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CONFORMAL RICCI SOLITON ON PARACONTACT METRIC (k,μ)-MANIFOLDS WITH SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. The main object of the present paper is to study conformal Ricci soliton on paracontact metric (k,μ)-manifolds with respect to Schouten-van Kampen connection. Further, we obtain the result when paracontact metric (k,μ)-manifolds with respect to Schouten-van Kampen connection satisfying the condition $\overset{\star}{C}(\xi,U)\cdot\overset{\star}{S}=0$. Finally we characterized concircular curvature tensor on paracontact metric (k,μ)-manifolds with respect to Schouten-van Kampen connection.

1. Introduction

In differential geometry of manifolds Schouten-van Kampen connection has been used for studying hyperdistributions in Riemannian manifolds as well as non-holonomic manifolds. In [9] Z. Olzak studied Schouten-van Kampen connection adapted to almost paracontact metric structure. Many authors investigated the hyperdistributions and some kind of affine connections adapted to these distributions [13–15]. Recently many authors studied Sasakian manifold, quasi Sasakian, Kenmotsu manifolds, f-Kenmotsu manifolds and trans-Sasakian manifolds with respect to Schouten-van Kampen connection [5, 8, 11, 16]. M. Manev [6] studied Schouten-van Kampen connection on almost contact B-metric structure which is counterpart of almost contact metric structure. S. Kaneyuki and F. L. Williams in [4] introduced paracontact metric structure as an odd dimensional counter part of para-Hermitian manifolds. S. Zamkovoy [19] defined a canonical paracontact connection on a paracontact metric manifold. A paracontact metric (k,μ)-manifold is a paracontact metric manifold for which the curvature tensor satisfies [7]

$$R(U,V)\xi=k(\eta(V)U-\eta(U)V)+\mu(\eta(V)hU-\eta(U)hV) \tag{1}$$

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for all vector fields U, V on manifold and k, μ are real constant. A. Yildiz and S. Y. Perktaş [18] studied h-projectively and ϕ-projectively semi-symmetric and S. Y. Perktaş et al. [10] studied some solitons on paracontact metric (k, μ)-manifold with respect to Schouten-van Kampen connection.

R. S. Hamilton [3] introduced the concept of Ricci flow in 1982 and the equation for Ricci flow is given by

$$\frac{\partial g}{\partial t} = -2S\ g.$$

In a Riemannian manifold (M, g), g is called a Ricci soliton if

$$L_V g + 2S + 2\lambda g = 0, \tag{2}$$

where L is the Lie derivative, S is the Ricci tensor, V is a vector field on M and λ is a real constant. It is well known that if λ is a smooth function, then the soliton is known as almost Ricci soliton. Further a Ricci soliton is expanding, steady and shrinking if λ is positive, zero and negative, respectively. A. E. Fisher modified Hamilton's Ricci flow equation by introducing conformal Ricci flow equation given by [2]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{2n+1}) = -p\ g,\ r(g) = -1,$$

where p is the conformal pressure and $r(g)$ is the scalar curvature of the manifold. Further N. Basu and A. Bhattacharyya generalizes the concept of Ricci soliton by introducing conformal Ricci soliton and given by the equation [1]

$$L_V g + 2S + (p + \frac{2}{2n+1} - 2\lambda)g = 0, \tag{3}$$

where λ is constant and p is conformal pressure.

Concircular curvature tensor on paracontact metric (k, μ)-manifolds with respect to Levi-Civita connection as well as Schouten-van Kampen connection is given as follows:

$$C(U, V)W = R(U, V)W - \frac{\tau}{2n(2n+1)}(g(V, W)U - g(U, W)V), \tag{4}$$

$$\overset{\star}{C}(U, V)W = \overset{\star}{R}(U, V)W - \frac{\overset{\star}{\tau}}{2n(2n+1)}(g(V, W)U - g(U, W)V), \tag{5}$$

where $R, \overset{\star}{R}$ $(\tau, \overset{\star}{\tau})$ are curvature tensors (scalar curvature tensors) with respect to Levi-Civita and Schouten-van Kampen connections.

2. Preliminaries

Let M be a $(2n+1)$-dimensional smooth manifold with the structure (ϕ, ξ, η) called almost para contact structure if it satisfies

$$\eta(\xi) = 1, \quad \phi^2(U) = U - \eta(U)\xi,$$
$$\phi\xi = 0, \quad g(U, \xi) = \eta(U),$$

for any vector fields U, V in M, where ϕ, ξ and η are the (1,1) tensor field, characteristic vector field and one-form, respectively. The (1,1) tensor field ϕ induces an almost paracomplex structure on each horizontal distribution $\mathrm{D} = ker(\eta)$, that is eigen distribution D^+, D^- have equal dimension n. If a pseudo Riemannian metric g satisfies

$$g(\phi U, \phi V) = -g(U, V) + \eta(U)\eta(V) \tag{6}$$

for all vector fields U, V in M, then (M, ϕ, ξ, η, g) is known as almost paracontact metric manifold. The fundamental two form Φ defined on M by $\Phi(U, V) = g(U, \phi V)$. If $d\eta(U, V) = g(U, \phi V)$ for all vector fields U, V on (M, ϕ, ξ, η, g), then almost paracontact metric manifold (M, ϕ, ξ, η, g) is called paracontact metric manifold. In a paracontact metric manifold, a traceless and symmetric operator h is defined as $h = \frac{1}{2}L_\xi \phi$ and it satisfies $h\xi = 0$ and $h\phi + \phi h = 0$. Moreover $h = 0$ if and only if ξ is a killing vector field and in this case manifold is known as K-paracontact manifold. If ∇ denotes the Levi-Civita connection on paracontact metric manifold, then the following relation satisfied [7]:

$$\nabla_U \xi = -\phi U + \phi h U. \tag{7}$$

Lemma 2.1. *For a paracontact metric* (k, μ)*-manifold* $M^{2n+1}(\phi, \xi, \eta, g)$ $(n > 1, k \neq -1)$*, the following relations hold* [7]:

$$h^2 = (1 + k)\phi^2, \tag{8}$$

$$(\nabla_U \phi)V = -g(U - hU, V)\xi + \eta(V)(U - hU), \tag{9}$$

$$\begin{aligned} S(U, V) = {} & [2(1 - n) + n\mu]g(U, V) + [2(n - 1) + \mu]g(hU, V) \\ & + [2(n - 1) + n(2k - \mu)]\eta(U)\eta(V), \end{aligned} \tag{10}$$

$$\begin{aligned} QU = {} & [2(1 - n) + n\mu]U + [2(n - 1) + \mu]hU \\ & + [2(n - 1) + n(2k - \mu)]\eta(U)\xi, \end{aligned} \tag{11}$$

$$S(U, \xi) = 2nk\eta(U), \tag{12}$$

$$Q\xi = 2nk\xi, \tag{13}$$

$$Q\phi - \phi Q = 2[2(n - 1) + \mu]h\phi \tag{14}$$

for all vector fields U, V *on* $M^{2n+1}(\phi, \xi, \eta, g)$*, where* Q *and* S *denote the Ricci operator and Ricci tensor of* $M^{2n+1}(\phi, \xi, \eta, g)$*, respectively.*

In [17] A. Yildiz and U. C. De proved the following identity on paracontact metric (k, μ)-manifold

$$\begin{aligned} & R(U, V)hW - hR(U, V)W \\ & = \mu(k + 1)\{g(V, W)\eta(U)\xi - g(U, W)\eta(V)\xi \\ & \quad + \eta(U)\eta(W)V - \eta(V)\eta(W)U\} \\ & \quad + k\{g(hV, W)\eta(U)\xi - g(hU, W)\eta(V)\xi \\ & \quad + \eta(U)\eta(W)hV - \eta(V)\eta(W)hU \end{aligned} \tag{15}$$

$$+ g(\phi V, W)\phi hU - g(\phi U, W)\phi hV\}$$
$$+ (\mu + k)\{g(\phi hU, W)\phi V - g(\phi hV, W)\phi U\}$$
$$+ 2\mu g(\phi U, V)\phi hW.$$

If ∇ and $\overset{\star}{\nabla}$ denote the Levi-Civita connection and Schouten-van Kampen connection, respectively, then on paracontact metric (k, μ)-manifold the two connections are related by [12]

$$\overset{\star}{\nabla}_U V = \nabla_U V + \eta(V)\phi U - \eta(V)\phi hU + g(U, \phi V)\xi - g(hU, \phi V)\xi. \tag{16}$$

Also, the torsion $\overset{\star}{T}$ of $\overset{\star}{\nabla}$ is defined by

$$\overset{\star}{T}(U, V) = \eta(U)\nabla_V \xi - \eta(V)\nabla_U \xi + 2d\eta(U, V)\xi. \tag{17}$$

Let R and $\overset{\star}{R}$ be the curvature tensors of M^{2n+1} with respect to Levi-Civita connection ∇ and the Schouten-van Kampen connection $\overset{\star}{\nabla}$. Then R and $\overset{\star}{R}$ are connected by the following formula [18]:

$$\begin{aligned} \overset{\star}{R}(U, V)W = R(U, V)W &+ g(U, \phi W)\phi V - g(V, \phi W)\phi U + g(hV, \phi W)\phi U \\ &- g(hU, \phi W)\phi V + g(V, \phi W)\phi hU - g(U, \phi W)\phi hV \\ &+ g(hU, \phi W)\phi hV - g(hV, \phi W)\phi hU \\ &+ k(g(U, W)\eta(V)\xi - g(V, W)\eta(U)\xi \\ &+ \eta(U)\eta(W)V - \eta(V)\eta(W)U) \\ &+ \mu(g(hU, W)\eta(V)\xi - g(hV, W)\eta(U)\xi \\ &+ \eta(U)\eta(W)hV - \eta(V)\eta(W)hU). \end{aligned} \tag{18}$$

Now taking the inner product in (18) with a vector field Z, then contracting with $U = Z = e_i$ we have

$$\overset{\star}{S}(V, W) = S(V, W) - 2nk\eta(V)\eta(W) - \mu g(hV, W), \tag{19}$$

where $\overset{\star}{S}$ and S denote the Ricci tensor of M^{2n+1} with respect to the connections $\overset{\star}{\nabla}$ and ∇, respectively. As a consequence of (19), we get for the Ricci operator

$$\overset{\star}{Q}V = QV - 2nk\eta(V)\xi - \mu hV. \tag{20}$$

Also if we take $V = W = e_i, \{i = 1, \ldots, 2n + 1\}$, in (19), we get

$$\overset{\star}{\tau} = \tau - 2nk, \tag{21}$$

where $\overset{\star}{\tau}$ and τ denote the scalar curvatures of M^{2n+1} with respect to the connections $\overset{\star}{\nabla}$ and ∇, respectively.

3. Conformal Ricci soliton on paracontact metric (k,μ)-manifold with Schouten-van Kampen connection

In this section we consider conformal Ricci soliton on a paracontact metric (k,μ)-manifold with respect to the Schouten-van Kampen connection. From (3), we have

$$(\overset{\star}{L}_V g + 2\overset{\star}{S} + (p + \frac{2}{2n+1} - 2\lambda)g)(U,W) = 0 \tag{22}$$

that is

$$g(\overset{\star}{\nabla}_U V, W) + g(U, \overset{\star}{\nabla}_W V) + 2\overset{\star}{S}(U,W) + (p + \frac{2}{2n+1} - 2\lambda)g(U,W) = 0. \tag{23}$$

Using equation (16) in (23) we obtain

$$\begin{aligned} &g(\nabla_U V, W) - \eta(V)g(\phi hU, W) + g(U, \phi V)\eta(W) - g(hU, \phi V)\eta(W) \\ &+ g(\nabla_W V, U) - \eta(V)g(\phi hW, U) + g(W, \phi V)\eta(U) - g(hW, \phi V)\eta(U) \\ &+ 2\overset{\star}{S}(U,W) + (p + \frac{2}{2n+1} - 2\lambda)g(U,W) = 0. \end{aligned} \tag{24}$$

Putting $V = \xi$ in (24) and using (7), we obtain

$$2\overset{\star}{S}(U,W) = -(p + \frac{2}{2n+1} - 2\lambda)g(U,W). \tag{25}$$

Also, from (19) and (25) we get

$$2S(U,W) = -(p + \frac{2}{2n+1} + 2\lambda)g(U,W) + 4nk\eta(U)\eta(W) + 2\mu g(hU,W).$$

Thus, we have the following result.

Theorem 3.1. *Let M be an paracontact metric (k,μ)-manifold bearing conformal Ricci soliton (g,V,λ) with respect to the Schouten-van Kampen connection. Then M is an Einstein manifold with respect to the Schouten-van Kampen connection and M is an η-Einstein manifold with respect to the Levi-Civita connection.*

Now we consider V is a pointwise collinear vector field with the structure vector field ξ, that is $V = f\xi$, where f is a smooth function on M. From (23) and using $V = f\xi$, we have

$$g(\overset{\star}{\nabla}_U f\xi, W) + g(U, \overset{\star}{\nabla}_W f\xi) + 2\overset{\star}{S}(U,W) + (p + \frac{2}{2n+1} - 2\lambda)g(U,W) = 0. \tag{26}$$

Then by using (7) and (16) in (26) we have

$$U[f]\eta(W) + W[f]\eta(U) + 2\overset{\star}{S}(U,W) + (p + \frac{2}{2n+1} - 2\lambda)g(U,W) = 0. \tag{27}$$

By virtue of (19), equation (27) becomes

$$U[f]\eta(W) + W[f]\eta(U) + 2\{S(U,W) - 2nk\eta(U)\eta(W) - \mu g(hU,W)\} \tag{28}$$

$$+(p+\frac{2}{2n+1}-2\lambda)g(U,W)=0.$$

Putting $W=\xi$ in (28) and using (12) we get

$$U[f]+(\xi f)\eta(U)+(p+\frac{2}{2n+1}-2\lambda)\eta(U)=0. \tag{29}$$

Taking $U=\xi$ in (29) gives

$$\xi f=\frac{p}{2}+\frac{1}{2n+1}-\lambda. \tag{30}$$

If we replace (30) in (29), we get

$$U[f]=(\frac{3p}{2}+\frac{3}{2n+1}-3\lambda)\eta(U) \tag{31}$$

which yields

$$df=(\frac{3p}{2}+\frac{3}{2n+1}-3\lambda)\eta. \tag{32}$$

Applying d on both sides in above equation and $p=\frac{-2}{2n+1}$, we have

$$\lambda=0. \tag{33}$$

Which implies

$$df=0,\ \text{that is } f=\text{constant}. \tag{34}$$

Thus using constancy of f in (27), we obtain

$$\overset{\star}{S}(U,W)=0. \tag{35}$$

Thus

$$S(U,W)=(k+2)g(U,W)-(k+2-2nk)\eta(U)\eta(W)+(\mu-1)g(hU,W)$$

for any $U,W\in TM$.

Hence we have the following theorem.

Theorem 3.2. *Let M be a paracontact metric (k,μ)-manifold with respect to the Schouten-van Kampen connection. If M admits conformal Ricci soliton (g,V,λ) with conformal pressure $p=\frac{-2}{2n+1}$ and V is pointwise collinear with the structure vector field ξ, then V is a constant multiple of the structure vector field, M is an η-Einstein manifold with respect to the Levi-Civita connection and the conformal Ricci soliton is steady.*

4. Paracontact metric (k, μ)-manifold with respect to Schouten-van Kampen connection satisfying $\overset{\star}{C}(\xi, U)\cdot \overset{\star}{S}= 0$

In this section we study paracontact metric (k, μ)-manifold with respect to Schouten-van Kampen connection satisfying $\overset{\star}{C}(\xi, U)\cdot \overset{\star}{S}= 0$. Therefore

$$\overset{\star}{S}(\overset{\star}{C}(\xi, U)V, W) + \overset{\star}{S}(V, \overset{\star}{C}(\xi, U)W) = 0. \tag{36}$$

From (18) we get

$$\overset{\star}{R}(\xi, U)V = \eta(U)\eta(V)\xi - g(U, V)\xi + g(hU, V)\xi. \tag{37}$$

Using (18) in (5) we have

$$\overset{\star}{C}(\xi, U)V = \overset{\star}{R}(\xi, U)V - \frac{\overset{\star}{\tau}}{2n(2n+1)}(g(U, V)\xi - \eta(V)U). \tag{38}$$

Making use of (37) and (38) we have

$$\begin{aligned}\overset{\star}{C}(\xi, U)V &= [\eta(U)\eta(V)\xi - g(U, V)\xi + g(hU, V)\xi] \\ &\quad - \frac{\overset{\star}{\tau}}{2n(2n+1)}(g(U, V)\xi - \eta(V)U).\end{aligned} \tag{39}$$

By virtue of (39), equation (36) becomes

$$\frac{\overset{\star}{\tau}}{2n(2n+1)}(\eta(V)\overset{\star}{S}(U, W)) + \frac{\overset{\star}{\tau}}{2n(2n+1)}(\eta(W)\overset{\star}{S}(U, V)) = 0. \tag{40}$$

Taking $V = \xi$ in (40) and using (19) we obtain

$$\frac{\overset{\star}{\tau}}{2n(2n+1)}\overset{\star}{S}(U, W) = 0. \tag{41}$$

From (41) we have the following theorem.

Theorem 4.1. *If M is a paracontact metric (k, μ)-manifold with respect to the Schouten-van Kampen connection satisfies $\overset{\star}{C}(\xi, U)\cdot \overset{\star}{S}= 0$, then M is Ricci flat with respect to the Schouten-van Kampen connection and η-Einstein manifold with respect to Levi-Civita connection provided $\frac{\overset{\star}{\tau}}{2n(2n+1)} \neq 0$.*

5. Concircular curvature tensor on paracontact metric (k, μ)-manifold with Schouten-van Kampen connection

In this section we study concircular curvature tensor on paracontact metric (k, μ)-manifold with respect to the Schouten-van Kampen connection.

Definition. A $(2n+1)$-dimensional semi-Riemannian manifold M, is said to be h-concircular semisymmetric with respect to Schouten-van Kampen connection if

$$\overset{\star}{C}(U, V)\cdot h = 0 \tag{42}$$

holds on M.

The above equation is equivalent to

$$\overset{\star}{C}(U,V)hW - h\overset{\star}{C}(U,V)W = 0 \tag{43}$$

for any $U, V, W \in \chi(M)$. Thus using (5) in (43) we get

$$\begin{aligned}[\overset{\star}{R}(U,V)hW - h\overset{\star}{R}(U,V)W] - \frac{\overset{\star}{\tau}}{2n(2n+1)}[g(V,hW)U \\ - g(V,W)hU - g(U,hW)V + g(U,W)hV] = 0.\end{aligned} \tag{44}$$

Next, Making use of (18) in (44), we have

$$\begin{aligned}&[R(U,V)hW - hR(U,V)W + g(U,\phi hW)\phi V - g(V,\phi hW)\phi U \\ &+ g(hV,\phi hW)\phi U + g(hU,h\phi W)\phi V + g(V,\phi hW)\phi hU \\ &- g(U,\phi hW)\phi hV - g(hU,h\phi W)\phi hV + g(hV,h\phi W)\phi hU \\ &+ (k+1)\{g(U,hW)\eta(V)\xi - g(V,hW)\eta(U)\xi\} \\ &+ (\mu-1)\{g(hU,hW)\eta(V)\xi - g(hV,hW)\eta(U)\xi\} \\ &- g(U,\phi W)h\phi V + g(V,\phi W)h\phi U - g(hV,\phi W)h\phi U \\ &+ g(hU,\phi W)h\phi V - g(V,\phi W)h\phi hU + g(U,\phi W)h\phi hV \\ &- g(hU,\phi W)h\phi hV + g(hV,\phi W)h\phi hU \\ &- k\{\eta(U)\eta(W)hV - \eta(V)\eta(W)hU\} \\ &- \mu\{\eta(U)\eta(W)h^2V - \eta(V)\eta(W)h^2U\}] \\ &- \frac{\overset{\star}{\tau}}{2n(2n+1)}[g(V,hW)U - g(V,W)hU \\ &- g(U,hW)V + g(U,W)hV] = 0.\end{aligned} \tag{45}$$

Next, using (15) in (45), we get

$$\begin{aligned}&[\mu(k+1)\{g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi + \eta(U)\eta(W)V \\ &- \eta(V)\eta(W)U\} + k\{g(hV,W)\eta(U)\xi - g(hU,W)\eta(V)\xi \\ &+ \eta(U)\eta(W)hV - \eta(V)\eta(W)hU - g(\phi V,W)h\phi U \\ &+ g(\phi U,W)h\phi V\} - (\mu+k)\{g(h\phi U,W)\phi V - g(h\phi V,W)\phi U\} \\ &- 2\mu g(\phi U,V)h\phi W + g(U,\phi hW)\phi V - g(V,\phi hW)\phi U \\ &+ g(hV,\phi hW)\phi U + g(hU,h\phi W)\phi V + g(V,\phi hW)\phi hU \\ &- g(U,\phi hW)\phi hV - g(hU,h\phi W)\phi hV + g(hV,h\phi W)\phi hU \\ &+ (k+1)\{g(U,hW)\eta(V)\xi - g(V,hW)\eta(U)\xi\} \\ &+ (\mu-1)\{g(hU,hW)\eta(V)\xi - g(hV,hW)\eta(U)\xi\} \\ &- g(U,\phi W)h\phi V + g(V,\phi W)h\phi U - g(hV,\phi W)h\phi U \\ &+ g(hU,\phi W)h\phi V - g(V,\phi W)h\phi hU + g(U,\phi W)h\phi hV\end{aligned} \tag{46}$$

$$
\begin{aligned}
&- g(hU,\phi W)h\phi hV + g(hV,\phi W)h\phi hU \\
&- k\{\eta(U)\eta(W)hV - \eta(V)\eta(W)hU\} \\
&- \mu\left\{\eta(U)\eta(W)h^2V - \eta(V)\eta(W)h^2U\right\} \\
&- \frac{\overset{\star}{\tau}}{2n(2n+1)}[g(V,hW)U - g(V,W)hU \\
&- g(U,hW)V + g(U,W)hV] = 0
\end{aligned}
$$

which gives to

$$
\begin{aligned}
&[\mu\{g(h\phi V,W)g(\phi U,X) - g(h\phi U,W)g(\phi V,X) + 2(U,\phi V)g(h\phi W,X)\} \\
&+ (k+1)\{g(V,W)\eta(U)\eta(X) - g(U,W)\eta(V)\eta(X)\} \\
&+ g(hU,W)\eta(V)\eta(X) - g(hV,W)\eta(U)\eta(X)] \\
&- \frac{\overset{\star}{\tau}}{2n(2n+1)}[g(V,hW)g(U,X) - g(V,W)g(hU,X) \\
&- g(V,W)g(U,hX) + g(U,W)g(hV,X)] = 0.
\end{aligned} \tag{47}
$$

Putting $U = X = e_i$ in (47), we get

$$
\begin{aligned}
&[\mu(k+1)g(hW,V) + \mu(k+1)\{g(V,W) - \eta(V)\eta(W)\} - g(hV,W)] \\
&+ 2n(k-2n)g(V,hW) - 2n(k+1)(2n-1)g(V,W) - S(hW,V) \\
&- S(hV,W) + (k+2)g(hV,W) + (k+2)g(hW,V) + 2g(hV,hW) \\
&- \frac{\overset{\star}{\tau}}{2n(2n+1)}\{(2n+1)g(V,hW) - g(hW,V) + g(hV,W)\} = 0.
\end{aligned} \tag{48}
$$

Again putting $V = hV$ in (48) and using $h^2 = (k+1)\phi^2$, we obtain

$$
\begin{aligned}
&(k+1)[\{\mu(k+1) - 1 + 2n(k-2n) + 2(k+\mu+1) - \frac{\overset{\star}{\tau}}{2n}\}g(V,W) \\
&- \{\mu(k+1) - 1 + 2n(k+2n) - (k+2\mu)\}\eta(V)\eta(W) \\
&+ \{\mu - 2n(2n-1)\}g(hV,W) - 2S(V,W)] = 0.
\end{aligned} \tag{49}
$$

From equation (10) we have

$$
\begin{aligned}
g(hV,W) = {}& \frac{1}{2(n-1)+\mu}S(V,W) - \frac{2(1-n)+n\mu}{2(n-1)+\mu}g(V,W) \\
&- \frac{2(n-1)+n(2k-\mu)}{2(n-1)+\mu}\eta(V)\eta(W).
\end{aligned} \tag{50}
$$

Hence using (50) in (49), we get

$$
\begin{aligned}
&(k+1)[\{\mu(k+1) - 1 + 2n(k-2n) + 2(k+\mu+1) \\
&- \frac{\overset{\star}{\tau}}{2n}g(V,W) + 2n(k+2n) - (k+2\mu)\}\eta(V)\eta(W)
\end{aligned} \tag{51}
$$

$$- \{\mu(k+1) - 1 + \mu - 2n(2n-1)\}\{\frac{1}{2(n-1)+\mu} S(V,W)$$
$$- \frac{2(1-n)+n\mu}{2(n-1)+\mu} g(V,W) - \frac{2(n-1)+n(2k-\mu)}{2(n-1)+\mu}\eta(V)\eta(W)\}$$
$$- 2S(V,W)] = 0.$$

Hence one can write

$$S(V,W) = \frac{A_1}{A_3} g(V,W) + \frac{A_2}{A_3}\eta(V)\eta(W), \tag{52}$$

where

$$A_1 = \mu(k+1) - 1 + 2n(k-2n) + 2(k+\mu+1) - \frac{\overset{\star}{\tau}}{2n} - \mu$$
$$- 2n(2n-1)\frac{2(1-n)+n\mu}{2(n-1)+\mu},$$
$$A_2 = \mu(k+1) - 1 + 2n(k+2n) - \{\mu - 2n(2n-1)\}\frac{2(1-n)+n\mu}{2(n-1)+\mu}$$
$$- (k+2\mu),$$
$$A_3 = 2 - \{\mu - 2n(2n-1)\}\frac{1}{2(n-1)+\mu}.$$

Therefore from (52) it follows that the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:

Theorem 5.1. *Let M be a $(2n+1)$-dimensional h-concircular semisymmetric paracontact (k,μ)-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then M is an η-Einstein manifold with respect to the Levi-Civita connection provided $\mu \neq 2(1-n)$.*

Example 5.2. Let G be a Lie group with Lie algebra g endowed with a basis $\{e_1, e_2, e_3, e_4, e_5\}$ and non-zero Lie brackets [10]:

$$[e_1, e_5] = e_1 + e_2, \quad [e_2, e_5] = e_1 + e_2,$$
$$[e_3, e_5] = -e_3 + e_4, \quad [e_4, e_5] = e_3 - e_4,$$
$$[e_1, e_2] = e_1 + e_2, \quad [e_1, e_3] = e_2 + e_4 - 2e_5,$$
$$[e_1, e_4] = e_2 + e_3, \quad [e_2, e_3] = e_1 - e_4,$$
$$[e_2, e_4] = e_1 - e_3 + 2e_5, \quad [e_3, e_4] = -e_3 + e_4.$$

Define on G a left invariant para contact metric structure (ϕ, ξ, η, g) such that $g(e_1, e_1) = g(e_4, e_4) = -g(e_2, e_2) = -g(e_3, e_3) = g(e_5, e_5) = 1$, $g(e_i, e_j) = 0$ for any $i \neq j$, and $\phi e_1 = e_3$, $\phi e_2 = e_4$, $\phi e_3 = e_1$, $\phi e_4 = e_2$, $\phi e_5 = 0$, $\xi = e_5$ and $\eta = g(\cdot, e_5)$. A straightforward computation shows that

$$\nabla_{e_1}\xi = e_1 - \phi e_1, \qquad \nabla_{e_2}\xi = e_2 - \phi e_2,$$
$$\nabla_{\phi e_1}\xi = -e_1 - \phi e_1, \qquad \nabla_{\phi e_2}\xi = -e_2 - \phi e_2,$$

$$\begin{aligned}
\nabla_\xi e_1 &= -e_2 - \phi e_1, & \nabla_\xi e_2 &= -e_1 - \phi e_2,\\
\nabla_\xi \phi e_1 &= -e_1 - \phi e_2, & \nabla_\xi \phi e_2 &= -e_2 - \phi e_1,\\
\nabla_{e_1} e_1 &= e_2 - e_5, & \nabla_{e_1} e_2 &= e_1,\\
\nabla_{e_1} \phi e_1 &= \phi e_2 - e_5, & \nabla_{e_1} \phi e_2 &= \phi e_1,\\
\nabla_{e_2} e_1 &= e_2, & \nabla_{e_2} e_2 &= -e_1 + e_5,\\
\nabla_{e_2} \phi e_1 &= -\phi e_2, & \nabla_{e_2} \phi e_2 &= -\phi e_1 + e_5,\\
\nabla_{\phi e_1} e_1 &= -e_2 + e_5, & \nabla_{\phi e_1} e_2 &= -e_1,\\
\nabla_{\phi e_1} \phi e_1 &= -\phi e_2 - \alpha e_5, & \nabla_{\phi e_1} \phi e_2 &= -\phi e_1,\\
\nabla_{\phi e_2} e_1 &= -e_2, & \nabla_{\phi e_2} e_2 &= -e_1 - e_5,\\
\nabla_{\phi e_2} \phi e_1 &= -\phi e_2, & \nabla_{\phi e_2} \phi e_2 &= -\phi e_1 + e_5.
\end{aligned}$$

Which implies that (G, ϕ, ξ, η, g) is a 5-dimensional paracontact metric manifold with $\kappa = -2$ and $\mu = 2$. Using (16), we have

$$\overset{\star}{\nabla}_{e_1} e_1 = e_2, \quad \overset{\star}{\nabla}_{e_1} e_2 = e_1, \quad \overset{\star}{\nabla}_{e_1} e_3 = e_4, \quad \overset{\star}{\nabla}_{e_1} e_4 = e_3,$$
$$\overset{\star}{\nabla}_{e_2} e_1 = -e_2, \quad \overset{\star}{\nabla}_{e_2} e_2 = -e_1, \overset{\star}{\nabla}_{e_2} e_3 = -e_4, \quad \overset{\star}{\nabla}_{e_2} e_4 = -e_3,$$
$$\overset{\star}{\nabla}_{e_3} e_1 = -e_2, \overset{\star}{\nabla}_{e_3} e_2 = -e_1, \quad \overset{\star}{\nabla}_{e_3} e_3 = -e_4, \quad \overset{\star}{\nabla}_{e_3} e_4 = -e_3,$$
$$\overset{\star}{\nabla}_{e_4} e_1 = -e_2, \quad \overset{\star}{\nabla}_{e_4} e_2 = -e_1, \quad \overset{\star}{\nabla}_{e_4} e_3 = -e_4,$$
$$\overset{\star}{\nabla}_{e_4} e_4 = -e_3, \quad \overset{\star}{\nabla}_{e_5} e_1 = -e_2 - e_3, \quad \overset{\star}{\nabla}_{e_5} e_2 = -\beta e_1 - e_4,$$
$$\overset{\star}{\nabla}_{e_5} e_3 = -e_1 - e_4, \quad \overset{\star}{\nabla}_{e_5} e_4 = -e_2 - e_3.$$

Now using (18), we can calculate the non-zero components of its curvature tensor with respect to the Schouten-van Kampen connection as follows:

$$\begin{aligned}
&\overset{\star}{R}(e_1, e_3) e_1 = -2e_3, \quad \overset{\star}{R}(e_1, e_3) e_2 = -2e_4,\\
&\overset{\star}{R}(e_1, e_3) e_3 = -2e_1, \quad \overset{\star}{R}(e_1, e_3) e_4 = -2e_2,\\
&\overset{\star}{R}(e_1, e_4) e_1 = 2e_2, \quad \overset{\star}{R}(e_1, e_4) e_2 = 2e_1,\\
&\overset{\star}{R}(e_1, e_4) e_3 = 2e_4, \quad \overset{\star}{R}(e_1, e_4) e_4 = 2e_3,\\
&\overset{\star}{R}(e_2, e_3) e_1 = -2e_2, \quad \overset{\star}{R}(e_2, e_3) e_2 = -2e_1,\\
&\overset{\star}{R}(e_2, e_3) e_3 = -2e_4, \quad \overset{\star}{R}(e_2, e_3) e_4 = -2e_3,\\
&\overset{\star}{R}(e_2, e_4) e_1 = 2e_3, \quad \overset{\star}{R}(e_2, e_4) e_2 = 2e_4,\\
&\overset{\star}{R}(e_2, e_4) e_3 = 2e_1, \quad \overset{\star}{R}(e_2, e_4) e_4 = 2e_2.
\end{aligned}$$

Thus the non-zero components of its Ricci tensor with respect to the Schouten-van Kampen connection as follows:

$$\overset{\star}{S}(e_1, e_1) = \overset{\star}{S}(e_4, e_4) = 2,\ \overset{\star}{S}(e_2, e_2) = \overset{\star}{S}(e_3, e_3) = -2. \tag{53}$$

From (35), (53) one can see that manifold is Ricci flat with respect to Schouten-van Kampen connection or η-Einstein manifold with respect to the Levi-Civita connection on such a 5-dimensional paracontact metric (κ, μ)-manifold with $\kappa = -2$.

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