SKEW BRACE ENHANCEMENTS AND VIRTUAL LINKS

Melody Chang and Sam Nelson

ABSTRACT. We use the structure of skew braces to enhance the biquandle counting invariant for virtual knots and links for finite biquandles defined from skew braces. We introduce two new invariants: a single-variable polynomial using skew brace ideals and a two-variable polynomial using the skew brace group structures. We provide examples to show that the new invariants are not determined by the counting invariant and hence are proper enhancements.

1. Introduction

Skew braces are a type of algebraic structure consisting of a set with two group operations (analogous to a ring) which interact via a kind of modified distributive law. They have been studied for the last decade or so in papers such as [1, 4–6]. Beyond their inherent algebraic interest, skew braces are important in knot theory because they provide solutions to the set-theoretic Yang-Baxter equation, which in turn lead to invariants of knots.

Set-theoretic Yang-Baxter solutions are also provided by *biquandles*, which have been studied for about the last two decades; see [3] and the references therein for more. Biquandle-based knot invariants include many examples of *enhancements*, invariants which specialize to the integer-valued biquandle counting invariant (i.e., the cardinality of the set of biquandle homomorphisms from the fundamental biquandle of the knot or link to a fixed finite biquandle) but which contain more information beyond merely the set's cardinality.

A skew brace defines a biquandle, and thus the notion of biquandle coloring extends to a notion of skew brace coloring, allowing for skew brace counting invariants and enhancements thereof. Like many biquandle-based invariants of classical knots and links, these skew brace invariants extend to the setting of virtual knot theory in a natural way by simply ignoring the virtual crossings,

i.e., letting skew brace colors remain constant when passing through virtual crossings.

In this paper we define two infinite families of new polynomial enhancements of the biquandle counting invariant for biquandles which come from skew braces. The paper is organized as follows. In Section 2 we collect a few definitions and observations about skew braces and their relationship with biquandles. We recall the biquandle counting invariant and define the skew brace counting invariant for finite biquandles and skew braces. In Section 3 we introduce the new enhanced invariants: the two-variable *skew brace enhanced polynomial* and the one-variable *skew brace ideal polynomial*. In Section 4 we provide examples to illustrate the computation of the invariant and to establish that the new invariants are proper enhancements, i.e., that they are not determined by the counting invariant. In Section 5 we end with some questions for future research.

2. Skew braces and skew brace colorings

We begin with a definition; see [1, 4–6] for more.

Definition 1. A *skew brace* is a set X with two group operations which we will denote by $(x, y) \mapsto x \circ y$ and $(x, y) \mapsto x * y$ with inverses denoted by x° and x^*, satisfying a modified distributive law

$$x \circ (y * z) = (x \circ y) * x^* * (x \circ z).$$

To familiarize ourselves a little with the algebra of skew braces, let us note the standard fact (see [1] for example) that this modified distributive law has the useful consequence that the two group identities are equal.

Lemma 1. *Let X be a skew brace. Then the identity elements with respect to both operations are the same.*

Proof. Let us temporarily denote the $*$-identity element by e_* and the $\circ$-identity element by $e_\circ$. Then we observe that for any $x, y \in X$ we have

$$x \circ y = x \circ (e_* * y) = (x \circ e_*) * x^* * (x \circ y)$$

and thus

$$(x \circ y) * (x \circ y)^* = (x \circ e_*) * x^* * (x \circ y) * (x \circ y)^*$$

so we have

$$e_* = (x \circ e_*) * x^*$$

which implies

$$x = x \circ e_*.$$

Then

$$x^\circ \circ x = x^\circ \circ x \circ e_*$$

which says $e_\circ = e_*$ as required.   □

For simplicity, we will denote the common identity element of both operations as e.

Let X be a skew brace and K an oriented classical or virtual knot or link diagram. A *skew brace coloring* of K, also called an *X-coloring of K*, is an assignment of elements in X to each semiarc in K such that at every classical crossing we have one of the following pictures:
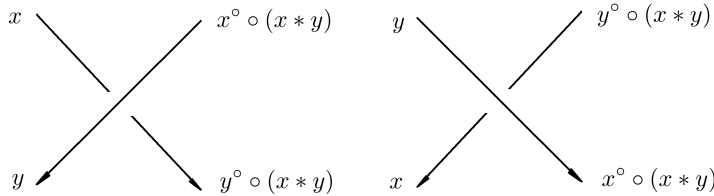
At virtual crossings, the colors on the semiarcs are not changed.

Next we recall a definition found in [3]:

Definition 2. A *biquandle* is a set X with binary operations $\underline{\triangleright}, \overline{\triangleright}$ satisfying

(i) For all $x \in X$,
$$x\underline{\triangleright}x = x\overline{\triangleright}x,$$

(ii) For all $x, y \in X$, the operations $\underline{\triangleright}$ and $\overline{\triangleright}$ are right-invertible and the map of pairs $S(x, y) = (y\overline{\triangleright}x, x\underline{\triangleright}y)$ is invertible, and

(iii) For all $x, y, z \in X$ the *exchange laws*
$$(x\underline{\triangleright}y)\underline{\triangleright}(z\underline{\triangleright}y) = (x\underline{\triangleright}z)\underline{\triangleright}(y\overline{\triangleright}z),$$
$$(x\underline{\triangleright}y)\overline{\triangleright}(z\underline{\triangleright}y) = (x\overline{\triangleright}z)\underline{\triangleright}(y\overline{\triangleright}z),$$
$$(x\overline{\triangleright}y)\overline{\triangleright}(z\overline{\triangleright}y) = (x\overline{\triangleright}z)\overline{\triangleright}(y\underline{\triangleright}z)$$

are satisfied.

As noted in [4], skew brace colorings provide solutions to the set-theoretic Yang Baxter equation. In fact, we have:

Theorem 2 (Guarnieri and Vendramin)**.** *A skew brace is a biquandle under the operations*
$$x\underline{\triangleright}y = y^{\circ} \circ (x * y) \quad and \quad x\overline{\triangleright}y = y^{\circ} \circ (y * x).$$

The result has been previously established in Corollary 3.3 of [7] using different notation; let us see how it works here using our notation.

Proof. We must verify that the biquandle axioms are satisfied. Checking, we have
$$x\underline{\triangleright}x = x^{\circ} \circ (x * x) = x\overline{\triangleright}x$$
as required, and axiom (i) is satisfied.

For axiom (ii), we note that the operations
$$x\underline{\triangleright}^{-1}y = (y \circ x) * y^{*} \quad and \quad x\overline{\triangleright}^{-1}y = y^{*} * (y \circ x)$$

are right-inverses for $\underline{\rhd}$ and $\overline{\rhd}$:

$$(x\underline{\rhd}^{-1}y)\underline{\rhd}y = y^\circ \circ ((y \circ x) * y^* * y) = x,$$
$$(x\underline{\rhd}y)\underline{\rhd}^{-1}y = (y \circ y^\circ \circ (x * y)) * y^* = x,$$
$$(x\overline{\rhd}^{-1}y)\overline{\rhd}y = y^\circ \circ (y * (y^* * (y \circ x))) = x \quad \text{and}$$
$$(x\overline{\rhd}y)\overline{\rhd}^{-1}y = y^* * (y \circ (y^\circ \circ (y * x))) = x$$

and that the map $S^{-1} : X \times X \to X \times X$ given by

$$S^{-1}(x, y) = (((x \circ y^\circ) * x^*)^\circ, ((x \circ y^\circ) * x^*)^\circ \circ x \circ y^\circ)$$

is the inverse of the map $S(x, y) = (x^\circ \circ (x * y), y^\circ \circ (x * y)) = (y\overline{\rhd}x, x\underline{\rhd}y)$.

Now, let us consider the exchange laws. We compute

$$
\begin{aligned}
(x\underline{\rhd}y)\underline{\rhd}(z\underline{\rhd}y) &= [y^\circ \circ (x * y)]\underline{\rhd}[y^\circ \circ (z * y)] \\
&= [y^\circ \circ (z * y)]^\circ \circ [(y^\circ \circ (x * y)) * (y^\circ \circ (z * y))] \\
&= [y^\circ \circ (z * y)]^\circ \circ [y^\circ \circ (x * y)] * [y^\circ \circ (z * y)]^{\circ *} \\
&\quad * [(y^\circ \circ (z * y))^\circ \circ (y^\circ \circ (z * y))] \\
&= [y^\circ \circ (z * y)]^\circ \circ [y^\circ \circ (x * y)] * [y^\circ \circ (z * y)]^{\circ *} \\
&= [y^\circ \circ (z * y)]^\circ \circ [y^\circ \circ (x * y)] * [(z * y)^\circ \circ y]^* \\
&= [(z * y)^\circ \circ y \circ y^\circ \circ (x * y)] * [(z * y)^\circ \circ y]^* \\
&= [(z * y)^\circ \circ (x * y)] * [(z * y)^\circ \circ y]^* \\
&= ((z * y)^\circ \circ x) * (z * y)^{\circ *} * [(z * y)^\circ \circ y] * [(z * y)^\circ \circ y]^* \\
&= ((z * y)^\circ \circ x) * (z * y)^{\circ *}
\end{aligned}
$$

and

$$
\begin{aligned}
(x\underline{\rhd}z)\underline{\rhd}(y\overline{\rhd}z) &= [z^\circ \circ (x * z)]\underline{\rhd}[z^\circ \circ (z * y)] \\
&= [z^\circ \circ (z * y)]^\circ \circ [(z^\circ \circ (x * z)) * (z^\circ \circ (z * y))] \\
&= [(z^\circ \circ (z * y))^\circ \circ (z^\circ \circ (x * z))] * [z^\circ \circ (z * y)]^{\circ *} \\
&\quad * [(z^\circ \circ (z * y))^\circ \circ (z^\circ \circ (z * y))] \\
&= [(z^\circ \circ (z * y))^\circ \circ (z^\circ \circ (x * z))] * [z^\circ \circ (z * y)]^{\circ *} \\
&= [(z * y)^\circ \circ z \circ z^\circ \circ (x * z)] * [z^\circ \circ (z * y)]^{\circ *} \\
&= [(z * y)^\circ \circ (x * z)] * [(z * y)^\circ \circ z]^* \\
&= ((z * y)^\circ \circ x) * (z * y)^{\circ *} * [(z * y)^\circ \circ z] * [(z * y)^\circ \circ z]^* \\
&= ((z * y)^\circ \circ x) * (z * y)^{\circ *}
\end{aligned}
$$

and the first exchange law is satisfied. The other two are similar and left to the reader. $\qquad\square$

A skew brace coloring is a biquandle coloring by the associated biquandle of the skew brace. Hence, it follows that the number of skew brace colorings of an oriented classical or virtual knot or link diagram K by a finite skew brace X is

a knot invariant, which we will denote by $\Phi_X^{\mathbb{Z}}(K) = |\mathcal{C}(K, X)|$, where $\mathcal{C}(K, X)$ denotes the set of X-colorings of K.

The set-theoretic Yang-Baxter solutions defined by skew braces with commutative $*$ operation are *involutive*, meaning that the vertical map of pairs

$$r(x, y) = (x^* * (x \circ y), \ (x^* * (x * y))^\circ \circ x \circ y)$$

satisfies $r^2 = \mathrm{Id}$. More precisely, we note that the first component of $r^2(x, y)$ is

$$(x^* * (x \circ y))^* * [(x^* * (x \circ y)) \circ ((x^* * (x * y))^\circ \circ x \circ y)]$$
$$= (x^* * (x \circ y))^* * [x \circ y]$$
$$= (x \circ y)^* * (x^*)^* * (x \circ y)$$
$$= (x \circ y)^* * x * (x \circ y)$$

which equals x if $*$ is commutative, and the second component is

$$[(x \circ y)^* * x * (x \circ y)]^\circ \circ (x^* * (x \circ y)) \circ (x^* * (x \circ y))^\circ \circ x \circ y$$
$$= [(x \circ y)^* * x * (x \circ y)]^\circ \circ x \circ y$$

which again reduces to y in the case that $*$ is commutative.

Remark 1. Various notational conventions for the skew brace operations are used in the literature, including commonly writing $+$ for $*$ and $-x$ for x^* even when $*$ is noncommutative. To avoid certain errors, e.g. drawing the conclusion that all skew braces are involutive, we will prefer to use the $*$ notation.

The fact that all groups of cardinality less than six are abelian implies that the coloring invariant and its enhancements for skew braces with fewer than six elements cannot distinguish between knots or links related by the 2-*move*:
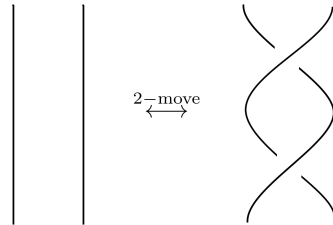
Since the 2-move can be combined with a Reidemeister II move to yield a crossing change,
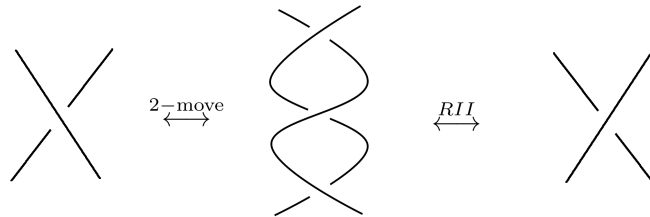
it follows that knots and links in various categories (classical, virtual, flat virtual, etc.) which are related by crossing change cannot be distinguished by involutory skew brace invariants. Indeed, involutory skew brace counting invariants and their enhancements are trivial for classical knots and links. However, virtual knots are links fall into several distinct classes under the virtual Reidemeister moves together with crossing changes; as we will show, involutory skew brace invariants and their enhancements can be effective at distinguishing these classes of virtual knots and links. Moreover, involutory skew brace invariants are well-defined for flat virtual knots.

Finite skew brace structures on a set X can be specified in various ways – algebraic formulas for the two group operations, using pairs of tuples as described in [5], etc. For our purposes it will be most useful to specify a skew brace on $X = \{1, 2, \ldots, n\}$ with a pair of operation tables for the two group operations, which we will call *structure tables*.

Example 1. The structure tables

$\circ$	1	2	3	3		$*$	1	2	3	3
1	1	2	3	4		1	1	2	3	4
2	2	1	4	3		2	2	3	4	1
3	3	4	1	2		3	3	4	1	2
4	4	3	2	1		4	4	1	2	3

define a skew brace with $\circ$-group isomorphic to the Klein 4-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $*$-group isomorphic to $\mathbb{Z}_4$.

If $*$ is nonabelian, then the skew brace's resulting set-theoretic Yang-Baxter solution r may not be involutive; these are the examples which can give us interesting invariants of classical knots and links.

Example 2. The skew brace structure on the set $X = \{1, 2, 3, 4, 5, 6\}$ defined by the structure tables

$\circ$	1	2	3	4	5	6		$*$	1	2	3	4	5	6
1	1	2	3	4	5	6		1	1	2	3	4	5	6
2	2	3	1	5	6	4		2	2	3	1	5	6	4
3	3	1	2	6	4	5		3	3	1	2	6	4	5
4	4	5	6	3	1	2		4	4	6	5	1	3	2
5	5	6	4	1	2	3		5	5	4	6	2	1	3
6	6	4	5	2	3	1		6	6	5	4	3	2	1

defines a non-involutive set-theoretic Yang-Baxter solution with for example $r(4, 3) = (2, 5) \neq (4, 3)$.

3. Skew brace enhancements

In this section we introduce new invariants of oriented virtual knots and links which enhance the skew brace counting invariant. More precisely, these

are polynomial invariants which specialize to the counting invariant when the variables are set equal to 1.

Recall that for any biquandle coloring $f \in \mathcal{C}(X, K)$, the *image* of f, denoted $\mathrm{Im}(f)$, is the biquandle closure of the set of biquandle elements used in f, i.e., the set of elements of X obtainable from biquandle elements used in f using the biquandle operations $\underline{\triangleright}$ and $\overline{\triangleright}$ (as well as their right inverses, though for finite biquandles the right inverse operations can be expressed in terms of the standard operations). Let us denote the group closures of $S \subset X$ under the group operations $*$ and $\circ$ by $\overline{S^*}$ and $\overline{S^{\circ}}$, respectively.

Definition 3. Let $(X, *, \circ)$ be a skew brace. For any oriented classical or virtual knot or link K, we define the *skew brace enhanced polynomial* of K to be

$$\Phi_X^{SB}(K) = \sum_{f \in \mathcal{C}(K, X)} u^{|\overline{Im(f)^{\circ}}|} v^{|\overline{Im(f)^{*}}|}.$$

Example 3. Let us illustrate the computation of the invariant for the *virtual Hopf link* on the left and compare it with the unlink of two components on the right:
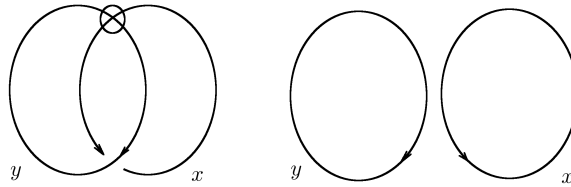
Colorings of the virtual Hopf link are pairs $x, y \in X$ satisfying $x \underline{\triangleright} y = x$ and $y \overline{\triangleright} x = y$ while colorings of the unlink are just pairs $x, y \in X$ without restriction. Then for example let X be the skew brace specified by the structure tables:

$\circ$	1	2	3	4		$*$	1	2	3	4
1	1	2	3	4		1	1	2	3	4
2	2	3	4	1		2	2	1	4	3
3	3	4	1	2		3	3	4	1	2
4	4	1	2	3		4	4	3	2	1

The coloring equations for the virtual Hopf link in terms of the skew brace

$$x \underline{\triangleright} y = y^{\circ} \circ (x * y) = x,$$
$$y \overline{\triangleright} x = x^{\circ} \circ (x * y) = y.$$

Of the sixteen potential colorings, as the reader can verify, there are four which do not satisfy the conditions: $(x, y) \in \{(2, 2), (2, 4), (4, 2), (4, 4)\}$. For the twelve valid colorings, there are eight which have closures of the entire set under both group operations, three which have closures of cardinality two under both group operations, and one with closures of cardinality 1 under both group operations. Hence, we have

$$\Phi_X^{SB}(\mathrm{vHopf}) = 8u^4 v^4 + 3u^2 v^2 + uv.$$

Repeating for the other colorings of the unlink, we have

$$\Phi_X^{SB}(\mathrm{U}_2) = 12u^4v^4 + 3u^2v^2 + uv.$$

Hence the invariant detects the non-triviality of the virtual Hopf link.

Next, we have a definition from [5].

Definition 4. Let X be a skew brace. A subset $I \subset X$ is an *ideal* if for all $x, y \in I$ and $z \in X$ the elements

$$y^\circ \circ x, \quad z^* * x * z, \quad z^\circ \circ x \circ z \quad \text{and} \quad z^* * (z \circ x)$$

are also in I.

Example 4. The skew brace with structure tables

$\circ$	1	2	3	4	5	6		$*$	1	2	3	4	5	6
1	1	2	3	4	5	6		1	1	2	3	4	5	6
2	2	3	4	5	6	1		2	2	1	6	5	4	3
3	3	4	5	6	1	2		3	3	4	5	6	1	2
4	4	5	6	1	2	3		4	4	3	2	1	6	5
5	5	6	1	2	3	4		5	5	6	1	2	3	4
6	6	1	2	3	4	5		6	6	5	4	3	2	1

has ideals including $\{1\}$, $\{1, 3, 5\}$ and $\{1, 2, 3, 4, 5, 6\}$.

Definition 5. Let X be a skew brace and K an oriented knot or link represented by a diagram D. Let $I(\mathrm{Im}(f))$ be the skew brace ideal generated by the image of f. We define the *skew brace ideal polynomial* of K with respect to X to be

$$\Phi_X^I(K) = \sum_{f \in \mathcal{C}(K, X)} u^{|I(\mathrm{Im}(f))|}.$$

We can now state our main theorem.

Theorem 3. *For any skew brace $(X, *, \circ)$, the skew brace enhanced polynomials and skew brace ideal polynomials are invariants of oriented classical and virtual knots and knots and links.*
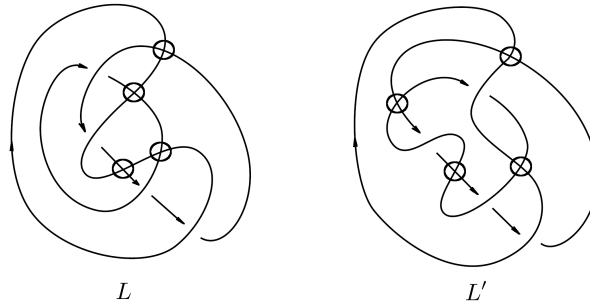
Proof. The image sub-biquandle $\mathrm{Im}(f)$ of a biquandle coloring is already an invariant for each coloring; it follows that the contributions to the polynomials from each coloring are not changed by Reidemeister moves. $\square$

4. Examples

In this section we collect some computations and examples of the new invariants.

Example 5. Let L and L' be the virtual links

$$L \qquad\qquad L'$$

and X the skew brace with structure tables:

∘	1	2	3	4	5	6	7	8		*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8		1	1	2	3	4	5	6	7	8
2	2	1	4	3	8	7	6	5		2	2	1	4	3	8	7	6	5
3	3	4	1	2	6	5	8	7		3	3	4	1	2	6	5	8	7
4	4	3	2	1	7	8	5	6		4	4	3	2	1	7	8	5	6
5	5	8	6	7	3	1	2	4		5	5	7	6	8	3	1	4	2
6	6	7	5	8	1	3	4	2		6	6	8	5	7	1	3	2	4
7	7	6	8	5	2	4	3	1		7	7	5	8	6	2	4	1	3
8	8	5	7	6	4	2	1	3		8	8	6	7	5	4	2	3	1

Then our `python` computations give skew brace enhanced polynomial values

$$\Phi_X^{SB}(L) = 144u^8v^8 + 154u^4v^4 + 21u^2v^2 + uv$$
$$\neq 168u^8v^8 + 130u^4v^4 + 21u^2v^2 + uv = \Phi_X^{SB}(L')$$

and

$$\Phi_X^I(L) = 144u^8 + 168u^4 + 7u^2 + u \neq 168u^8 + 144u^4 + 7u^2 + u = \Phi_X^I(L').$$

Since both links have 320 X-colorings, this example shows that both enhancements are proper and not determined by the counting invariant.

Example 6. We computed the two-variable invariant for the sets of prime virtual knots with up to four classical crossings as found at the knot atlas [2] with respect to the skew brace X with structure tables:

∘	1	2	3	4	5	6	7	8	*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	1	1	2	3	4	5	6	7	8
2	2	1	6	5	4	3	8	7	2	2	1	6	8	7	3	5	4
3	3	6	1	8	7	2	5	4	3	3	6	1	7	8	2	4	5
4	4	7	8	3	2	5	6	1	4	4	8	7	1	6	5	3	2
5	5	8	7	6	1	4	3	2	5	5	7	8	6	1	4	2	3
6	6	3	2	7	8	1	4	5	6	6	3	2	5	4	1	8	7
7	7	4	5	2	3	8	1	6	7	7	5	4	3	2	8	1	6
8	8	5	4	1	6	7	2	3	8	8	4	5	2	3	7	6	1

The results are collected in the table:

$\Phi_X^{SB}(L)$	L
$5u^2v^2 + uv$	3.1, 3.2, 3.4, 4.10, 4.11, 4.15, 4.17, 4.19, 4.20, 4.22, 4.23, 4.24, 4.29, 4.32, 4.34, 4.35, 4.38, 4.39, 4.42, 4.49, 4.50, 4.57, 4.62, 4.63, 4.66, 4.67, 4.70, 4.78, 4.79
$2u^8v^8 + 5u^2v^2 + uv$	2.1, 3.2, 3.5, 3.6, 3.7, 4.3, 4.6, 4.12, 4.13, 4.14, 4.18, 4.21, 4.25, 4.26, 4.27, 4.28, 4.30, 4.31, 4.36, 4.37, 4.40, 4.41, 4.43, 4.44, 4.45, 4.46, 4.47, 4.48, 4.51, 4.53, 4.54, 4.59, 4.60, 4.61, 4.64, 4.65, 4.68, 4.69, 4.71, 4.73, 4.74, 4.75, 4.80, 4.81, 4.82, 4.83, 4.84, 4.86, 4.87, 4.88, 4.91, 4.92, 4.93, 4.94, 4.95, 4.96, 4.97, 4.99, 4.100, 4.101, 4.102, 4.103, 4.104, 4.105, 4.106, 4.108
$6u^8v^8 + 5u^2v^2 + uv$	4.9, 4.16, 4.33, 4.52, 4.58, 4.72
$14u^8v^8 + 5u^2v^2 + uv$	4.1, 4.2, 4.4, 4.5, 4.7, 4.8, 4.55, 4.56, 4.76, 4.77, 4.85, 4.89, 4.90, 4.98, 4.107

We note that the enhancement information provides additional information beyond the counting invariant in that it filters the colorings into classes. Each virtual knot in this example has a uv term with coefficient 1 coming from the monochromatic coloring by the identity element and a u^2v^2 term with coefficient 5; the differences are in the coefficients of the surjective u^8v^8 colorings, which range from zero to 14. In particular, the virtual knots in the table with invariant values other than $2u^8v^8 + 5u^2v^2 + uv$ cannot be unknotted using crossing changes together with virtual Reidemeister moves.

Example 7. Provided $*$ is noncommutative, skew brace invariants can be effective at distinguishing classical knots and links as well as virtual and flat knots and links. For example, the skew brace X with noncommutative $*$ operation in Example 2 distinguishes the trefoil knot 3_1 from the figure eight knot 4_1 via the counting invariant, with

$$\Phi_X^{\mathbb{Z}}(3_1) = 12 \neq 6 = \Phi_X^{\mathbb{Z}}(4_1).$$

Our enhancements further refine this information into

$$\Phi_X^I(3_1) = 9u^6 + 2u^3 + u \neq 3u^6 + 2u^3 + u = \Phi_X^I(4_1)$$

and

$$\Phi_X^{SB}(3_1) = 8u^6v^6 + 2u^3v^3 + u^2v^2 + uv \neq 2u^6v^6 + 2u^3v^3 + u^2v^2 + uv = \Phi_X^{SB}(4_1).$$

5. Questions

We conclude this short paper with a list of questions for future study.

- What additional enhancements of the skew brace counting invariant using the skew brace structure are possible?
- In our examples, the powers on u and v are always the same; is this true in general, or a consequence of the small cardinality of our example skew braces?

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MELODY CHANG
DEPARTMENT OF MATHEMATICAL SCIENCES
CLAREMONT MCKENNA COLLEGE
850 COLUMBIA AVE.
CLAREMONT, CA 91711, USA
Email address: `mchang2353@scrippscollege.edu`

SAM NELSON
DEPARTMENT OF MATHEMATICAL SCIENCES
CLAREMONT MCKENNA COLLEGE
850 COLUMBIA AVE.
CLAREMONT, CA 91711, USA
Email address: `Sam.Nelson@cmc.edu`