ and

$$xf = \begin{cases} 1 & \text{if } x \in X_1, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $f, g \in T(X, \mathcal{P})$ and $\chi^{(f)} = \chi^{(g)} = \mathrm{id}_I$, so $f, g \in \mathcal{B}(X, \mathcal{P})$. Notice that $\mathcal{P}_f$ has exactly one block of cardinality 2, while $\mathcal{P}_g$ has two blocks of cardinality 2. Therefore there does not exist any $\alpha \in \mathrm{Sym}(I)$ such that $|X_i f| = |X_{i\alpha} g|$ for all $i \in I$. Hence $(f, g) \notin \mathcal{D}$ in $\mathcal{B}(X, \mathcal{P})$ by Theorem 4.5.

Finally, we prove that $(f, g) \in \mathcal{J}$ in $\mathcal{B}(X, \mathcal{P})$. Consider $\alpha, \beta \in T(I)$ defined by $i\alpha = i$ and

$$i\beta = \begin{cases} 6 & \text{if } i = 1, \\ 3 & \text{if } i = 3, \\ 1 & \text{if } i = 5, \\ 5(k-1) & \text{if } i = 5k, \text{ where } k \geq 2, \\ 5(k+1)+1 & \text{if } i = 5k+1, \text{ where } k \geq 1. \end{cases}$$

Observe that $\alpha, \beta \in \mathrm{Sym}(I)$. For every $i \in I$, it is also routine to verify that $|X_i f| \leq |X_{i\alpha} g|$ and $|X_i g| \leq |X_{i\beta} f|$. Therefore $(f, g) \in \mathcal{J}$ in $\mathcal{B}(X, \mathcal{P})$ by Theorem 4.7, and thus $(f, g) \in \mathcal{J} \setminus \mathcal{D}$ in $\mathcal{B}(X, \mathcal{P})$.

In the following proposition, we give a sufficient condition under which the relations $\mathcal{D}$ and $\mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$ do not coincide.

Proposition 4.10. *Let $\mathcal{P} = \{X_i : i \in I\}$ be a partition of X. If $J = \{i \in I : |X_i| \geq 3\}$ is an infinite subset of I, then $\mathcal{D} \neq \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$.*

Proof. Since J is infinite, there exists a proper subset K of J such that $|K| = |J \setminus K| = |J|$. Let $k \in K$, and fix distinct $y_k, y'_k \in X_k$. Also, fix distinct $u_i, v_i, w_i \in X_i$ for each $i \in K \setminus \{k\}$; fix $z_i \in X_i$ for each $i \in I \setminus K$. Consider mappings $f, g : X \to X$ defined by

$$xf = \begin{cases} y_k & \text{if } x = y_k, \\ y'_k & \text{if } x \in X_k \setminus \{y_k\}, \\ u_i & \text{if } x = u_i \in X_i, \text{ where } i \in K \setminus \{k\}, \\ v_i & \text{if } x = v_i \in X_i, \text{ where } i \in K \setminus \{k\}, \\ w_i & \text{if } x \in X_i \setminus \{u_i, v_i\}, \text{ where } i \in K \setminus \{k\}, \\ z_i & \text{if } x \in X_i, \text{ where } i \in I \setminus K, \end{cases}$$

and

$$xg = \begin{cases} y_k & \text{if } x \in X_k, \\ u_i & \text{if } x = u_i \in X_i, \text{ where } i \in K \setminus \{k\}, \\ v_i & \text{if } x = v_i \in X_i, \text{ where } i \in K \setminus \{k\}, \\ w_i & \text{if } x \in X_i \setminus \{u_i, v_i\}, \text{ where } i \in K \setminus \{k\}, \\ z_i & \text{if } x \in X_i, \text{ where } i \in I \setminus K. \end{cases}$$

For every $i \in I$, we observe that $X_i f \subseteq X_i$ and $X_i g \subseteq X_i$. Therefore $f, g \in T(X, \mathcal{P})$ and $\chi^{(f)} = \mathrm{id}_I = \chi^{(g)}$. It follows that $f, g \in \mathcal{B}(X, \mathcal{P})$. Notice that $|X_k f| = 2$, while $|X_i g| \neq 2$ for all $i \in I$. Therefore there does not exist $\alpha \in \mathrm{Sym}(I)$ such that $|X_k f| = |X_{k\alpha} g|$. Hence $(f, g) \notin \mathcal{D}$ in $\mathcal{B}(X, \mathcal{P})$ by Theorem 4.5.

Finally, we prove that $(f, g) \in \mathcal{J}$ in $\mathcal{B}(X, \mathcal{P})$. For this, we first fix $k' (\neq k) \in K$. Since K is infinite, we see that $|K \setminus \{k\}| = |K \setminus \{k, k'\}|$. Therefore there is a bijection $\varphi : K \setminus \{k\} \to K \setminus \{k, k'\}$. Recall that J is an infinite subset of I. Since $|J \setminus K| = |J|$ and $J \setminus K \subseteq I \setminus K$, it follows that $I \setminus K$ is infinite. Therefore $|I \setminus K| = |(I \setminus K) \cup \{k\}|$, so there is a bijection $\psi : I \setminus K \to (I \setminus K) \cup \{k\}$. Consider mappings $\alpha, \beta \in T(I)$ defined by $i\beta = i$ and

$$i\alpha = \begin{cases} k' & \text{if } i = k, \\ i\varphi & \text{if } i \in K \setminus \{k\}, \\ i\psi & i \in I \setminus K. \end{cases}$$

Notice that $\alpha, \beta \in \mathrm{Sym}(I)$. For every $i \in I$, it is routine to verify that $|X_i f| \leq |X_{i\alpha} g|$ and $|X_i g| \leq |X_{i\beta} f|$. Therefore $(f, g) \in \mathcal{J}$ in $\mathcal{B}(X, \mathcal{P})$ by Theorem 4.7. Thus $\mathcal{D} \neq \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$. □

Now we establish an alternative characterization of the relation $\mathcal{D}$ on $\mathcal{B}(X, \mathcal{P})$ in Proposition 4.11. Before that, we introduce the following terminology.

Let $f \in \mathcal{B}(X,\mathcal{P})$. Recall that $\mathcal{P}_f = \{X_i f\colon i \in I\}$ is a partition of the range set Xf of f. For any cardinal λ, let $I_\lambda^f = \{i \in I\colon |X_i f| = \lambda\}$ and $n_\lambda^f = |I_\lambda^f|$. Then it is clear that the collection $\{I_\lambda^f\colon \lambda \leq |X| \text{ such that } I_\lambda^f \neq \varnothing\}$ is a partition of I.

Proposition 4.11. *Let $f, g \in \mathcal{B}(X,\mathcal{P})$. Then $(f,g) \in \mathcal{D}$ in $\mathcal{B}(X,\mathcal{P})$ if and only if $n_\lambda^f = n_\lambda^g$ for all cardinals λ.*

Proof. Suppose that $(f,g) \in \mathcal{D}$ in $\mathcal{B}(X,\mathcal{P})$. Then by Theorem 4.5, there exists $\alpha \in \mathrm{Sym}(I)$ such that $|X_i f| = |X_{i\alpha} g|$ for all $i \in I$. To prove the desired result, let λ be a cardinal. It is clear that $n_\lambda^f = 0$ if and only if $n_\lambda^g = 0$, since $|X_i f| \neq \lambda$ for all $i \in I$ if and only if $|X_i g| \neq \lambda$ for all $i \in I$. Assume that $n_\lambda^f \geq 1$, and let $i \in I_\lambda^f$. Then $|X_{i\alpha} g| = |X_i f| = \lambda$, so $i\alpha \in I_\lambda^g$. Since α is injective, we thus get $n_\lambda^f \leq n_\lambda^g$. Now, let $i \in I_\lambda^g$. Then $|X_{i\alpha^{-1}} f| = |X_{(i\alpha^{-1})\alpha} g| = |X_i g| = \lambda$, so $i\alpha^{-1} \in I_\lambda^f$. Since α^{-1} is injective, we thus get $n_\lambda^f \geq n_\lambda^g$. Hence $n_\lambda^f = n_\lambda^g$ as required.

Conversely, suppose that the given condition holds. In view of Theorem 4.5, in order to prove $(f,g) \in \mathcal{D}$ in $\mathcal{B}(X,\mathcal{P})$, it suffices to construct $\alpha \in \mathrm{Sym}(I)$ such that $|X_i f| = |X_{i\alpha} g|$ for all $i \in I$. For this, consider a cardinal λ. Note by hypothesis that $I_\lambda^f = \varnothing$ if and only if $I_\lambda^g = \varnothing$. If $I_\lambda^f \neq \varnothing$, then $|I_\lambda^f| = |I_\lambda^g|$ by hypothesis. Therefore there exists a bijection $\alpha_\lambda\colon I_\lambda^f \to I_\lambda^g$. Define $\alpha \in T(I)$ by $i\alpha = i\alpha_\lambda$ whenever $i \in I_\lambda^f$. Clearly, α is well-defined, since λ is arbitrary. Also, since every α_λ is bijective and $\bigcup_\lambda I_\lambda^f = I = \bigcup_\lambda I_\lambda^g$, we see that $\alpha \in \mathrm{Sym}(I)$. It is also routine to verify that $|X_i f| = |X_{i\alpha} g|$ for all $i \in I$. Hence $(f,g) \in \mathcal{D}$ by Theorem 4.5. □

In the following theorem, we give a sufficient condition for $D_f = J_f$ in $\mathcal{B}(X,\mathcal{P})$.

Theorem 4.12. *Let $f \in \mathcal{B}(X,\mathcal{P})$. If there exists two consecutive cardinals λ_1, λ_2 such that $n_\lambda^f = 0$ for all cardinals $\lambda \notin \{\lambda_1, \lambda_2\}$, then $D_f = J_f$ in $\mathcal{B}(X,\mathcal{P})$.*

Proof. Clearly, $D_f \subseteq J_f$, since $\mathcal{D} \subseteq \mathcal{J}$. For the reverse inclusion, let $g \in J_f$. Then by Theorem 4.2, there exist $\alpha, \beta \in \mathrm{Sym}(I)$ such that for all $i \in I$,

$$\text{(i)} \qquad |X_i f| \leq |X_{i\alpha} g| \qquad \text{and} \qquad \text{(ii)} \qquad |X_i g| \leq |X_{i\beta} f|.$$

First, we claim that $n_\lambda^g = 0$ for all cardinals $\lambda \notin \{\lambda_1, \lambda_2\}$. Suppose to the contrary that there exists $\lambda \notin \{\lambda_1, \lambda_2\}$ such that $n_\lambda^g \neq 0$. Then $|X_i g| = \lambda$ for some $i \in I$. Consider the following two possible cases:

Case 1: Suppose $\lambda < \lambda_1$. By inequality (i), we obtain $|X_{i\alpha^{-1}} f| \leq |X_{(i\alpha^{-1})\alpha} g| = |X_i g| = \lambda$. This yields $|X_{i\alpha^{-1}} f| < \lambda_1$, which is a contradiction of our hypothesis.

Case 2: Suppose $\lambda > \lambda_2$. By inequality (ii), we obtain $\lambda = |X_i g| \leq |X_{i\beta} f|$. This yields $|X_{i\beta} f| > \lambda_2$, which is a contradiction of our hypothesis.

In either case, we get a contradiction. Hence $n^g_\lambda = 0$ for all cardinals $\lambda \notin \{\lambda_1, \lambda_2\}$, and thus $n^f_\lambda = 0 = n^g_\lambda$ for all cardinals $\lambda \notin \{\lambda_1, \lambda_2\}$.

Next, we prove that $n^f_{\lambda_1} = n^g_{\lambda_1}$. Consider the following two possible cases:

Case 1: Suppose $I^f_{\lambda_1} = \varnothing$. Suppose to the contrary that $I^g_{\lambda_1} \neq \varnothing$, and let $i \in I^g_{\lambda_1}$. Then $|X_i g| = \lambda_1$, and so $|X_{i\alpha^{-1}} f| \leq |X_{(i\alpha^{-1})\alpha} g| = |X_i g| = \lambda_1$ by inequality (i). This yields $|X_{i\alpha^{-1}} f| = \lambda_1$ by hypothesis, and so $i\alpha^{-1} \in I^f_{\lambda_1}$. This is a contradiction, because $I^f_{\lambda_1} = \varnothing$. Hence $I^g_{\lambda_1} = \varnothing$, and thus $n^f_{\lambda_1} = n^g_{\lambda_1}$.

Case 2: Suppose $I^f_{\lambda_1} \neq \varnothing$. Let $i \in I^f_{\lambda_1}$. Then $|X_i f| = \lambda_1$, and so $|X_{i\beta^{-1}} g| \leq |X_{(i\beta^{-1})\beta} f| = |X_i f| = \lambda_1$ by inequality (ii). Since $n^g_\lambda = 0$ for all cardinals $\lambda \notin \{\lambda_1, \lambda_2\}$, it follows that $|X_{i\beta^{-1}} g| = \lambda_1$ whence $i\beta^{-1} \in I^g_{\lambda_1}$. Since β^{-1} is injective, we thus get $n^f_{\lambda_1} \leq n^g_{\lambda_1}$. Similarly, we can show that $i\alpha^{-1} \in I^f_{\lambda_1}$ for every $i \in I^g_{\lambda_1}$, and further prove that $n^g_{\lambda_1} \leq n^f_{\lambda_1}$ by using the fact that α^{-1} is injective. Thus $n^f_{\lambda_1} = n^g_{\lambda_1}$.

In either case, we have $n^f_{\lambda_1} = n^g_{\lambda_1}$.

Next, we prove that $n^f_{\lambda_2} = n^g_{\lambda_2}$. Consider the following two possible cases:

Case 1: Suppose $I^f_{\lambda_2} = \varnothing$. Suppose to the contrary that $I^g_{\lambda_2} \neq \varnothing$, and let $i \in I^g_{\lambda_2}$. Then $|X_i g| = \lambda_2$, and so $\lambda_2 = |X_i g| \leq |X_{i\beta} f|$ by inequality (ii). This yields $|X_{i\beta} f| = \lambda_1$ by hypothesis, and so $i\beta \in I^f_{\lambda_2}$. This is a contradiction, because $I^f_{\lambda_2} = \varnothing$. Hence $I^g_{\lambda_2} = \varnothing$, and thus $n^f_{\lambda_2} = n^g_{\lambda_2}$.

Case 2: Suppose $I^f_{\lambda_2} \neq \varnothing$. Let $i \in I^f_{\lambda_2}$. Then $|X_i f| = \lambda_2$, and so $\lambda_2 = |X_i f| \leq |X_{i\alpha} g|$ by inequality (i). Since $n^g_\lambda = 0$ for all cardinals $\lambda \notin \{\lambda_1, \lambda_2\}$, it follows that $|X_{i\alpha} g| = \lambda_2$ whence $i\alpha \in I^g_{\lambda_2}$. Since α is injective, we thus get $n^f_{\lambda_2} \leq n^g_{\lambda_2}$. Similarly, we can show that $i\beta \in I^f_{\lambda_2}$ for every $i \in I^g_{\lambda_2}$, and further prove that $n^g_{\lambda_2} \leq n^f_{\lambda_2}$ by using the fact that β is injective. Thus $n^f_{\lambda_2} = n^g_{\lambda_2}$.

In either case, we have $n^f_{\lambda_2} = n^g_{\lambda_2}$. Thus, since $n^f_\lambda = n^g_\lambda$ for all cardinals λ, we conclude from Proposition 4.11 that $(f, g) \in \mathcal{D}$ in $\mathcal{B}(X, \mathcal{P})$. Hence $g \in D_f$ as required. □

As an immediate consequence of Theorem 4.12, we get:

Corollary 4.13. *Let $f \in \mathcal{B}(X, \mathcal{P})$. If $n^f_\lambda = 0$ for all cardinals $\lambda \geq 3$, then $D_f = J_f$ in $\mathcal{B}(X, \mathcal{P})$.*

In the following theorem, we give a necessary and sufficient condition for $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$.

Theorem 4.14. *Let $\mathcal{P} = \{X_i : i \in I\}$ be a partition of X. Then $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$ if and only if $J = \{i \in I : |X_i| \geq 3\}$ is finite.*

Proof. Suppose that $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$. If $\mathcal{P}$ is finite, then there is nothing to prove. Assume that $\mathcal{P}$ is infinite. Suppose to the contrary that $J = \{i \in$

$I\colon |X_i| \geq 3\}$ is infinite. Then $\mathcal{D} \neq \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$ by Proposition 4.10, which is a contradiction. Hence $J = \{i \in I\colon |X_i| \geq 3\}$ is finite.

Conversely, suppose that $J = \{i \in I\colon |X_i| \geq 3\}$ is finite. If $\mathcal{P}$ is finite, then $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$ by Proposition 4.8. Assume that $\mathcal{P}$ is infinite. In general, we have $\mathcal{D} \subseteq \mathcal{J}$. For the reverse inclusion, let $(f, g) \in \mathcal{J}$ in $\mathcal{B}(X, \mathcal{P})$. Then by Theorem 4.7, there exist $\alpha, \beta \in \mathrm{Sym}(I)$ such that for all $i \in I$,

$$\text{(i)} \qquad |X_i f| \leq |X_{i\alpha} g| \qquad \text{and} \qquad \text{(ii)} \qquad |X_i g| \leq |X_{i\beta} f|.$$

In view of Proposition 4.11, in order to show that $(f, g) \in \mathcal{D}$ in $\mathcal{B}(X, \mathcal{P})$, it suffices to prove that $n^f_\lambda = n^g_\lambda$ for all cardinals λ. Consider the following two possible cases:

Case 1: Suppose $n^f_\lambda = 0$ for all $\lambda \geq 3$. Then $(f, g) \in \mathcal{D}$ by Corollary 4.13.

Case 2: Suppose $n^f_\lambda \neq 0$ for some cardinal $\lambda \geq 3$. Let $i \in I^f_\lambda$. Then $|X_i f| = \lambda$, and so $3 \leq |X_{i\alpha} g|$ by inequality (i). Therefore $n^g_\lambda \neq 0$ for some cardinal $\lambda \geq 3$.

Let $\lambda_1, \ldots, \lambda_k$, where $\lambda_1 > \cdots > \lambda_k$, be the cardinalities of blocks in $\mathcal{P}_f$ of cardinalities at least three; let $\mu_1, \ldots, \mu_t$, where $\mu_1 > \cdots > \mu_t$, be the cardinalities of blocks in $\mathcal{P}_g$ of cardinalities at least three. Note by hypothesis that both k and t are finite.

First, we prove that $\lambda_1 = \mu_1$ and subsequently $n^f_{\lambda_1} = n^g_{\mu_1}$. For this, let $i \in I^f_{\lambda_1}$. Then by inequality (i), we have $\lambda_1 = |X_i f| \leq |X_{i\alpha} g| \leq \mu_1$. Now, let $i \in I^g_{\mu_1}$. Then by inequality (ii), we have $\mu_1 = |X_i g| \leq |X_{i\beta} f| \leq \lambda_1$. Thus $\lambda_1 = \mu_1$. To prove $n^f_{\lambda_1} = n^g_{\mu_1}$, we first observe for every $i \in I^f_{\lambda_1}$ that $|X_{i\alpha} g| = \lambda_1$, whence $i\alpha \in I^g_{\lambda_1}$. Therefore, since α is injective, we get $n^f_{\lambda_1} \leq n^g_{\lambda_1}$. Similarly, by using the facts that β is injective and $i\beta \in I^f_{\lambda_1}$ for every $i \in I^g_{\lambda_1}$, we get $n^g_{\lambda_1} \leq n^f_{\lambda_1}$. Thus $n^f_{\lambda_1} = n^g_{\lambda_1}$. Moreover, recall by hypothesis that both $I^f_{\lambda_1}$ and $I^g_{\lambda_1}$ are finite. Therefore, since α, β are injective, both maps $\alpha_{\lambda_1}\colon I^f_{\lambda_1} \to I^g_{\lambda_1}$ and $\beta_{\lambda_1}\colon I^g_{\lambda_1} \to I^f_{\lambda_1}$ defined by $i\alpha_{\lambda_1} = i\alpha$ and $i\beta_{\lambda_1} = i\beta$, respectively, are bijective.

Next, we prove that $\lambda_2 = \mu_2$ and subsequently $n^f_{\lambda_2} = n^g_{\mu_2}$. For this, let $i \in I^f_{\lambda_2}$. Recall that $\alpha_{\lambda_1}\colon I^f_{\lambda_1} \to I^g_{\lambda_1}$ is bijective, and so $i\alpha \notin I^g_{\lambda_1}$. Therefore $|X_{i\alpha} g| \leq \mu_2$, which gives $\lambda_2 = |X_i f| \leq |X_{i\alpha} g| \leq \mu_2$ by inequality (i). Now, let $i \in I^g_{\mu_2}$. Recall that $\beta_{\lambda_1}\colon I^g_{\lambda_1} \to I^f_{\lambda_1}$ is bijective, and so $i\beta \notin I^f_{\lambda_1}$. Therefore $|X_{i\alpha} f| \leq \lambda_2$, which gives $\mu_2 = |X_i g| \leq |X_{i\beta} f| \leq \lambda_2$ by inequality (ii). Thus $\lambda_2 = \mu_2$. To prove $n^f_{\lambda_2} = n^g_{\mu_2}$, we first observe for every $i \in I^f_{\lambda_2}$ that $|X_{i\alpha} g| = \lambda_2$, whence $i\alpha \in I^g_{\lambda_2}$. Therefore, since α is injective, we get $n^f_{\lambda_2} \leq n^g_{\lambda_2}$. Similarly, by using the facts that β is injective and $i\beta \in I^f_{\lambda_2}$ for every $i \in I^g_{\lambda_2}$, we get $n^g_{\lambda_2} \leq n^f_{\lambda_2}$. Thus $n^f_{\lambda_2} = n^g_{\lambda_2}$. Moreover, recall by hypothesis that both $I^f_{\lambda_2}$ and $I^g_{\lambda_2}$ are finite. Therefore, since α, β are injective, both maps $\alpha_{\lambda_2}\colon I^f_{\lambda_2} \to I^g_{\lambda_2}$ and $\beta_{\lambda_2}\colon I^g_{\lambda_2} \to I^f_{\lambda_2}$ defined by $i\alpha_{\lambda_2} = i\alpha$ and $i\beta_{\lambda_2} = i\beta$, respectively, are bijective.

We can prove in similar way that $\lambda_j = \mu_j$ and $n^f_{\lambda_j} = n^g_{\mu_j}$ for all $j = 3, \ldots, \min\{k, t\}$. Moreover, we observe for every $j = 3, \ldots, \min\{k, t\}$ that both maps $\alpha_{\lambda_j} : I^f_{\lambda_j} \to I^g_{\lambda_j}$ and $\beta_{\lambda_j} : I^g_{\lambda_j} \to I^f_{\lambda_j}$ defined by $i\alpha_{\lambda_j} = i\alpha$ and $i\beta_{\lambda_j} = i\beta$, respectively, are bijective.

We now claim that $k = t$. Suppose to the contrary that $k \neq t$. Assume without loss of generality that $k < t$. Then $\min\{k, t\} = k$, and so $\lambda_j = \mu_j$ for all $j \in [k]$. Now, let $i \in I^g_{\mu_t}$. Then we see that $i\beta \notin I^f_{\lambda_j}$ for all $j \in [k]$, since $\beta_{\lambda_j} : I^g_{\lambda_j} \to I^f_{\lambda_j}$ is bijective for all $j \in [k]$. Therefore $i\beta \in I^f_1 \cup I^f_2$, and so $|X_{i\beta} f| \leq 2$. Recall that $|X_i g| = \mu_t \geq 3$. By inequality (ii), we therefore obtain $3 \leq |X_i g| \leq |X_{i\beta} f| \leq 2$, which is a contradiction. Hence $k = t$.

Next, we prove that $n^f_2 = n^g_2$. Consider the following two possible cases:
Case I: Suppose $I^f_2 = \varnothing$. Suppose to the contrary that $I^g_2 \neq \varnothing$, and let $i \in I^g_2$. Then $|X_i g| = 2$. Notice that $i\beta \notin I^f_{\lambda_j}$ for all $j \in [k]$, since $\beta_{\lambda_j} : I^g_{\lambda_j} \to I^f_{\lambda_j}$ is bijective for all $j \in [k]$. Therefore $i\beta \in I^f_1$, and so $|X_{i\beta} f| = 1$. By inequality (ii), we obtain $2 = |X_i g| \leq |X_{i\beta} f| = 1$, which is a contradiction. Hence $I^g_2 = \varnothing$, and thus $n^f_2 = n^g_2$.
Case II: Suppose $I^f_2 \neq \varnothing$. Let $i \in I^f_2$. Then $|X_i f| = 2$, and so $2 \leq |X_{i\alpha} g|$ by inequality (i). Notice that $i\alpha \notin I^g_{\lambda_j}$ for all $j \in [k]$, since $\alpha_{\lambda_j} : I^f_{\lambda_j} \to I^g_{\lambda_j}$ is bijective for all $j \in [k]$. Therefore $i\alpha \in I^g_2$. Since α is injective, we thus get $n^f_2 \leq n^g_2$. Now, let $i \in I^g_2$. Then $|X_i g| = 2$, and so $2 \leq |X_{i\beta} f|$ by inequality (ii). Notice that $i\beta \notin I^f_{\lambda_j}$ for all $j \in [k]$, since $\beta_{\lambda_j} : I^g_{\lambda_j} \to I^f_{\lambda_j}$ is bijective for all $j \in [k]$. Therefore $i\beta \in I^f_2$. Since β is injective, we thus get $n^f_2 \geq n^g_2$. Hence $n^f_2 = n^g_2$.

In either case, we have $n^f_2 = n^g_2$. Finally, we prove that $n^f_1 = n^g_1$. Consider the following two possible cases:
Case I: Suppose $I^f_1 = \varnothing$. Suppose to the contrary that $I^g_1 \neq \varnothing$, and let $i \in I^g_1$. Then $|X_i g| = 1$, and so $|X_{i\alpha^{-1}} f| \leq |X_{(i\alpha^{-1})\alpha} g| = |X_i g|$ by inequality (i). It follows that $|X_{i\alpha^{-1}} f| = 1$, and so $i\alpha^{-1} \in I^f_1$, which is a contradiction. Hence $I^g_1 = \varnothing$, and thus $n^f_1 = n^g_1$.
Case II: Suppose $I^f_1 \neq \varnothing$. Let $i \in I^f_1$. Then $|X_i f| = 1$, and so $|X_{i\beta^{-1}} g| \leq |X_{(i\beta^{-1})\beta} f| = |X_i f|$ by inequality (ii). It follows that $|X_{i\beta^{-1}} g| = 1$, and so $i\beta^{-1} \in I^g_1$. Since β^{-1} is injective, we thus get $n^f_1 \leq n^g_1$. Now, let $i \in I^g_1$. Then $|X_i g| = 1$, and so $|X_{i\alpha^{-1}} f| \leq |X_{(i\alpha^{-1})\alpha} g| = |X_i g|$ by inequality (i). It follows that $|X_{i\alpha^{-1}} f| = 1$, and so $i\alpha^{-1} \in I^f_1$. Since α^{-1} is injective, we thus get $n^f_1 \geq n^g_1$. Hence $n^f_1 = n^g_1$.

In either case, we have $n^f_1 = n^g_1$. In addition, we note for every cardinal $\lambda \notin \{1, 2, \lambda_k, \ldots, \lambda_1\}$ that $n^f_\lambda = 0 = n^g_\lambda$. Thus we conclude from Proposition 4.11 that $(f, g) \in \mathcal{D}$ in $\mathcal{B}(X, \mathcal{P})$ as required. $\square$

Note that $\mathcal{D} = \mathcal{J}$ on $\mathcal{B}(X, \mathcal{P})$ if $\mathcal{P}$ is a trivial partition of X. In connections with Proposition 4.8, Theorem 4.12, and Theorem 4.14, we end this section with the following conjecture.

Conjecture. Let $\mathcal{P} = \{X_i \colon i \in I\}$ be a partition of X and $f \in \mathcal{B}(X, \mathcal{P})$. Then $D_f = J_f$ in $\mathcal{B}(X, \mathcal{P})$ if and only if there exist two consecutive cardinals λ_1, λ_2 and a finite subset K of I such that $|X_i f| \in \{\lambda_1, \lambda_2\}$ for all $i \in I \setminus K$.

References

[1] J. Araújo, W. Bentz, J. D. Mitchell, and C. Schneider, *The rank of the semigroup of transformations stabilising a partition of a finite set*, Math. Proc. Cambridge Philos. Soc. **159** (2015), no. 2, 339–353. https://doi.org/10.1017/S0305004115000389

[2] J. Araújo and C. Schneider, *The rank of the endomorphism monoid of a uniform partition*, Semigroup Forum **78** (2009), no. 3, 498–510. https://doi.org/10.1007/s00233-008-9122-0

[3] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups. Vol. I*, Mathematical Surveys, No. 7, Amer. Math. Soc., Providence, RI, 1961.

[4] H. D'Alarcao, *Factorizable as a finiteness condition*, Semigroup Forum **20** (1980), no. 3, 281–282. https://doi.org/10.1007/BF02572687

[5] L. Z. Deng, *On certain semigroups of transformations that preserve double direction equivalence*, Bull. Iranian Math. Soc. **42** (2016), no. 4, 1015–1024.

[6] L. Z. Deng, J. W. Zeng, and B. Xu, *Green's relations and regularity for semigroups of transformations that preserve double direction equivalence*, Semigroup Forum **80** (2010), no. 3, 416–425. https://doi.org/10.1007/s00233-009-9204-7

[7] L. Z. Deng, J. W. Zeng, and T. J. You, *Green's relations and regularity for semigroups of transformations that preserve order and a double direction equivalence*, Semigroup Forum **84** (2012), no. 1, 59–68. https://doi.org/10.1007/s00233-011-9332-8

[8] I. Dolinka and J. East, *Idempotent generation in the endomorphism monoid of a uniform partition*, Comm. Algebra **44** (2016), no. 12, 5179–5198. https://doi.org/10.1080/00927872.2016.1149186

[9] I. Dolinka, J. East, and J. D. Mitchell, *Idempotent rank in the endomorphism monoid of a nonuniform partition*, Bull. Aust. Math. Soc. **93** (2016), no. 1, 73–91. https://doi.org/10.1017/S0004972715000751

[10] V. H. Fernandes and T. M. Quinteiro, *The cardinal of various monoids of transformations that preserve a uniform partition*, Bull. Malays. Math. Sci. Soc. (2) **35** (2012), no. 4, 885–896.

[11] V. H. Fernandes and T. M. Quinteiro, *On the ranks of certain monoids of transformations that preserve a uniform partition*, Comm. Algebra **42** (2014), no. 2, 615–636. https://doi.org/10.1080/00927872.2012.720322

[12] P. M. Higgins, J. M. Howie, and N. Ruškuc, *Generators and factorisations of transformation semigroups*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), no. 6, 1355–1369. https://doi.org/10.1017/S0308210500027360

[13] J. M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs. New Series, 12, The Clarendon Press, Oxford University Press, New York, 1995.

[14] H. Pei, *Equivalences, α-semigroups and α-congruences*, Semigroup Forum **49** (1994), no. 1, 49–58. https://doi.org/10.1007/BF02573470

[15] H. Pei, *Regularity and Green's relations for semigroups of transformations that preserve an equivalence*, Comm. Algebra **33** (2005), no. 1, 109–118. https://doi.org/10.1081/AGB-200040921

[16] H. Pei, *On the rank of the semigroup $T_E(X)$*, Semigroup Forum **70** (2005), no. 1, 107–117. https://doi.org/10.1007/s00233-004-0135-z

[17] H. Pei, L. Sun, and H. Zhai, *Green's relations for the variants of transformation semigroups preserving an equivalence relation*, Comm. Algebra **35** (2007), no. 6, 1971–1986. https://doi.org/10.1080/00927870701247112

[18] H. Pei and H. Zhou, *Abundant semigroups of transformations preserving an equivalence relation*, Algebra Colloq. **18** (2011), no. 1, 77–82. https://doi.org/10.1142/S1005386711000034

[19] P. Purisang and J. Rakbud, *Regularity of transformation semigroups defined by a partition*, Commun. Korean Math. Soc. **31** (2016), no. 2, 217–227. https://doi.org/10.4134/CKMS.2016.31.2.217

[20] J. Rakbud, *Regularity of a particular subsemigroup of the semigroup of transformations preserving an equivalence*, Kyungpook Math. J. **58** (2018), no. 4, 627–635. https://doi.org/10.5666/KMJ.2018.58.4.627

[21] M. Sarkar and S. N. Singh, *On certain semigroups of transformations that preserve a partition*, Comm. Algebra **49** (2021), no. 1, 331–342. https://doi.org/10.1080/00927872.2020.1799224

[22] M. Sarkar and S. N. Singh, *Regular, unit-regular, and idempotent elements of semigroups of transformations that preserve a partition*, Semigroup Forum **104** (2022), no. 1, 148–165. https://doi.org/10.1007/s00233-021-10238-2

[23] M. Sarkar and S. N. Singh, *Regular and unit-regular elements in various monoids of transformations*, J. Algebra Appl. **22** (2023), no. 11, 2350244, 16pp. https://doi.org/10.1142/S0219498823502444

[24] L. Sun and J. Sun, *A partial order on transformation semigroups that preserve double direction equivalence relation*, J. Algebra Appl. **12** (2013), no. 8, 1350041, 12 pp. https://doi.org/10.1142/S0219498813500412

Mosarof Sarkar
Department of Mathematics
Central University of South Bihar
Gaya–824236(Bihar), India
Email address: mosarofsarkar@cusb.ac.in

Shubh N. Singh
Department of Mathematics
Central University of South Bihar
Gaya–824236(Bihar), India
Email address: shubh@cub.ac.in