THE KERNELS OF THE LINEAR MAPS OF
FINITE GROUP ALGEBRAS

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Abstract. Let $G$ be a finite group, $K$ a split field for $G$, and $L$ a linear map from $K[G]$ to $K$. In our paper, we first give sufficient and necessary conditions for $\text{Ker } L$ and $\text{Ker } L \cap Z(K[G])$, respectively, to be Mathieu-Zhao spaces for some linear maps $L$. Then we give equivalent conditions for $\text{Ker } L$ to be Mathieu-Zhao spaces of $K[G]$ in terms of the degrees of irreducible representations of $G$ over $K$ if $G$ is a finite Abelian group or $G$ has a normal Sylow $p$-subgroup $H$ and $L$ are class functions of $G/H$. In particular, we classify all Mathieu-Zhao spaces of the finite Abelian group algebras if $K$ is a split field for $G$.

1. Introduction

Throughout this paper, we will write $K$ for a field without specific note and $K[G]$ for the group algebra of $G$ over $K$. $V_G$ is the $K$-subspace of the group algebra $K[G]$ consisting of all the elements of $K[G]$ whose coefficient of the identity element $1_G$ of $G$ is equal to zero. It is easy to see that $V_G$ is a subspace of $K[G]$ with codimension one. Let $L$ be a linear map from $K[G]$ to $K$ and $L|_H$ means restricting $L$ to $H$, where $H$ is a subgroup of $G$. We call $H$ a $p'$-subgroup of $G$ if $p \nmid |H|$. Let $	au : K[H] \to K$

such that $\tau(\sum a_xx) = \sum a_x$. Then $w(K[H]) := \text{Ker } \tau$, which is called the augmentation ideal of $K[H]$. It’s equal to $\sum_{h_i \in H} (h_i - 1)K[H]$ for any subgroup $H$ of $G$ and $w(K[H])K[G] = \sum_{h_i \in H} (h_i - 1)K[G]$. The Mathieu-Zhao space was introduced by W. Zhao in [7], which is a natural generalization of ideals, motivated by a conjecture of O. Mathieu. The term Mathieu-Zhao space was suggested and used by A. van den Essen. We recall

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the definitions of Mathieu-Zhao spaces of $K[G]$ and the radical of a subspace of $K[G]$. We say that a $K$-subspace $M$ of $K[G]$ is called a Mathieu-Zhao space of $K[G]$ if for any $a, b \in K[G]$ with $a^m \in M$ for all $m \geq 1$, we have $ba^m \in M$ when $m \gg 0$. Let $S$ be a $K$-subspace of $K[G]$. The radical of $S$ is the set of all elements $a \in K[G]$ such that $a^m \in S$ when $m \gg 0$. We say that a subspace of $K[G]$ has MZ-property if it is a Mathieu-Zhao space of $K[G]$. In [1], J. J. Duistermaat and W. van der Kallen proved the Mathieu conjecture for the case of tori, which can be re-stated as follows.

**Theorem 1.1.** Let $z = (z_1, z_2, \ldots, z_m)$ be $m$ commutative free variables and $V$ be a subspace of the Laurent polynomial algebra $\mathbb{C}[z^{-1}, z]$ consisting of the Laurent polynomials with no constant term. Then $V$ is a Mathieu-Zhao space of $\mathbb{C}[z^{-1}, z]$.

Let $G$ be the free Abelian group $\mathbb{Z}^m$ ($m \geq 1$). Then the Laurent polynomial algebra $\mathbb{C}[z^{-1}, z]$ can be identified with the group algebra $\mathbb{C}[G]$. Under this identification, the subspace of $V$ in the theorem is $V_G$. In [9], W. Zhao and R. Willems proved that $V_G$ is a Mathieu-Zhao space of $K[G]$ if $G$ is a finite group and $\text{char } K = 0$ or $\text{char } K = p > |G|$. For finite Abelian group, they proved that if $K$ contains a primitive $d$-th root of unity and $\text{char } K = p$, then $V_G$ is a Mathieu-Zhao space of $K[G]$ if and only if $\text{char } K = p > d$, where $|G| = p^d$, $p \nmid d$. In [10], W. Zhao and the author give a sufficient and necessary condition for $V_G$ to be a Mathieu-Zhao space of $K[G]$ if $G$ is a finite group and $K$ is a split field for $G$. Since $V_G$ is just one subspace of $K[G]$ with codimension one, we first want to consider all subspaces of $K[G]$ with codimension one. Then we want to consider all subspaces of $K[G]$. Hence it is natural to ask the following question.

**Problem 1.2.** Let $G$ be a finite group with $|G| = n$, $L = (L_1, L_2, \ldots, L_r)$ and $L_i$ be a linear map from $K[G]$ to $K$ such that $L_i(g_j) = l_{ij}$ for all $1 \leq i \leq r, 1 \leq j \leq n$. Suppose that $L_1, L_2, \ldots, L_r$ are linearly independent over $K$. Then under what conditions on $L$ and $K$, $\text{Ker } L$ forms a Mathieu-Zhao space of the group algebra $K[G]$?

It’s easy to see that if $r \geq n$, then $\text{Ker } L = 0$. If $r \leq n - 1$, then $\dim_K \text{Ker } L = n - r$ and every codimension $r$ subspace of $K[G]$ is $\text{Ker } L$ for some linear map $L$. Hence $\text{Ker } L$ are all the codimension $r$ subspaces of $K[G]$.

In our paper, we first prove some properties of $L$ and $K$ in Section 2. In Section 3, we give sufficient and necessary conditions for $\text{Ker } L$ and $\text{Ker } L \cap Z(K[G])$, respectively, to be Mathieu-Zhao spaces for some linear maps $L$. Then we classify all Mathieu-Zhao spaces of $K[G]$ if $G$ is a finite Abelian group and $K$ a split field for $G$ in Section 4. Thus, we solve Problem 1.2 if $G$ is a finite Abelian group. In Section 5, we give equivalent conditions for $\text{Ker } L$ to be Mathieu-Zhao spaces of $K[G]$ in term of the degrees of irreducible representations of $G$ over $K$ if $G$ has a normal Sylow $p$-subgroup $H$ and $L$ are class functions of $G/H$ or $L_1, \ldots, L_{r-1}$ are class functions of $G/H$ and
\[ L_r(\tilde{g}_j h_2) = L_r(\tilde{g}_j h_3) = \cdots = L_r(\tilde{g}_j h_t) \text{ for all } 1 \leq j \leq d, \text{ where } G = \bigcup_{j=1}^{d} \tilde{g}_j H, \]
\[ H = \{1_H, h_2, \ldots, h_t\}. \]

2. Some properties of \( \text{Ker}L \) and \( \text{Ker}L \cap Z(K[G]) \)

**Proposition 2.1.** Let \( L = (L_1, L_2, \ldots, L_r) \) and \( L_i \) be a linear map from \( K[G] \) to \( K \) such that \( L_i(g_j) = l_{i,j} \) for all \( 1 \leq i \leq r, 1 \leq j \leq n \). \( K, G \) be as in Problem 1.2 and \( g_1 \) be the identity 1 of \( G \). Then we have the following statements:

1. If all the \( l_{i,j} \) are equal for all \( 1 \leq i \leq r, 1 \leq j \leq n \), then \( \text{Ker}L \) is an ideal of \( K[G] \).

2. If \( \text{Ker}L \) is a Mathieu-Zhao space of \( K[G] \), then there exists \( i_0 \in \{1, 2, \ldots, r\} \) such that \( l_{i_0,1} \neq 0 \).

**Proof.** (1) Let \( l := l_{i,j} \) for all \( 1 \leq i \leq r, 1 \leq j \leq n \). Then \( \text{Ker}L = \{ \sum_{j=1}^{n} c_j g_j \in K[G] \mid l \sum_{j=1}^{n} c_j = 0 \} \). Since \( l \neq 0 \), we have that \( \text{Ker}L = \{ \sum_{j=1}^{n} c_j g_j \in K[G] \mid \sum_{j=1}^{n} c_j = 0 \} \). It is easy to check that \( \text{Ker}L \) is an ideal of \( K[G] \).

(2) If \( l_{1,1} = \cdots = l_{r,1} = 0 \), then \( 1_G \in \text{Ker}L \). If \( \text{Ker}L \) is a Mathieu-Zhao space of \( K[G] \), then \( \text{Ker}(L) \) is a Mathieu-Zhao space of \( K[G] \). That is, \( L = 0 \), which is a contradiction. Then the conclusion follows. \( \square \)

**Remark 2.2.** We can see from Proposition 2.1 that we can assume \( l_{i_0,1} \neq 0 \) for some \( i_0 \in \{1, 2, \ldots, r\} \) in the following arguments. If \( r = 1 \) and \( l_{1,2} = l_{1,3} = \cdots = l_{1,n} = 0, l_{1,1} \neq 0 \), then \( \text{Ker}L = V_G \), which is discussed in [9] and [10].

**Proposition 2.3.** Let \( R \) be any commutative ring and \( G \) any group. Suppose that \( L = (L_1, L_2, \ldots, L_r) \) is a linear map from \( R[G] \) to \( R \). If \( \text{Ker}L \) is a Mathieu-Zhao space of \( R[G] \), then \( \text{Ker}(L) \) is a Mathieu-Zhao space of \( R[H] \), where \( H \) is any subgroup of \( G \).

**Proof.** Assume otherwise. Then there exist \( u, v_1, v_2 \in R[H] \) such that \( u^m \in \text{Ker}(L) \) for all \( m \geq 1 \) and \( v_1 u^m v_2 \notin \text{Ker}(L) \) for infinitely many \( m \geq 1 \). Since \( R[H] \subseteq R[G] \), we have \( u, v_1, v_2 \in R[G] \) and \( u^m \in \text{Ker}L \) for all \( m \geq 1 \) and \( v_1 u^m v_2 \notin \text{Ker}L \) for infinitely many \( m \geq 1 \). Otherwise, \( v_1 u^m v_2 \in \text{Ker}L \cap R[H] = \text{Ker}(L) \), which is a contradiction. Hence \( \text{Ker}L \) is not a Mathieu-Zhao space of \( R[G] \), which is a contradiction. Then the conclusion follows. \( \square \)

**Corollary 2.4.** Let \( L, G \) be as in Problem 1.2 and \( K \) a field of characteristic \( p \), \( H \) a normal subgroup of \( G \). If \( H \) is a \( p' \)-subgroup and \( \text{Ker}L \) is a Mathieu-Zhao space of \( K[G] \), then \( \text{Ker}(L)_{G/H} \) is a Mathieu-Zhao space of \( K[G/H] \).

**Proof.** Let \( \varphi \) be the natural surjective homomorphism from \( K[G] \) to \( K[G/H] \) and \( E_H = 1_H \sum_{j=1}^{H} h_j \). Then \( (1 - E_H)K[G] = \text{Ker} \varphi \) and \( E_H K[G] \cong K[G/H] \). Thus, we have \( K[G] \cong (1 - E_H)K[G] \oplus K[G/H] \). Therefore, \( K[G/H] \) can be seen as a subalgebra of \( K[G] \). It follows from the arguments of Proposition 2.3 that \( \text{Ker}(L)_{G/H} \) is a Mathieu-Zhao space of \( K[G/H] \). \( \square \)
Corollary 2.9. Let $K$ and $L$ be as in Problem 1.2 and $G$ a finite group with $G = \{g_1, g_2, \ldots, g_n\}$, $g_1 = 1_G$. If there exists $i \in \{1, 2, \ldots, r\}$ such that $\det M_{L_i} \neq 0$, then $\ker L$ is a Mathieu-Zhao space of $K[G]$ if and only if all elements of $r(\ker L)$ are nilpotent, where $M_{L_i} = (l_{i,j_1,j_2})_{n \times n}$ and $l_{i,j_1,j_2} = L_i(g_j^{-1} g_{j_2})$ for $1 \leq j_1, j_2 \leq n$.

Proof. ($\Leftarrow$) It follows from the definition of Mathieu-Zhao spaces.

($\Rightarrow$) Let $u \in r(\ker L)$. Replacing $u$ by a positive power of $u$, if necessary, we may assume that $u^m \in \ker L$ for all $m \geq 1$. Since $G$ is finite, by definition of Mathieu-Zhao space, there exists $N \geq 1$ such that $g_j^{-1} u^m \in \ker L$ for all $g_j \in G$ and $m \geq N$. Let $u^N = \sum_{j=1}^n d_j g_j$. Then we have $g_j^{-1} u^N \in \ker L$ for all $1 \leq j_1 \leq n$. That is,

\[
M_{L_i} \cdot \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = 0
\]

for all $1 \leq i \leq r$. Since there exists $i \in \{1, 2, \ldots, r\}$ such that $\det M_{L_i} \neq 0$, we have that $d_1 = \cdots = d_n = 0$. That is, $u^N = 0$. Thus, $u$ is nilpotent. □

Remark 2.6. If $l_{i,1} = 1$ and $l_{i,2} = \cdots = l_{i,n} = 0$ for some $i \in \{1, 2, \ldots, r\}$, then $M_{L_i}$ is the identity matrix. Thus, we have $\det M_{L_i} = 1$ in this case. It is easy to see that $\det M_{L_i}$ is the group determinant of $G$ up to a sign for $1 \leq i \leq r$.

Corollary 2.7. Let $K$ and $L$ be as in Problem 1.2 and $G$ a finite group with $G = \{g_1, g_2, \ldots, g_n\}$, $g_1 = 1_G$. If there exists $i \in \{1, 2, \ldots, r\}$ such that $\det M_{L_i} \neq 0$, then $\ker L \cap Z(K[G])$ is a Mathieu-Zhao space of $K[G]$ if and only if all elements of $r(\ker L \cap Z(K[G]))$ are nilpotent, where $M_{L_i} = (l_{i,j_1,j_2})_{n \times n}$ and $l_{i,j_1,j_2} = L_i(g_j^{-1} g_{j_2})$ for $1 \leq j_1, j_2 \leq n$.

Proof. The conclusion follows from the arguments of Proposition 2.5 by replacing $\ker L$ with $\ker L \cap Z(K[G])$. □

Proposition 2.8. Let $K$, $L$ and $G$ be as in Problem 1.2. If there exists $i \in \{1, 2, \ldots, r\}$ such that $\det M_{L_i} \neq 0$, then $\ker L$ is a Mathieu-Zhao space of $K[G]$ if and only if $\ker L$ contains no nonzero idempotent of $K[G]$.

Proof. ($\Rightarrow$) Let $e \in \ker L$ be an idempotent. Then $e^m = e \in \ker L$ for all integers $m \geq 1$, whence $e \in r(\ker L)$. It follows from Proposition 2.5 that $e$ is nilpotent. Thus, we have $e = e^N = 0$ for some $N \in \mathbb{N}$. Thus, the conclusion follows.

($\Leftarrow$) Since $G$ is finite, we have that $K[G]$ is algebraic over $K$. In particular, the radical $r(\ker L)$ is algebraic over $K$. It follows from Theorem 4.2 in [8] that $\ker L$ is a Mathieu-Zhao space of $K[G]$. □

Corollary 2.9. Let $K$, $L$ and $G$ be as in Problem 1.2. If there exists $i \in \{1, 2, \ldots, r\}$ such that $\det M_{L_i} \neq 0$, then $\ker L \cap Z(K[G])$ is a Mathieu-Zhao
space of $K'[G]$ if and only if $\ker L \cap Z(K[G])$ contains no nonzero idempotent of $K[G]$.

**Proof.** The conclusion follows from the arguments of Proposition 2.8 by replacing $\ker L$ with $\ker L \cap Z(K[G])$. □

**Remark 2.10.** If $\ker L (\ker L \cap Z(K[G]))$ contains no nonzero idempotent of $K[G]$, then $\ker L (\ker L \cap Z(K[G]))$ is a Mathieu-Zhao space of $K'[G]$ without the condition that $\det M_{L_i} \neq 0$ for some $i \in \{1, 2, \ldots, r\}$ in Proposition 2.8 (Corollary 2.9).

**Corollary 2.11.** Let $K$ be a field of characteristic $p$ and $G$ a $p$-group. Then $\ker L$ is a Mathieu-Zhao space of $K[G]$.

**Proof.** Note that $K[G]$ is a local $K$-algebra. Hence $K[G]$ does not contain nontrivial idempotent. Thus, $\ker L$ contains no nonzero idempotent of $K'[G]$. Then the conclusion follows from Proposition 2.8 and Remark 2.10. □

**Remark 2.12.** Corollary 2.11 can also be deduced from Theorem 7.6 in [8].

**Lemma 2.13.** Let $L$ and $G$ be as in Problem 1.2. Then $\ker L = \{ \beta \in K[G] \mid \text{Tr} \beta \alpha_i = 0 \text{ for all } 1 \leq i \leq r \}$, where $\alpha_i = \sum_{j=1}^{n_i} g_j l_{i,j}^{-1}$ for all $1 \leq i \leq r$.

**Proof.** Let $\beta = \sum_{j=1}^{n_i} c_j g_j$. Then $L_i(\beta) = \sum_{j=1}^{n_i} c_j l_{i,j} = \text{Tr} \beta \alpha_i$ for all $1 \leq i \leq r$. Hence the conclusion follows.

**Theorem 2.14.** Let $L$ and $G$ be as in Problem 1.2 and $K$ a field of characteristic zero or a field of characteristic $p$ and $p \mid |G|$. If $K$ is a split field for $G$, then

$$\ker L \cong \{ (A_1, \ldots, A_s) \in A \mid \sum_{j=1}^{s} n_j \text{Tr}(C_{i,j} A_j) = 0 \text{ for all } 1 \leq i \leq r \},$$

where $A = M_{n_1}(K) \times \cdots \times M_{n_s}(K)$ is the product of matrices and $C_{i,j} = \rho_j(\alpha_i) \in M_{n_j}(K)$, $\alpha_i$ be as in Lemma 2.13, $\rho_j$ is an irreducible representation of $G$, $n_j = \rho_j(1)$ for $1 \leq j \leq s$, $1 \leq i \leq r$ and $s$ is the number of distinct (up to isomorphism) irreducible representations of $G$.

**Proof.** Since char $K = 0$ or char $K = p$ and $p \mid |G|$, we have that $K[G]$ is semi-simple. Since $K$ is a split field for $G$, we have that

$$K'[G] \cong M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_s}(K),$$

where $M_{n_j}(K)$ is the ring of $n_j \times n_j$ matrices over $K$ for $1 \leq j \leq s$. Let $\bar{\rho}$ be the regular representation of $K[G]$. Then $\text{Tr}(\bar{\rho}(\beta)) = 0$ for all $\beta \in K[G]$. Let $\rho = (\rho_1, \rho_2, \ldots, \rho_s)$. Then $\rho$ is a ring isomorphism from $K[G]$ to $A$. Let $\beta$ be any element in $K[G]$. Then

$$\rho(\alpha_1 \beta) = (\rho_1(\alpha_1 \beta), \rho_2(\alpha_1 \beta), \ldots, \rho_s(\alpha_1 \beta)) = (\rho_1(\alpha_1)\rho_1(\beta), \ldots, \rho_s(\alpha_1)\rho_s(\beta)).$$
Suppose that
\[
\rho(\alpha_i) = (\rho_1(\alpha_i), \ldots, \rho_s(\alpha_i)) = \begin{pmatrix}
C_{i,1} & 0 & \cdots & 0 \\
0 & C_{i,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_{i,s}
\end{pmatrix} \in A
\]
and
\[
\rho(\beta) = (\rho_1(\beta), \ldots, \rho_s(\beta)) = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_s
\end{pmatrix} \in A
\]
for all 1 \leq i \leq r. Then we have that
\[
\rho(\alpha_i \beta) = \begin{pmatrix}
C_{i,1}A_1 & 0 & \cdots & 0 \\
0 & C_{i,2}A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_{i,s}A_s
\end{pmatrix} \in A.
\]
Thus, we have the following commutative diagram:
\[
K[G] \xrightarrow{\sim} Mn_1(K) \times Mn_2(K) \times \cdots \times Mn_s(K) \\
\rho(\alpha, \beta) \downarrow \quad \phi(\rho(\alpha, \beta)) \downarrow \\
K[G] \xrightarrow{\sim} \begin{pmatrix}
Mn_1(K) & 0 & \cdots & 0 \\
0 & Mn_2(K) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & Mn_s(K)
\end{pmatrix},
\]
where \(\phi\) is the natural isomorphism between the two algebras. Thus, we have that
\[\Tr(\rho(\alpha, \beta)) = 0\] if and only if \(\Tr(\phi(\rho(\alpha, \beta))) = 0\). Since \(\Tr(\phi(\rho(\alpha, \beta))) = n_1 \Tr(C_{i,1}A_1) + n_2 \Tr(C_{i,2}A_2) + \cdots + n_s \Tr(C_{i,s}A_s)\), we have that \(\Tr(\alpha, \beta) = 0\) if and only if \(n_1 \Tr(C_{i,1}A_1) + n_2 \Tr(C_{i,2}A_2) + \cdots + n_s \Tr(C_{i,s}A_s) = 0\) for all 1 \leq i \leq r. Thus, we have that \(\text{Ker } L \cong V\), where
\[V = \{(A_1, A_2, \ldots, A_s) \in A | \sum_{j=1}^s n_j \Tr C_{i,j}A_j = 0 \text{ for all } 1 \leq i \leq r\}.\]

**Corollary 2.15.** Let \(L\) and \(G\) be as in Problem 1.2 and \(K\) a field of characteristic zero or a field of characteristic \(p\) and \(p \nmid |G|\). If \(K\) is a split field for \(G\) and \(r = 1\), then \(\text{Ker } L\) is a Mathieu-Zhao space of \(K[G]\) if and only if \(n_1 \lambda_1d_1 + n_2 \lambda_2d_2 + \cdots + n_t \lambda_td_t \neq 0\) for all non-zero vectors \(d = (d_1, \ldots, d_t)\), \(d_j \in \{0,1,\ldots,n_j\}\) for 1 \leq j \leq t, where \(n_j \lambda_j = \Tr \rho_j(\alpha_1)\), \(\alpha_1\) is as in Lemma 2.13, \(\rho_j\) is an irreducible representation of \(G\) for 1 \leq j \leq s and \(s\) is the number of distinct (up to isomorphism) irreducible representations of \(G\) and \(t \in \{1,2,\ldots,s\}\).
Proof. It follows from Theorem 2.14 that \( \ker L \cong V \), where \( V = \{(A_1, \ldots, A_s) \in A | \sum_{j=1}^s n_j \operatorname{Tr}(C_{1,j} A_j) = 0 \} \) and \( C_{1,j} = \rho_j(\alpha_1) \in M_{n_j}(K) \) for \( 1 \leq j \leq s \). Let \( \rho \) be as in Theorem 2.14. Since \( \alpha_1 \neq 0 \) and \( \rho \) is an isomorphism, we have that \( \rho(\alpha_1) \neq 0 \). We can assume that \( C_{1,1}, \ldots, C_{1,t} \) are not equal to zero and \( C_{1,t+1} = \cdots = C_{1,s} = 0 \) for some \( t \in \{1, 2, \ldots, s\} \) by reordering the \( \rho_j \) for \( 1 \leq j \leq s \). It follows from Theorem 5.8.1 in [2] or Theorem 4.4 in [4] that \( V \) is a Mathieu-Zhao space of \( A \) if and only if \( C_{1,j} = \lambda_j I_{n_j} \) and \( n_1 \lambda_1 d_1 + \cdots + n_t \lambda_t d_t \neq 0 \) for all nonzero vectors \( \tilde{d} = (d_1, \ldots, d_t) \) and \( d_j \in \{0, 1, \ldots, n_j\} \) for \( 1 \leq j \leq t \). Then the conclusion follows. \( \square \)

Proposition 2.16. Let \( L \) and \( G \) be as in Problem 1.2 and \( K \) a field of characteristic zero or a field of characteristic \( p \) and \( p \nmid |G| \). If \( K \) is a split field for \( G \) and \( r = 1 \), then the following two statements are equivalent:

1. \( \ker L \) is a Mathieu-Zhao space of \( K[G] \).
2. There exist \( \mu_1, \ldots, \mu_s \in K \) such that \( L_1 = \mu_1 \chi_1 + \mu_2 \chi_2 + \cdots + \mu_s \chi_s \) and \( \mu_1 d_1 + \cdots + \mu_s d_s \neq 0 \) for all nonzero vectors \( \tilde{d} = (d_1, d_2, \ldots, d_s) \), \( d_j \in \{0, 1, \ldots, n_j\} \) for \( 1 \leq j \leq t \), where \( \chi_1, \chi_2, \ldots, \chi_s \) are the non-isomorphic irreducible characters of \( G \) and \( \mu_j = n^{-1} n_j \lambda_j \), \( n_j = \chi_j(1), n \lambda_j = \operatorname{Tr} \rho_j(\alpha_1), \alpha_1 \) is as in Lemma 2.13 and \( \rho_j \) is an irreducible representation of \( G \) with character \( \chi_j \) for \( 1 \leq j \leq s \), \( s \) is the number of distinct (up to isomorphism) irreducible representations of \( G \) and \( t \in \{1, 2, \ldots, s\} \). In particular, \( L_1 \) is a class function of \( G \).

Proof. (1) \( \Rightarrow \) (2) Since \( L_1(\beta) = \operatorname{Tr}(\alpha_1 \beta) \) for any \( \beta \in K[G] \), where \( \alpha_1 \) is as in Lemma 2.13, we have that

\[
n \operatorname{Tr}(\alpha_1 \beta) = \operatorname{Tr} \tilde{\rho}(\alpha_1 \beta) = \operatorname{Tr} \phi(\alpha_1 \beta)
\]

by following the arguments of Theorem 2.14, where \( \tilde{\rho} \) is as in Theorem 2.14. Since \( \ker L \) is a Mathieu-Zhao space of \( K[G] \), it follows from Corollary 2.15 that \( C_{1,j} = \lambda_j I_{n_j} \) for \( \lambda_j \in K \) and for all \( 1 \leq j \leq s \). We can assume that \( \lambda_1 \cdots \lambda_t \neq 0 \) and \( \lambda_{t+1} = \cdots = \lambda_s = 0 \) for some \( t \in \{1, 2, \ldots, s\} \) by reordering \( \chi_1, \chi_2, \ldots, \chi_s \).

Thus, it follows from Lemma 2.13 that

\[
L_1(\beta) = \operatorname{Tr}(\alpha_1 \beta) = n^{-1}(n_1 \lambda_1 \operatorname{Tr} A_1 + n_2 \lambda_2 \operatorname{Tr} A_2 + \cdots + n_t \lambda_t \operatorname{Tr} A_t).
\]

Since \( \operatorname{Tr} A_j = \chi_j(\beta) \) for all \( 1 \leq j \leq s \), we have that

\[
L_1 = n^{-1}(n_1 \lambda_1 \chi_1 + n_2 \lambda_2 \chi_2 + \cdots + n_t \lambda_t \chi_t).
\]

It follows from Corollary 2.15 that \( n_1 \lambda_1 d_1 + \cdots + n_t \lambda_t d_t \neq 0 \) for all nonzero vectors \( \tilde{d} = (d_1, d_2, \ldots, d_t) \), \( d_j \in \{0, 1, \ldots, n_j\} \) for \( 1 \leq j \leq t \). Let \( \mu_j = n^{-1} n_j \lambda_j \) for all \( 1 \leq j \leq s \). Then the conclusion follows.

(2) \( \Rightarrow \) (1) Since \( \ker L = \{ \beta \in K[G] | L_1(\beta) = 0 \} = \{ \beta \in K[G] | \mu_1 \chi_1(\beta) + \cdots + \mu_s \chi_s(\beta) = 0 \} \) and there exists \( A_j \in M_{n_j}(K) \) such that \( \operatorname{Tr} A_j = \chi_j(\beta) \) for
all $1 \leq j \leq t$, we have that

$$\text{Ker } L = \{(A_1, \ldots, A_t) \in M_{n_1}(K) \times \cdots \times M_{n_t}(K) \mid \sum_{j=1}^{t} \mu_j \text{ Tr } A_j = 0\}.$$ 

Then the conclusion follows from Theorem 5.8.1 in [2] or Theorem 4.4 in [4]. □

Remark 2.17. To prove that Ker $L$ is a Mathieu-Zhao space of $K[G]$ for $r = 1$ if $n_1 \lambda_1 d_1 + n_2 \lambda_2 d_2 + \cdots + n_t \lambda_t d_t \neq 0$ for all nonzero vectors $\bar{d} = (d_1, d_2, \ldots, d_t)$ and $d_j \in \{0, 1, \ldots, n_j\}$ for $1 \leq j \leq t$, we don’t need the condition that $K$ is a split field for $G$ in Corollary 2.15 by following the arguments Theorem 5.8.1 in [2], because an idempotent matrix can be conjugated to a diagonal matrix with only 0 and 1 on the diagonal over division rings.

If $L = \mu_j \chi_j$ for some $j \in \{1, 2, \ldots, t\}$, $\mu_j \in K^*$, then it follows from the arguments of Proposition 2.16 that the condition $n_1 \lambda_1 d_1 + n_2 \lambda_2 d_2 + \cdots + n_t \lambda_t d_t \neq 0$ in Theorem 2.14 is equivalent to $n_j d_j \neq 0$ for all $1 \leq d_j \leq n_j$, which is clearly true if char $K = 0$. If char $K = p$, then the condition is equivalent to $p > n_j$. To see this, we can assume that $p \mid n_j d_j$ for some $d_j \in \{1, 2, \ldots, n_j\}$, then $p \mid n_j$ or $p \mid d_j$, which contradicts with $p > n_j$. Thus, if $p > n_j$, then $n_j d_j \neq 0$ mod $p$ for all $1 \leq d_j \leq n_j$. Conversely, suppose that $p \leq n_j$. Then let $d_j = p \in \{1, 2, \ldots, n_j\}$, we have that $n_j p = 0$ mod $p$, which is a contradiction. Thus, if $n_j d_j \neq 0$ mod $p$ for all $1 \leq d_j \leq n_j$, then $p > n_j$. Therefore, the conclusion is the same as Theorem 5.1 in [8] in this situation.

3. The MZ-property of Ker$L$ and Ker$L \cap Z(K[G])$

Condition 1: Let $L$ and $G$ be as in Problem 1.2 and $K$ a field of characteristic $p$, $H$ a normal $p$-subgroup of $G$, $G = \bigcup_{j=1}^{k} \bar{g}_j H$, $H = \{h_1, h_2, \ldots, h_t\}$ for $\ell = p^\ell$ for some $\ell \in \mathbb{N}$ and $L_i(\bar{g}_j h_2) = \cdots = L_i(\bar{g}_j h_\ell)$ for all $1 \leq i \leq r$, $1 \leq j \leq k$.

Proposition 3.1. Let $L$, $G$, $K$, $H$ be as in Condition 1 and $L_i(\bar{g}_j h_1) = L_i(\bar{g}_j h_2) = \cdots = L_i(\bar{g}_j h_\ell)$ for all $1 \leq i \leq r$, $1 \leq j \leq k$. Then Ker $L$ is a Mathieu-Zhao space of $K[G]$ if and only if Ker$(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$.

Proof. Let $\varphi$ be the natural surjective homomorphism from $K[G]$ to $K[G/H]$. Since $L_i(\bar{g}_j h_1) = L_i(\bar{g}_j h_2) = \cdots = L_i(\bar{g}_j h_\ell)$ for all $1 \leq i \leq r$, $1 \leq j \leq k$, there exists a linear map $\bar{L}$ from $K[G/H]$ to $K$ such that $L = \varphi^{-1}(\bar{L})$, where $\bar{L} = L|_{G/H}$. Since $\varphi$ is surjective and Ker $\varphi = \{K[H]K[G] = \sum_{j=1}^{k} (h_j - 1)K[G]\}$, we have Ker $\varphi \subseteq$ Ker $L$. Then it follows from Theorem 5.2.19 in [2] that Ker $L$ is a Mathieu-Zhao space of $K[G]$ if and only if Ker$(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$. □

Corollary 3.2. Let $L$, $G$, $K$, $H$ be as in Condition 1, $|G| = p^\ell d$, $p \nmid d$, $\ell = a$, $k = d$ and $H$ a normal Sylow $p$-subgroup of $G$. If $r = 1$, then the following two statements are equivalent:

1. Ker $L$ is a Mathieu-Zhao space of $K[G]$. 

Proposition 3.4. Let \( L = \mu_1 \chi_1 + \mu_2 \chi_2 + \cdots + \mu_t \chi_t \) and \( \mu_1 d_1 + \cdots + \mu_t d_t \neq 0 \) for all nonzero vectors \( d = (d_1, d_2, \ldots, d_t) \), \( d_j \in \{0, 1, \ldots, n_j\} \) for \( 1 \leq j \leq t \), where \( \chi_1, \chi_2, \ldots, \chi_n \) are the distinct (up to isomorphism) irreducible characters of \( K[G] \) and \( \mu_j = d^{-1} n_j \lambda_j, n_j = \chi_j(1) \), \( n_j \lambda_j = \text{Tr} \rho_j(\alpha_1), \alpha_1 = \sum_{j=1}^t l_{1,j} \bar{\gamma}_j^{-1} \) and \( \rho_j \) is an irreducible representation of \( K[G] \) with character \( \chi_j \) for \( 1 \leq j \leq t \) and \( t \in \{1, 2, \ldots, s\} \).

Proof. It follows from Proposition 3.1 that \( \text{Ker} L \) is a Mathieu-Zhao space of \( K[G] \) if and only if \( \text{Ker}(L_{G/H}) \) is a Mathieu-Zhao space of \( K[G/H] \). Since \( p \nmid |G/H| \), the conclusion follows from Proposition 2.16.

\[ \square \]

Remark 3.3. Let the notations be the same as Corollary 3.2. Then \( J(K[G]) = w(K[H])K[G] \subseteq \text{Ker} L \) if and only if \( L_i(\bar{g}_j h_1) = L_i(\bar{g}_j h_2) = \cdots = L_i(\bar{g}_j h_t) \) for all \( 1 \leq i \leq r, 1 \leq j \leq d \).

Proposition 3.4. Let \( L, G, K, H \) be as in Condition 1 and \( h_1 = 1_H \). Then we have the following statements:

1. If there exists \( i \in \{1, 2, \ldots, r\} \) such that \( \det M_{E_{G/H}} \neq 0 \) and \( \text{Ker}(L_{G/H}) \) is a Mathieu-Zhao space of \( K[G/H] \), then \( \text{Ker} L \) is a Mathieu-Zhao space of \( K[G] \), where \( M_{E_{G/H}} = (\tilde{e}_{i,j1,k})_{k \times k} \) and \( \tilde{e}_{i,j1,k} = e_{i,j1,k} \) for \( 1 \leq j_1, j_2 \leq k \).

2. If there exists \( i \in \{1, 2, \ldots, r\} \) such that \( \det M_{E_i} \neq 0 \) and \( \text{Ker} L \) is a Mathieu-Zhao space of \( K[G] \), then \( \text{Ker}(L_{G/H}) \) is a Mathieu-Zhao space of \( K[G/H] \), where \( M_{E_i} = (e_{i,j1,k})_{k \times n} \) and \( e_{i,j1,k} = e_{i,j1,k} \) for \( 1 \leq j_1, j_2 \leq n \).

Proof. Let \( \varphi \) be the natural surjective homomorphism from \( K[G] \) to \( K[G/H] \).

1. Let \( E \) be an idempotent of \( \text{Ker} L \). Then
   \[ E = \bar{g}_1 \cdot a_1(h) + \bar{g}_2 \cdot a_2(h) + \cdots + \bar{g}_k \cdot a_k(h), \]
   where \( a_i(h) \in K[H], h = (h_1, h_2, \ldots, h_t), \bar{g}_j \notin H \) for \( 2 \leq j \leq k \) and \( \bar{g}_1 = 1_{G/H} \). Let \( b \in H \) and \( b \neq 1_H \). Then \( b \) is a \( p \)-element. Thus, it follows from Lemma 2.7 in [6] that the sum of coefficients in \( E \) of the \( G \)-conjugacy class of \( b \) is equal to zero. Then \( \varphi(E) = \bar{g}_1 \cdot a_1(1) + \bar{g}_2 \cdot a_2(1) + \cdots + \bar{g}_k \cdot a_k(1) \).
   Let \( a_j(h) = a_j(h_1 + a_1 h_2 + \cdots + a_k h_t) \) for \( 1 \leq j \leq k \). Then we have that \( a_j(1) = a_{j_1} \) and \( L_i(\bar{g}_j \cdot a_j(h)) = a_{j_1} L_i(\bar{g}_j) \) for \( 1 \leq i \leq r, 1 \leq j \leq k \). Thus, we have that \( L_i(\bar{g}_j h_1) = a_{j_1} L_i(\bar{g}_j) + a_{j_2} L_i(\bar{g}_j) + \cdots + a_{j_k} L_i(\bar{g}_j) = L_i(\varphi(E)) \) for all \( 1 \leq i \leq r \). Therefore, we have that \( E \in \text{Ker} L \) if and only if \( \varphi(E) \in \text{Ker}(L_{G/H}) \).
   That is, \( E = \varphi(E) \) is an idempotent of \( \text{Ker}(L_{G/H}) \). Since \( \text{Ker}(L_{G/H}) \) is a Mathieu-Zhao space of \( K[G/H] \), it follows from Proposition 2.8 that \( \varphi(E) = 0 \) in \( K[G/H] \). That is, \( E \in \text{Ker} \varphi = w(K[H])K[G] \). It follows from Lemma 2.8 in [6] that \( E \) is nilpotent. Thus, we have \( E = 0 \). Hence it follows from Proposition 2.8 and Remark 2.10 that \( \text{Ker} L \) is a Mathieu-Zhao space of \( K[G] \).

2. Since \( \text{Ker} \varphi = w(K[H])K[G] \), it follows from Lemma 2.8 in [6] that \( w(K[H])K[G] \) is a nilpotent ideal and \( K[G/H] \cong K[G]/\text{Ker} \varphi \). Let \( u \) be any idempotent of \( \text{Ker}(L_{G/H}) \). Then there exists a \( u \in K[G] \) such that \( \bar{u} = \varphi(u) \).
   It follows from Lemma 3.7(i) of Chapter 2 in [6] that there exists an idempotent
\[ e = \bar{u}b \bar{u} \text{ such that } \varphi(e) = \bar{u} \text{ for some } \bar{b} \in K[G]. \] We have that \( e \in \text{Ker } L \) by following the arguments of Proposition 3.4(1). Since \( \text{Ker } L \) is a Mathieu-Zhao space of \( K[G] \), it follows from Proposition 2.8 that \( e = 0 \). Thus, we have \( \bar{u} = \varphi(e) = 0 \). Hence it follows from Proposition 2.8 and Remark 2.10 that \( \text{Ker}(L|_{G/H}) \) is a Mathieu-Zhao space of \( K[G/H] \).

\[ \square \]

**Proposition 3.5.** Let \( L, G, K, H \) be as in Condition 1, \( |G| = p^n d \), \( p \nmid d \), \( \tilde{r} = a, k = d \) and \( H \) a normal Sylow \( p \)-subgroup of \( G \), \( h_1 = 1_R \) and \( K \) is a split field for \( G/H \). If there exist \( i, i \in \{1, 2, \ldots, r\} \) such that \( \text{det } M_{l_i} \neq 0 \) and \( \text{det } M_{l_i} \neq 0 \) and \( r = 1 \), then \( \text{Ker } L \) is a Mathieu-Zhao space of \( K[G] \) if and only if \( n_1 \lambda_1 d_1 + n_2 \lambda_2 d_2 + \cdots + n_t \lambda_t d_t \neq 0 \) for all nonzero vectors \( \tilde{a} = (d_1, d_2, \ldots, d_t) \) and \( d_j \in \{0, 1, \ldots, n_j\} \) for \( 1 \leq j \leq t \), where \( n_1 \lambda_1 = \text{Tr } \rho_1(\alpha_1) \) for \( 1 \leq j \leq s \) and \( \alpha_1 = \sum_{j=1}^{d} l_{i,j} \tilde{g}_j^{-1} \), \( \rho_1, \ldots, \rho_s \) are distinct (up to isomorphism) irreducible representations of \( K[G] \) and \( t \in \{1, 2, \ldots, s\} \), \( M_{l_i} \mid_{G/H} = (\tilde{t}_{i,j_1})_{d \times d} \) and \( \tilde{t}_{i,j_1,2} = L_l(\tilde{g}_{j_1}^{-1} \tilde{g}_{j_2}) \) for \( 1 \leq j_1, j_2 \leq d \), \( M_{l_i} = (l_{i,j_1,2})_{n \times n} \) and \( l_{i,j_1,2} = l_{i}(\tilde{g}_{j_1}^{-1} \tilde{g}_{j_2}) \) for \( 1 \leq j_1, j_2 \leq n \).

**Proof.** It follows from Proposition 3.4 that \( \text{Ker } L \) is a Mathieu-Zhao space of \( K[G] \) if and only if \( \text{Ker}(L|_{G/H}) \) is a Mathieu-Zhao space of \( K[G/H] \). Since \( p \nmid |G/H| \), the conclusion follows from Corollary 2.15.

\[ \square \]

**Theorem 3.6.** Let \( L, G, K, H \) be as in Problem 1.2 and \( K \) a field of characteristic zero or a field of characteristic \( p \) with \( g_1 = 1_G \) and \( \chi_1, \ldots, \chi_s \) are the distinct (up to isomorphism) irreducible characters of \( K[G] \). Then we have the following statements:

1. If there exists \( i, j \in \{1, 2, \ldots, r\} \) such that \( \sum_{i=1}^{n} (\sum_{j=1}^{d} \chi_{i_j}(1) \chi_{i_j}(g_i^{-1})) l_{q_1, \ldots, q_i} \neq 0 \) for all \( 1 \leq i_1 < i_2 < \cdots < i_j \leq s \), \( l \in \{1, 2, \ldots, s\} \), then \( \text{Ker } L \cap Z(K[G]) \) is a Mathieu-Zhao space of \( K[G] \).

2. If there exists \( i \in \{1, 2, \ldots, r\} \) such that \( \text{det } M_{l_i} \neq 0 \) and \( \text{Ker } L \cap Z(K[G]) \) is a Mathieu-Zhao space of \( K[G] \), then there exists \( q_1, \ldots, q_i \in \{1, 2, \ldots, r\} \) such that \( \sum_{i=1}^{n} (\sum_{j=1}^{d} \chi_{i_j}(1) \chi_{i_j}(g_i^{-1})) l_{q_1, \ldots, q_i} \neq 0 \) for all \( 1 \leq i_1 < i_2 < \cdots < i_j \leq s \).

**Proof.** (1) Let \( e_k = \frac{1}{n} \sum_{g \in G} \chi_k(1) \chi_k(g^{-1}) g \) for \( 1 \leq k \leq s \). Then it follows from Theorem 2.12 in [3] that \( e_1, e_2, \ldots, e_s \) are the primitive orthogonal idempotents of \( Z(K[G]) \). It follows from Theorem 3.11 in [5] that every idempotent of \( Z(K[G]) \) is some sum of \( e_1, e_2, \ldots, e_s \). Since \( \sum_{j=1}^{n} (\sum_{j=1}^{d} \chi_{i_j}(1) \chi_{i_j}(g_i^{-1})) l_{q_1, \ldots, q_i} \neq 0 \), we have that \( L_{q_1, \ldots, q_i}(e_1 + e_2 + \cdots + e_s) \neq 0 \) for all \( 1 \leq i_1 < i_2 < \cdots < i_j \leq s, l \in \{1, 2, \ldots, s\} \). That is, any nonzero idempotent of \( Z(K[G]) \) is not in \( \text{Ker } L \). Thus, \( \text{Ker } L \cap Z(K[G]) \) has no nonzero idempotent. It follows from Corollary 2.9 and Remark 2.10 that \( \text{Ker } L \cap Z(K[G]) \) is a Mathieu-Zhao space of \( K[G] \).

(2) It follows from Corollary 2.9 that \( \text{Ker } L \cap Z(K[G]) \) has no nonzero idempotent. Hence there exists \( q_1, \ldots, q_i \in \{1, 2, \ldots, r\} \) such that \( L_{q_1, \ldots, q_i}(e_1 + e_2 + \cdots + e_s) \neq 0 \).
such that \( i_k \neq 0 \) for all \( 1 \leq i_1 < i_2 < \cdots < i_t \leq s, l \in \{1, 2, \ldots, s\} \). That is, 
\[
\sum_{i=1}^{n}(\sum_{j=1}^{l} x_{ij}(1) \chi_{ij}(g_{j}^{-1})) \cdot l_{q_{i_1\ldots i_t}} \neq 0 \text{ for all } 1 \leq i_1 < i_2 < \cdots < i_t \leq s, l \in \{1, 2, \ldots, s\}.
\]

**Proposition 3.7.** Let \( L, G, K, H \) be as in Condition 1 and \( h_1 = 1_H \). If there exists \( i \in \{1, 2, \ldots, r\} \) such that \( \det L_{iG/H} \neq 0 \) and \( \ker(L_{iG/H}) \cap Z(K[G/H]) \) is a Mathieu-Zhao space of \( K[G/H] \), then \( \ker L \cap Z(K[G]) \) is a Mathieu-Zhao space of \( K[G] \), where \( L_{iG/H} = (\tilde{l}_{i,j}, 2) \) and \( \tilde{l}_{i,j} = L_i(g_{j1}^{-1} g_{j2}) \) for \( 1 \leq j_1, j_2 \leq k \).

**Proof.** Let \( \varphi \) be the natural surjective homomorphism from \( K[G] \) to \( K[G/H] \). Then it's easy to check that if \( E \in Z(K[G]) \), then \( \varphi(E) \in Z(K[G/H]) \). Thus, the conclusion follows by following the arguments of Proposition 3.4(1). \( \square \)

**Corollary 3.8.** Let \( L, G, K, H \) be as in Condition 1, \( |G| = p^d, p \nmid d, r = a, k = d \) and \( H \) a normal Sylow \( p \)-subgroup of \( G \) and \( h_1 = 1_H \). If there exists \( i \in \{1, 2, \ldots, r\} \) such that \( \det M_{iG/H} \neq 0 \) and there exists \( q_{1}, \ldots, q_{r} \in \{1, 2, \ldots, r\} \) such that \( \sum_{i=1}^{d}(\sum_{j=1}^{l} x_{ij}(1) \chi_{ij}(g_{j}^{-1})) \cdot l_{q_{i_1\ldots i_t}} \neq 0 \text{ for all } 1 \leq i_1 < i_2 < \cdots < i_t \leq s, l \in \{1, 2, \ldots, s\}, \) then \( \ker L \cap Z(K[G]) \) is a Mathieu-Zhao space of \( K[G] \), where \( 1, \ldots, \chi_s \) are the distinct (up to isomorphism) irreducible characters of \( K[G] \) if and only if \( q_{1}, \ldots, q_{r} \) are the distinct (up to isomorphism) irreducible characters of \( K[G/H] \).

**Proof.** The conclusion follows from Theorem 3.6(1) and Proposition 3.7. \( \square \)

4. Mathieu-Zhao spaces of finite Abelian group algebras

**Proposition 4.1.** Let \( B = K \times \cdots \times K \) be a \( K \)-algebra and
\[
V = \{(a_1, a_2, \ldots, a_n) \in B \mid \sum_{j=1}^{n} \gamma_{i,j} a_j = 0 \text{ for all } 1 \leq i \leq r, \}
\]
where \( \gamma_{i,j} \in K \) for all \( 1 \leq i \leq r, 1 \leq j \leq n \). If at least one of \( \gamma_{i,j} \) is nonzero for all \( 1 \leq i \leq r, 1 \leq j \leq n \), then \( V \) is a Mathieu-Zhao space of \( B \) if and only if \( \gamma_{i_1} d_1 + \gamma_{i_2} d_2 + \cdots + \gamma_{i_t} d_t \neq 0 \) for some \( i \in \{1, 2, \ldots, r\} \) for all nonzero vectors \( d = (d_1, d_2, \ldots, d_t) \) and \( d_i, \in \{0, 1\} \) for \( 1 \leq j_1 < j_2 < j_3 < \cdots < j_t \leq t, t \in \{1, \ldots, n\} \).

**Proof.** We can assume that \( \gamma_{i,j} \neq 0 \) for all \( 1 \leq j \leq t \) for some \( i \in \{1, 2, \ldots, r\} \) and \( \gamma_{i,j} = 0 \) for all \( 1 \leq i \leq r \) and \( t + 1 \leq j \leq n \) by reordering \( \gamma_{i,j} \) for \( 1 \leq i \leq r, 1 \leq j \leq n \) and then we have
\[
\text{t columns}\begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} \times K \times \cdots \times K \subseteq V
\]
and
\[
\text{n-t columns}\begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} \times K \times 0 \cdots 0 \not\subset V,
\]
where \( t = \max\{t_1, t_2, \ldots, t_r\} \).
(⇒) Suppose that \( \gamma_{i_1}d_1 + \gamma_{i_2}d_2 + \cdots + \gamma_{i_r}d_r = 0 \) for some nonzero vector \( \vec{d} = (d_1, d_2, \ldots, d_r) \), \( d_{j_i} = 0 \) or 1 for \( 1 \leq j_i \leq t_i \) for all \( 1 \leq i \leq r \), then \( e = (d_1, d_2, 0, \ldots, 0) \) is an idempotent of \( V \). Since \( V \) is a Mathieu-Zhao space of \( B \), we have that \( Be = Kd_1 \times \cdots \times Kd_r \times 0 \cdots \times 0 \subseteq V \), which is a contradiction. Then the conclusion follows.

(⇐) Let \( I = 0 \times \cdots \times 0 \times K \times \cdots \times K \). Then \( I \) is an ideal of \( B \). We claim that \( V/I \) contains no nonzero idempotent. Suppose that \( e \) is a nonzero idempotent of \( V/I \). Then we have \( e = (e_1, e_2, \ldots, e_t) \), where \( e_j = 0 \) or 1 for \( 1 \leq j \leq t \). Let \( \vec{d} = (d_1, \ldots, d_t) = e \neq (0, \ldots, 0) \). Then \( \gamma_{i_1}d_1 + \gamma_{i_2}d_2 + \cdots + \gamma_{i_r}d_r = 0 \) for all \( 1 \leq i \leq r \), which is a contradiction. It follows from Theorem 4.2 in [8] that \( V/I \) is a Mathieu-Zhao space of \( B/I \). Then it follows from Proposition 2.7 in [8] that \( V \) is a Mathieu-Zhao space of \( B \).

**Remark 4.2.** In Proposition 4.1, if \( \gamma_{i,j} = 0 \) for all \( 1 \leq i \leq r, 1 \leq j \leq n \), then \( V = B \). Clearly, \( V \) is a Mathieu-Zhao space of \( B \).

**Corollary 4.3.** Let \( L \) and \( G \) be as in Problem 1.2 and \( K \) a field of characteristic zero or a field of characteristic \( p \) and \( p \nmid |G| \). If \( K \) is a split field for \( G \) and \( G \) is Abelian, then \( \text{Ker} L \) is a Mathieu-Zhao space of \( K[G] \) if and only if \( \gamma_{i_1}d_1 + \gamma_{i_2}d_2 + \cdots + \gamma_{i_t}d_t \neq 0 \) for some \( i \in \{1, 2, \ldots, r\} \) for all nonzero vectors \( \vec{d} = (d_1, d_2, \ldots, d_t) \) and \( d_{j_i} \in \{0, 1\} \) for \( 1 \leq j_i \leq t_i, t_i \in \{1, \ldots, n\} \), where \( \gamma_{i,t} = \rho_j(a_i) \) for all \( 1 \leq i \leq r, 1 \leq j \leq n \) and \( \rho_j \) is an irreducible representation of \( G \) for \( 1 \leq j \leq n \) and \( a_i \) be as in Lemma 2.13 for \( 1 \leq i \leq r \).

**Proof.** Since \( G \) is Abelian, we have that all the irreducible representations of \( G \) are of degree one. It follows from Theorem 2.14 that \( \text{Ker} L \cong \{ (a_1, a_2, \ldots, a_n) \in A \mid \sum_{j=1}^{n} \gamma_{i,j}a_j = 0 \text{ for all } 1 \leq i \leq r \} \), where \( A \) is \( n \) times product of \( K \), \( \gamma_{i,j} = \rho_j(a_i) = \text{Tr} \rho_j(a_i) \in K \) for all \( 1 \leq i \leq r, 1 \leq j \leq n \). Since \( L \neq 0 \), we have that at least one of \( \gamma_{i,j} \) is nonzero for \( 1 \leq i \leq r, 1 \leq j \leq n \). Then the conclusion follows from Proposition 4.1.

**Lemma 4.4.** Let \( R \) be an integral domain of characteristic \( p \) and \( G \) a finite Abelian group with \( |G| = p^a, p \nmid d \). Then every idempotent of \( R[G] \) is also an idempotent of \( R[\hat{G}] \), where \( G = H \times \hat{G} \) and \( |H| = p^a \). In particular, the idempotent elements of \( R[G] \) are the same as the idempotent elements of \( R[\hat{G}] \).

**Proof.** Since \( G \) is a finite Abelian group, we have that \( G = H \times \hat{G} \) and \( |\hat{G}| = d \). Let \( e \) be an idempotent of \( R[G] \). Then \( e \) can be written as

\[
eq \sum_{h \in H} \alpha_h h
\]

with \( \alpha_h \in R[\hat{G}] \) for each \( h \in H \). Since \( |H| = p^a \), we have \( h^{q^m} = 1 \) for any \( m \geq 1, h \in H \), where \( q = p^a \). Thus, we have

\[
eq e^{q^m} = \sum_{h \in H} \alpha_h^{q^m} \in R[\hat{G}].
\]
Then the conclusion follows.

\textbf{Theorem 4.5.} Let \( L \) and \( G \) be as in Problem 1.2 and \( K \) a field of characteristic \( p \). If \( K \) is a split field for \( G \) and \( G \) is Abelian with \( |G| = p^d, p \nmid d \), then the following statements are equivalent:

1. \( \text{Ker} \, L \) is a Mathieu-Zhao space of \( K[G] \).
2. \( \gamma_{1,i}d_1 + \gamma_{2,d_2} + \cdots + \gamma_{i,t}d_i \neq 0 \) for some \( i \in \{1, 2, \ldots, r\} \) for all nonzero vectors \( \vec{d} = (d_1, d_2, \ldots, d_k) \) and \( d_{ji} \in \{0, 1\} \) for \( 1 \leq j_i \leq t_i, t_i \in \{1, \ldots, d\} \), where \( \gamma_{i,j} \) is the \( \rho_j(\alpha_i) = \text{Tr} \rho_j(\alpha_i) \) for \( 1 \leq i \leq r, 1 \leq j \leq d \) and \( \rho_j \) is an irreducible representation of \( G/H \) for \( 1 \leq j \leq d \). \( H \) is a Sylow \( p \)-subgroup of \( G \) and \( \alpha_i \) is as in Lemma 2.13 by replacing \( G \) with \( G/H \) for \( 1 \leq i \leq r \). \( \langle l_1, l_2, \ldots, l_n \rangle \) satisfy the following equations:

\[
\begin{align*}
\chi_j(\tilde{g}_1^{-1})l_{i,1} + \chi_j(\tilde{g}_2^{-1})l_{i,1} + \cdots + \chi_j(\tilde{g}_d^{-1})l_{i,(d-1)p^a+1} &= 0, \\
\chi_j(\tilde{g}_1^{-1})l_{i,2} + \chi_j(\tilde{g}_2^{-1})l_{i,2} + \cdots + \chi_j(\tilde{g}_d^{-1})l_{i,(d-1)p^a+2} &= 0, \\
&\vdots \\
\chi_j(\tilde{g}_1^{-1})l_{i,p^a} + \chi_j(\tilde{g}_2^{-1})l_{i,2p^a} + \cdots + \chi_j(\tilde{g}_d^{-1})l_{i,dp^a} &= 0
\end{align*}
\]

for all \( 1 \leq i \leq r \) and \( t+1 \leq j \leq d \), where \( \chi_j \) is the irreducible character according to \( \rho_j \) for \( t+1 \leq j \leq d \) and \( G = \bigcup_{k=1}^d \tilde{g}_k H \) with \( \tilde{g}_1 = 1 \) for \( H = \{h_1, h_2, \ldots, h_p\} \) with \( h_1 = 1 \) and \( L_i(h_k) = l_{i,k} \) and \( L_i(\tilde{g}_k h_q) = l_{i, (k-1)p^a + q} \) for all \( 1 \leq i \leq r, 1 \leq k \leq d, 1 \leq q \leq p^a \) and \( t = \max\{t_1, t_2, \ldots, t_r\} \).

\textbf{Proof.} Since \( G \) is Abelian, we have that \( G = H \times \tilde{G} \), where \( \tilde{G} \cong G/H \) and \( |\tilde{G}| = d \).

Note that

\[
\gamma_{i,j} = \text{Tr} \rho_j(\alpha_i) = \sum_{k=1}^d \text{Tr} \rho_j(\tilde{g}_k^{-1})\tilde{g}_k^{-1}l_{i,(k-1)p^a+1} = \sum_{k=1}^d \chi_j(\tilde{g}_k^{-1})\tilde{g}_k^{-1}l_{i,(k-1)p^a+1}
\]

for all \( 1 \leq i \leq r, 1 \leq j \leq d \). Let \( e_j = d^{-1}\sum_{k=1}^d \chi_j(\tilde{g}_k^{-1})\tilde{g}_k \) for \( 1 \leq j \leq d \). Then it follows from Theorem 2.12 in [3] that \( e_1, e_2, \ldots, e_d \) are the primitive orthogonal idempotents of \( K[\tilde{G}] \). Without loss of generality, we can assume that \( \gamma_{1,1} = 0 \) for all \( 1 \leq i \leq r, t+1 \leq j \leq d \) and \( \gamma_{1,1} \neq 0 \) for all \( 1 \leq j \leq t \) for some \( i \in \{1, 2, \ldots, r\} \) by reordering \( \rho_j(\alpha_i) \) for all \( 1 \leq i \leq r, 1 \leq j \leq d \).

(1) \implies (2) It’s easy to see that if \( \gamma_{1,1} = 0 \) for all \( 1 \leq i \leq r, t+1 \leq j \leq d \), then \( e_{1}, \ldots, e_d \) belong to \( \text{Ker} (L_{\tilde{G}}) \subseteq \text{Ker} L \). Thus, the ideal \( I \) generated by \( e_{1}, \ldots, e_d \) belongs to \( \text{Ker} L \). Since \( \tilde{G} \) is Abelian, it is easy to check that \( e_j \tilde{g}_k = \chi_j(\tilde{g}_k) e_j \) for all \( 1 \leq k, j \leq d \). Hence we have \( e_j \tilde{g}_k \in \text{Ker} L \) for all \( t+1 \leq j \leq d, 1 \leq k \leq d \). Note that \( e_j h_q \in \text{Ker} L \) for all \( t+1 \leq j \leq d, 1 \leq q \leq p^a \). Then we have equations (4.1) for all \( 1 \leq i \leq r, t+1 \leq j \leq d, 1 \leq q \leq p^a \). It follows from Proposition 2.3 that \( \text{Ker} (L_{\tilde{G}}) \) is a Mathieu-Zhao space of \( K[\tilde{G}] \). This is, \( \text{Ker} (L_{\tilde{G}}) \) is a Mathieu-Zhao space of \( K[\tilde{G}] \). Since \( p \nmid |G/H| \), the conclusion follows from Corollary 4.3.
(2) \( \Rightarrow \) (1) If \( \gamma_{i,j} = 0 \) for all \( 1 \leq i \leq r, \ t + 1 \leq j \leq d \), then \( e_{t+1}, \ldots, e_d \in \text{Ker}(L_1G) \subseteq \text{Ker}L \). It is easy to check that 
\[ e_j \tilde{g}_k = \chi_j(\tilde{g}_k)e_j \quad \text{and} \quad e_j \tilde{g}_k \tilde{h}_q = \chi_j(\tilde{g}_k)e_j \tilde{h}_q \quad \text{for all} \quad t + 1 \leq j \leq d, \ 1 \leq k \leq d, \ 1 \leq q \leq p^t. \]
Therefore, we have 
\[ I \subseteq \text{Ker}L, \quad \text{where} \quad I \text{ is an ideal generated by} \quad e_{t+1}, \ldots, e_d. \]
Since \( e_1, \ldots, e_d \) are the primitive orthogonal idempotent elements of \( K[G] \) and there are \( 2^d \) idempotent elements in \( K[G] \), we have that any idempotent of \( K[G] \) is a sum of some of the \( e_j \) for \( 1 \leq j \leq d \). Note that the condition that 
\[ \gamma_{i,1}d_1 + \gamma_{i,2}d_2 + \cdots + \gamma_{i,t}d_t, \neq 0 \]
for some \( i \in \{1, 2, \ldots, r\} \) for all nonzero vectors \( \tilde{d} = (d_1, d_2, \ldots, d_t) \) and \( d_j, \in \{0,1\} \) is equivalent to that any sum of some of the \( e_j \) is not in \( \text{Ker}(L_1G) \) except zero for all \( 1 \leq j \leq t \). Hence any sum of some of the \( e_j \) is not in \( \text{Ker}(L_1G) \) for all \( 1 \leq j \leq d \) if it contains \( e_{j_0} \) for some \( j_0 \in \{1, 2, \ldots, t\} \). Thus, any sum of some of the \( e_j \) is not in \( \text{Ker}L \) for all \( 1 \leq j \leq d \) if it contains \( e_{j_0} \) for some \( j_0 \in \{1, 2, \ldots, t\} \). Otherwise, the sum of \( e_j \) belong to \( \text{Ker}L \cap K[G] = \text{Ker}(L_1G) \) for \( 1 \leq j \leq d \), which is a contradiction. It follows from Lemma 4.4 that 
\[ K[G] \text{ and } K[G_1] \text{ have the same idempotents}. \]
Hence \( \text{Ker}L/I \) has no nonzero idempotents. It follows from Theorem 4.2 in [8] that \( \text{Ker}L/I \) is a Mathieu-Zhao space of \( K[G]/I \). Hence it follows from Proposition 2.7 in [8] that \( \text{Ker}L \) is a Mathieu-Zhao space of \( K[G] \).

\bigskip

Remark 4.6. If \( G \) is cyclic in Theorem 4.5, then all the primitive orthogonal idempotent elements of \( K[G] \) are 
\[ e_j = d^{-1}(1 + (\xi^{d-1})^{-1}\tilde{g}^{d-1} + \cdots + \xi^{-1}\tilde{g}^{d-1}) \]
for \( 1 \leq j \leq d \), where \( \xi \) is a \( d \)-th root of unity and \( G \) is generated by \( \tilde{g} \), where \( G \) be as in Theorem 4.5.

5. The kernels of the class functions of finite group algebras

Condition 2: Let \( L \) and \( G \) be as in Problem 1.2 and \( K \) a field of characteristic zero or a field of characteristic \( p, \ p \nmid |G| \). Then \( L_2, \ldots, L_r \) are class functions of \( G \) and \( K \) is a split field for \( G \).

Proposition 5.1. Let \( L, G, K \) be as in Condition 2 and \( L_1 \) is a class functions of \( G \). Then the following statements are equivalent:

(1) \( \text{Ker}L \) is a Mathieu-Zhao space of \( K[G] \).

(2) \[ a_{i,1}d_1 + a_{i,2}d_2 + \cdots + a_{i,t}d_t, \neq 0 \]
for some \( i \in \{1, 2, \ldots, r\} \) for all nonzero vectors \( \tilde{d} = (d_1, d_2, \ldots, d_t) \) and \( d_j, \in \{0,1, \ldots, n_j\} \) for \( 1 \leq j \leq t \), \( t \in \{1, \ldots, s\} \), where \( L_i = \sum_{j=1}^{s} a_{i,j}\chi_j \) and \( \chi_1, \ldots, \chi_s \) are the distinct (up to isomorphism) irreducible characters of \( G \) and \( n_j = \chi_j(1) \), \( a_{i,j} \in K \) for all \( 1 \leq i \leq r, \ 1 \leq j \leq s \).

Proof. Since \( L_1, \ldots, L_r \) are class functions of \( G \), we have 
\[ L_i = \sum_{j=1}^{s} a_{i,j}\chi_j, \]
where \( a_{i,j} \in K \) for all \( 1 \leq i \leq r, \ 1 \leq j \leq s \). Hence we have 
\[ \text{Ker}L = \{ \beta \in K[G] | \sum_{j=1}^{s} a_{i,j}\chi_j(\beta) = 0 \text{ for all } 1 \leq i \leq r \}. \]
Since $K'[G]$ is semi-simple and $K$ is a split field for $G$, $K'[G]$ can be written as the product of matrices over $K$. That is, $K'[G] \cong M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_s}(K) := A$. It’s easy to see that there exists $A_j \in M_{n_j}(K)$ such that $\text{Tr } A_j = \chi_j(\beta)$ for $1 \leq j \leq s$. Then we have

$$\text{Ker } L = \{(A_1, \ldots, A_s) \in A \mid \sum_{j=1}^s a_{i,j} \text{Tr } A_j = 0 \text{ for all } 1 \leq i \leq r\}.$$ 

We can assume that $a_{i,j} \neq 0$ for all $1 \leq j \leq t$ for some $i \in \{1, 2, \ldots, r\}$ and $a_{i,j} = 0$ for all $1 \leq i \leq r$, $t+1 \leq j \leq s$ by reordering $\chi_j$ for $1 \leq j \leq s$. Then we have $0 \times \cdots \times 0 \times M_{n_{t+1}}(K) \times \cdots \times M_{n_s}(K) \subseteq \text{Ker } L$ and

$$0 \times \cdots \times M_{n_k}(K) \times 0 \times \cdots \times 0 \times \cdots \times 0 \not\subseteq \text{Ker } L,$$

where $t = \max\{t_1, t_2, \ldots, t_r\}$.

(1) $\Rightarrow$ (2) Suppose that $a_{1,1}d_1 + a_{1,2}d_2 + \cdots + a_{1,t}d_t = 0$ for some nonzero vectors $\vec{d} = (d_1, d_2, \ldots, d_{t_i})$ and $d_{j_i} \in \{0, 1, \ldots, n_{j_i}\}$ for $1 \leq j_i \leq t_i$ for all $1 \leq i \leq r$. Then $e = (A_1, \ldots, A_t, 0, \ldots, 0)$ is an idempotent of $\text{Ker } L$, where

$$A_k = \begin{pmatrix} I_{d_k} & 0 \\ 0 & 0 \end{pmatrix}$$

and $\text{Tr } A_k = d_k$ for all $1 \leq k \leq t$. Since $\text{Ker } L$ is a Mathieu-Zhao space of $K[G]$, we have $K'[G] \subseteq K'[G] \subseteq \text{Ker } L$. That is, $(M_{n_1}(K)A_1M_{n_1}(K), \ldots, M_{n_s}(K)A_sM_{n_s}(K), 0, \ldots, 0) \subseteq \text{Ker } L$. Since $M_{n_k}(K)A_kM_{n_k}(K)$ is a submodule of $M_{n_k}(K)$ and $M_{n_k}(K)$ is simple, we have $M_{n_k}(K)A_kM_{n_k}(K) = 0$ or $M_{n_k}(K)$. Without loss of generality, we can assume that $A_1 \neq 0$. Then we have $M_{n_1}(K)A_1M_{n_1}(K) = M_{n_1}(K)$. That is, $M_{n_1}(K) \times 0 \times \cdots \times 0 \subseteq \text{Ker } L$, which is a contradiction. Then the conclusion follows.

(2) $\Rightarrow$ (1) Let $I = 0 \times \cdots \times 0 \times M_{n_{t+1}}(K) \times \cdots \times M_{n_s}(K)$. Then $I$ is an ideal of $A$. We claim that $\text{Ker } L/I$ has no nonzero idempotent. Suppose that $e$ is a nonzero idempotent of $\text{Ker } L/I$. Then we have $e = (A_1, \ldots, A_t, 0, \ldots, 0)$ and $\bar{A}_k$ is similar to $A_k$ for all $1 \leq k \leq t$, where $\bar{A}_k$ is defined as above. Thus, we have $\text{Tr } \bar{A}_k \in \{0, 1, \ldots, n_{j_i}\}$ for all $1 \leq k \leq t$ and at least one of $\text{Tr } \bar{A}_k$ is nonzero for $1 \leq k \leq t$. Let $\vec{d} = (d_1, d_2, \ldots, d_{t_i}) = (\text{Tr } \bar{A}_1, \text{Tr } \bar{A}_2, \ldots, \text{Tr } \bar{A}_t) \neq (0, \ldots, 0)$. Then $a_{1,1}d_1 + a_{1,2}d_2 + \cdots + a_{1,t}d_t = 0$ for all $1 \leq i \leq r$, which is a contradiction. Hence the claim follows. It follows from Theorem 4.2 in [8] that $\text{Ker } L/I$ is a Mathieu-Zhao space of $A/I$. Then it follows from Proposition 2.7 in [8] that $\text{Ker } L$ is a Mathieu-Zhao space of $K[G]$. \qed

**Corollary 5.2.** Let $K$ be a field and $V = \{(A_1, \ldots, A_s) \in A \mid \sum_{j=1}^s a_{i,j} \text{Tr } A_j = 0 \text{ for all } 1 \leq i \leq r\}$, where $A = M_{n_1}(K) \times \cdots \times M_{n_s}(K)$. Then $V$ is a Mathieu-Zhao space of $A$ if and only if $a_{1,1}d_1 + a_{1,2}d_2 + \cdots + a_{1,t_i}d_{t_i} \neq 0$ for some $i \in \{1, 2, \ldots, r\}$ for all nonzero vectors $\vec{d} = (d_1, d_2, \ldots, d_{t_i})$ and $d_{j_i} \in \{0, 1, \ldots, n_{j_i}\}$ for $1 \leq j_i \leq t_i$, $t_i \in \{1, \ldots, s\}$. 


Proof. The conclusion follows from the proof of Proposition 5.1.

**Theorem 5.3.** Let $L$ and $G$ be as in Problem 1.2 and $K$ a field of characteristic $p$. If $G$ has a normal Sylow $p$-subgroup $H$ and $L_1, \ldots, L_r$ are class functions of $G/H$ and $K$ is a split field for $G/H$, then the following statements are equivalent:

1. $\ker L$ is a Mathieu-Zhao space of $K[G]$.
2. $a_{i,1}d_1 + a_{i,2}d_2 + \cdots + a_{i,r}d_r \neq 0$ for some $i \in \{1,2,\ldots,r\}$ for all nonzero vectors $d = (d_1, d_2, \ldots, d_r)$ and $d_j \in \{0,1,\ldots,n_j\}$ for $1 \leq j \leq t$, $t_i \in \{1,\ldots,s\}$, where $L_i = \sum_{j=1}^s a_{i,j} \chi_j$ and $\chi_1, \ldots, \chi_s$ are the distinct (up to isomorphism) irreducible characters of $G/H$ and $n_j = \chi_j(1)$, $a_{i,j} \in K$ for all $1 \leq i \leq r, 1 \leq j \leq s$.

Proof. Let $|G| = p^d$, $p \nmid d$ and $G = \bigcup_{j=1}^r g_j H$, $H = \{h_1, h_2, \ldots, h_t\}$ with $i = p^d$. Then we have $L_i(g_j)h_1 = L_i(g_j)h_2 = \cdots = L_i(g_j)h_t$ for all $1 \leq i \leq r, 1 \leq j \leq d$. Hence the conclusion follows from Proposition 3.1 and Proposition 5.1.

**Remark 5.4.** It’s easy to see that $\ker L = V_G$ if $L = n_1 \chi_1 + n_2 \chi_2 + \cdots + n_s \chi_s$ and $\chi_1, \ldots, \chi_s$ are the distinct (up to isomorphism) irreducible characters of $K[G]$. If $G$ has a normal Sylow $p$-subgroup $H$, then Theorem 5.3 implies Theorem 1.5 in [10].

**Proposition 5.5.** Let $L$, $G$, $K$ be as in Condition 2. Then the following two statements are equivalent:

1. $\ker L$ is a Mathieu-Zhao space of $K[G]$.
2. For all $0 \neq b = (b_1, \ldots, b_s) \in \{0,1,\ldots,n_1\} \times \cdots \times \{0,1,\ldots,n_s\}$ with $a_{i,1}b_1 + a_{i,2}b_2 + \cdots + a_{i,s}b_s = 0$ for all $1 \leq i \leq r - 1$, the following are true:
   (a) there exists a $\lambda_m \in K$ such that $C_m = \lambda_m l_{m,n}$, for all $m \in T_b$,
   (b) $\sum_{m \in T_b} n_m \lambda_m l_{m,n} + \sum_{m \in S_b} n_m Tr(C_m) \neq 0$,
where $L_i = \sum_{j=1}^s a_{i,j} \chi_j$, $\chi_1, \ldots, \chi_s$ are the distinct (up to isomorphism) irreducible characters of $K[G]$ and $n_j = \chi_j(1)$, $C_j = p_j(a_r)$, $\alpha_r = \sum_{j=1}^n l_{r,j} \chi_j^{-1}$, $G = \{g_1, \ldots, g_n\}$, $\rho_j$ is an irreducible representation according to $\chi_j$, $a_{i,j} \in K^*$ for all $1 \leq i \leq r - 1, 1 \leq j \leq s$ and $T_b := \{1 \leq m \leq s | b_m \neq 0, n_m\}$, $S_b := \{1 \leq m \leq s | b_m = n_m\}$.

Proof. Since $L_1, \ldots, L_{r-1}$ are class functions of $G$, we have

$L_i = \sum_{j=1}^r a_{i,j} \chi_j$

for all $1 \leq i \leq r - 1$. Since $a_{i,j} \in K^*$ for all $1 \leq i \leq r - 1, 1 \leq j \leq s$, we have

$\ker L = \{\beta \in K[G] \mid \sum_{j=1}^s a_{i,j} \chi_j(\beta) = 0 \text{ and } L_r(\beta) = 0 \text{ for all } 1 \leq i \leq r - 1\}$. 
Since $K[G]$ is semi-simple, we have $K[G] \cong M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_s}(K) := A$. It’s easy to see that there exists $A_j \in M_{n_i}(K)$ such that $\text{Tr} A_j = \chi_j(\beta)$ for all $1 \leq j \leq s$. It follows from Lemma 2.13 and Theorem 2.14 that $L_r(\beta) = 0$ if and only if

$$\sum_{j=1}^s n_j \text{Tr}(C_j A_j) = 0,$$

where $A_j = \rho_j(\beta)$ and $C_j = \rho_j(\alpha_r)$, $\alpha_r = \sum_{j=1}^n \ell_{r,j} g_j^{-1}$ for all $1 \leq j \leq s$.

(2) $\Rightarrow$ (1) Since $\text{Ker} L \cong V$ and

$$V = \{(A_1, \ldots, A_s) \in A | \sum_{j=1}^s a_{i,j} \text{Tr} A_j = 0 \text{ and } \sum_{j=1}^s n_j \text{Tr}(C_j A_j) = 0 \text{ for all } 1 \leq i \leq r-1\}$$

and for all $0 \neq b = (b_1, b_2, \ldots, b_s) \in \{0,1,\ldots,n_1\} \times \cdots \times \{0,1,\ldots,n_s\}$ with $a_{i,1} b_1 + a_{i,2} b_2 + \cdots + a_{i,s} b_s = 0$ for all $1 \leq i \leq r-1$, we have that:

(a) there exists a $\lambda_m \in K$ such that $C_m = \lambda_m I_{n_m}$ for all $m \in T_b$,

(b) $\sum_{m \in T_b} n_m \lambda_m b_m + \sum_{m \in S_b} n_m \text{Tr}(C_m) \neq 0$.

Now suppose that $V$ contains a nonzero idempotent $(E_1, \ldots, E_s)$ and $b_j = \text{Tr}(E_j)$ for $1 \leq j \leq s$. Then we have that $a_{i,1} b_1 + \cdots + a_{i,s} b_s = 0$ for all $1 \leq i \leq r-1$ and (a), (b) hold. Hence we have

$$\sum_{m \in T_b} n_m \lambda_m b_m + \sum_{m \in S_b} n_m \text{Tr}(C_m) \neq 0,$$

which contradicts with $(E_1, \ldots, E_s) \in V$. Thus, $V$ does not contain any nonzero idempotent and hence is Mathieu-Zhao space of $K[G]$. Then the conclusion follows.

(1) $\Rightarrow$ (2) Suppose that $\text{Ker} L$ is a Mathieu-Zhao space of $K[G]$ and there exists a $0 \neq b = (b_1, \ldots, b_s) \in \{0,1,\ldots,n_1\} \times \cdots \times \{0,1,\ldots,n_s\}$ with $a_{i,1} b_1 + \cdots + a_{i,s} b_s = 0$ for all $1 \leq i \leq r-1$ such that (a) does not hold. Then there is an $m \in T_b$ such that $C_m$ is not a multiple of the identity matrix. Let $E_j$ be the matrix with ones on the first $b_j$ diagonal entries and zeros on all other entries for all $1 \leq j \leq s$ with $j \neq m$. Then $E_j$ is an idempotent of rank $b_j$. It follows from Lemma 4.6 in [4] that there exists an idempotent $E_m$ of rank $b_m \neq 0, n_m$ such that

$$\text{Tr}(C_m E_m) = -\frac{1}{n_m} \sum_{j \neq m} n_j \text{Tr}(C_j E_j).$$

Since $\text{Tr} E_j = \text{rank} E_j$ for all $1 \leq j \leq s$, we have that $(E_1, E_2, \ldots, E_s)$ is a nonzero idempotent which contained in $V$. This contradicts with that $V$ is a Mathieu-Zhao space of $A$.

Suppose that there exists a $0 \neq b = (b_1, \ldots, b_s) \in \{0,1,\ldots,n_1\} \times \cdots \times \{0,1,\ldots,n_s\}$ with $a_{i,1} b_1 + \cdots + a_{i,s} b_s = 0$ for all $1 \leq i \leq r-1$ such that (1) does hold but (2) does not hold. Let $E_j$ be the matrix with ones on the
first \( b_j \) diagonal entries and zero on all other entries. Then \( E_j \) is an idempotent of rank \( b_j \). Since \( \text{Tr} \ E_j = \text{rank} \ E_j \) for all \( 1 \leq j \leq s \), we have

\[
\sum_{m \in T_b} n_m \lambda_m b_m + \sum_{m \in S_b} n_m \text{Tr}(C_m) = 0,
\]

which exactly means that \((E_1, \ldots, E_s)\) is contained in \( V \). As \( b \neq 0 \), we have that \( V \) contains a nonzero idempotent, which contradicts with that \( V \) is a Mathieu-Zhao space of \( A \). Then the conclusion follows. \( \Box \)

We can remove the condition that \( a_{i,j} \in K^* \) for all \( 1 \leq i \leq r - 1, 1 \leq j \leq s \) in Proposition 5.5 by introducing a new set \( X := \{a_{i,j} | \text{there exists } i_j \in \{1, 2, \ldots, r - 1\} \text{ such that } a_{i,j} \neq 0 \text{ for } 1 \leq i \leq r - 1, 1 \leq j \leq s \} \). Then we have the following theorem.

**Theorem 5.6.** Let \( L, G, K \) be as in Condition 2. Then the following two statements are equivalent:

1. \( \text{Ker} \ L \) is a Mathieu-Zhao space of \( K[G] \).
2. For all \( 0 \neq b = (b_{k_1}, b_{k_2}, \ldots, b_{k_t}) \in \{0, 1, \ldots, n_k_1\} \times \cdots \times \{0, 1, \ldots, n_k_t\} \) with \( a_{i,k_1} b_{k_1} + a_{i,k_2} b_{k_2} + \cdots + a_{i,k_t} b_{k_t} = 0 \) for all \( 1 \leq i \leq r - 1, 1 \leq j \leq s \), the following two statements are true:
   
   (a) there exists a \( \lambda_m \in K \) such that \( C_m = \lambda_m I_{m_m} \) for all \( m \in T_b \cap X \),
   
   (b) \( \sum_{m \in T_b \cap X} n_m \lambda_m b_m + \sum_{m \in S_b \cap X} n_m \text{Tr}(C_m) \neq 0 \),

where \( L_i = \sum_{j=1}^s a_{i,j} \chi_j \), \( \chi_1, \ldots, \chi_s \) are the distinct (up to isomorphism) irreducible characters of \( K[G] \), \( a_{i,j} \in K \) and \( n_j = \chi_j(1) \), \( C_j = \rho_j(\alpha_r) \), \( \alpha_r = \sum_{j=1}^s \rho_j \chi_j^{-1} \cdot G = \{g_1, \ldots, g_n\} \), \( \rho_j \) is an irreducible representation according to \( \chi_j \) for all \( 1 \leq i \leq r - 1, 1 \leq j \leq s \) and \( T_b := \{1 \leq m \leq s | b_m \neq 0, n_m \} \), \( S_b := \{1 \leq m \leq s | b_m = n_m \} \), \( X = \{a_{i,j} | \text{there exists } i_j \in \{1, 2, \ldots, r - 1\} \text{ such that } a_{i,j} \neq 0 \text{ for } 1 \leq i \leq r - 1, 1 \leq j \leq s \} = \{a_{i,k_1}, \ldots, a_{i,k_t} \text{ for } 1 \leq i \leq r - 1\} \).

**Proof.** The conclusion follows by following the arguments of Proposition 5.5. \( \Box \)

**Proposition 5.7.** Let \( L \) and \( G \) be as in Problem 1.2 and \( K \) a field of characteristic \( p \), \( |G| = p^d \), \( H \) a normal Sylow \( p \)-subgroup of \( G \) and \( G = \cup_{j=1}^d \tilde{g}_jH \), \( H = \{1_H, h_2, \ldots, h_t\} \) for \( \tilde{t} = p^u \). Suppose that \( L_0(\tilde{g}_j h_2) = L_0(\tilde{g}_j h_3) = \cdots = L_0(\tilde{g}_j h_t) \) for all \( 1 \leq j \leq d \) and \( K \) is a split field for \( G/H \). If there exist \( i, i \in \{1, 2, \ldots, r \} \) such that \( \det M_{L_i|G/H} \neq 0 \) and \( \det M_{L_i} \neq 0 \) and \( L_1, \ldots, L_{r-1} \) are class functions of \( G/H \), then the following two statements are equivalent:

1. \( \text{Ker} \ L \) is a Mathieu-Zhao space of \( K[G] \).
2. For all \( 0 \neq b = (b_{k_1}, b_{k_2}, \ldots, b_{k_t}) \in \{0, 1, \ldots, n_k_1\} \times \cdots \times \{0, 1, \ldots, n_k_t\} \) with \( a_{i,k_1} b_{k_1} + a_{i,k_2} b_{k_2} + \cdots + a_{i,k_t} b_{k_t} = 0 \) for all \( 1 \leq i \leq r - 1 \), the following two statements are true:
   
   (a) there exists a \( \lambda_m \in K \) such that \( C_m = \lambda_m I_{m_m} \) for all \( m \in T_b \cap X \),
(b) \[ \sum_{m \in T \cap X} n_m \lambda_m b_m + \sum_{m \in S \cap X} n_m \text{Tr}(C_m) \neq 0, \]
where \( L_i = \sum_{j=1}^s a_{i,j} \chi_j, \chi_1, \ldots, \chi_s \) are the distinct (up to isomorphism) irreducible characters of \( K[G] \), \( a_{i,j} \in K \) and \( n_j = \chi_j(1) \), \( C_j = \rho_j(a_r) \), \( a_r = \sum_{j=1}^d b_{r,j} g_j^{-1} \), \( G/H = \{ g_1, \ldots, g_d \} \), \( \rho_j \) is an irreducible representation according to \( \chi_j \) for all \( 1 \leq i \leq r-1 \), \( 1 \leq j \leq s \) and \( M_{r,j}|G/H \), \( M_{L_i} \) be as in Proposition 3.4; \( T_b, S_b, X \) be as in Theorem 5.6.

Proof. Since \( L_1, \ldots, L_{r-1} \) are class functions of \( G/H \), we have \( L_i(\tilde{g}_1 h_1) = L_i(\tilde{g}_1 h_2) = \cdots = L_i(\tilde{g}_1 h_\tilde{t}) \) for all \( 1 \leq i \leq r-1 \). Then it follows from Proposition 3.4 that \( \text{Ker} L \) is a Mathieu-Zhao space of \( K[G] \) if and only if \( \text{Ker}(L|G/H) \) is a Mathieu-Zhao space of \( K[G/H] \). Since \( p \nmid |G/H| \), the conclusion follows from Theorem 5.6. \( \square \)

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