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ON THE GENERALIZED ORNSTEIN-UHLENBECK OPERATORS WITH REGULAR AND SINGULAR POTENTIALS IN WEIGHTED L^p-SPACES

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Abstract. In this paper, we give sufficient conditions for the generalized Ornstein-Uhlenbeck operators perturbed by regular potentials and inverse square potentials

$$A_{\Phi,G,V,c} = \Delta - \nabla\Phi \cdot \nabla + G \cdot \nabla - V + c|x|^{-2}$$

with a suitable domain generates a quasi-contractive, positive and analytic C_0-semigroup in $L^p(\mathbb{R}^N, e^{-\Phi(x)}dx)$, $1 < p < \infty$. The proofs are based on an L^p-weighted Hardy inequality and perturbation techniques. The results extend and improve the generation theorems established by Metoui [7] and Metoui–Mourou [8].

1. Introduction

Generalized Ornstein-Uhlenbeck operators have been widely investigated in literrature by using different methods, see for instance [1, 3–12]. The main motivation comes from the study of Metafune–Prüss–Rhandi–Schnaubelt [6] in which they dealt with the operator

$$A_{\Phi,G} = \Delta - \nabla\Phi \cdot \nabla u + G \cdot \nabla$$

in the space $L^p(\mathbb{R}^N, d\mu)$, where $d\mu = e^{-\Phi(x)}dx$, $1 < p < \infty$. More precisely, under appropriate conditions on Φ and G, they established that $A_{\Phi,G}$ with the domain $W^{2,p}_\mu(\mathbb{R}^N)$ generates an analytic C_0-semigroup on $L^p(\mathbb{R}^N, d\mu)$, $1 < p < \infty$. Afterwards, Kojima–Yokota [4] also Sobajima–Yokota [12] studied the operator $A_{\Phi,G}$ perturbed by a positive potential $V \in C^1(\mathbb{R}^N)$. By using different methods and some conditions on Φ, G and V, they proved that the operator $A_{\Phi,G} - V$ endowed with the domain

$$W^{2,p}_V(\mathbb{R}^N, d\mu) = \left\{u \in W^{2,p}_\mu(\mathbb{R}^N) : Vu \in L^p_\mu(\mathbb{R}^N)\right\}$$

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generates a quasi-contractive analytic C_0-semigroup on $L^p_\mu(\mathbb{R}^N)$ for $1 < p < \infty$. Besides, several recent studies concerned with $A_{\Phi,G}$ perturbed by singular potentials [1, 3, 7, 8]. In [1], Durante–Rhandi considered the case where $p = 2$, $G(x) = 0$, $\Phi(x) = \frac{1}{2}\langle Mx, x\rangle$ and $V = c|x|^{-2}$. More specifically, they showed that

$$A_{M,c} = \Delta - Mx \cdot \nabla + c|x|^{-2}$$

is essentially selfadjoint in $L^2(\mathbb{R}^N, d\mu)$ if $c \leq \frac{(N-2)^2}{4} - 1$ and $N > 4$, where

$$d\mu = (2\Pi)^{-\frac{N}{2}}(\det M)^{\frac{1}{2}} e^{-\frac{1}{2}\langle Mx,x\rangle} dx$$

and M is a real, symmetric $N \times N$-matrix. Their result was generalized by Fornaro–Rhandi [3] to L^p-setting, $1 < p < \infty$. Subsequently, the operator $A_{\Phi,G}$ perturbed by a nonnegative singular potential νV in the space $L^p(\mathbb{R}^N, d\mu)$, $1 < p < \infty$, has been investigated by Metoui–Mourou [8]. They showed that $A_{\Phi,G} - \nu V$ generates a quasi-contractive and positive analytic C_0-semigroup in $L^p(\mathbb{R}^N, d\mu)$. More recently, Metoui [7] proved under sufficient conditions on Φ, G, V and c that

$$A_{\Phi,G,V,c} = \delta - \nabla\Phi \cdot \nabla u + G \cdot \nabla - V + c|x|^{-2}$$

generates a positive C_0-semigroup in $L^2(\mathbb{R}^N, d\mu)$.

To complete the picture, we investigate the perturbation of $A_{\Phi,G,V}$ with the inverse square potential $c|x|^{-2}$ in the weighted space $L^p(\mathbb{R}^N, d\mu)$, $1 < p < \infty$. We focus on the accretivity and dispersivity of such operator. Moreover, we provide sufficient conditions on Φ, G, V and c ensuring that $A_{\Phi,G,V,c}$ endowed with a suitable domain generates an analytic semigroup on the weighted spaces $L^p_\mu(\mathbb{R}^N)$, $1 < p < \infty$. Our proofs based on an L^p-weighted Hardy's inequality and on the following perturbation results.

Theorem 1.1 ([11, Theorem 1.6])**.** *Let A and B be linear m-accretive operators in a Banach space X with uniformly convex X^*. Let D be a core of A. Assume that there are constants $a, b, d \geq 0$ such that for all $u \in D$ and $\epsilon > 0$,*

$$\mathrm{Re}\langle Au, \|B_\varepsilon u\|_p^{2-p}|B_\varepsilon u|^{p-2}B_\varepsilon u\rangle \geq -b\|B_\varepsilon u\|_p^2 - d\|u\|_p^2 - a\|B_\varepsilon u\|_p\|u\|_p,$$

where $B_\epsilon = B(I + \epsilon B)^{-1}$ denotes the Yosida approximation.

If $\nu > b$, then $A + \nu B$ with domain $D(A) \cap D(B)$ is m-accretive and $D(A) \cap D(B)$ is core for A.

Moreover, $A + bB$ is essentially m-accretive on $D(A) \cap D(B)$.

Theorem 1.2 ([11, Theorem 1.7])**.** *Let A and B be linear m-accretive operators in a Banach space X with uniformly convex X^*. Let D be a core of A. Assume that*

(i) *there are constants $d, a \geq 0$ and $k_1 > 0$ such that for all $u \in D$ and $\varepsilon > 0$,*

$$\mathrm{Re}\langle Au, \|B_\varepsilon u\|_p^{2-p}|B_\varepsilon u|^{p-2}B_\varepsilon u\rangle \geq k_1\|B_\varepsilon u\|_p^2 - d\|u\|_p^2 - a\|B_\varepsilon u\|_p\|u\|_p,$$

where B_ε denote the Yosida approximation of B.

(ii) *Re*$\langle u, \|B_\varepsilon u\|_p^{2-p}|B_\varepsilon u|^{p-2}B_\varepsilon u\rangle \geq 0$ *for all* $u \in X$ *and* $\varepsilon > 0$.

(iii) *there is* $k_2 > 0$ *such that* $A - k_2 B$ *is accretive.*

Set $k = \min\{k_1, k_2\}$. *If* $t > -k$, *then* $A + tB$ *with domain* $D(A+tB) = D(A)$ *is* m*-accretive and any core of* A *is also core for* $A+tB$. *Furthermore,* $A - kB$ *is essentially* m*-accretive on* $D(A)$.

Now, we introduce the following conditions on Φ, G and V:

(A1) The function $\Phi \in C^2(\mathbb{R}^N, \mathbb{R})$ and satisfies that for every $\tau \in (0, \frac{1}{2N})$, there is a constant $C_\tau > 0$ such that

$$|D^2\Phi| \leq \tau |\nabla \Phi|^2 + C_\tau.$$

(A2) The function $G \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfies

$$|G| \leq \kappa \Big(|\nabla \Phi|^2 + V + \lambda_1 \Big)^{\frac{1}{2}}$$

for some constants $\kappa \geq 0$ and $\lambda_1 \geq 0$.

(A3) There are constants $\theta < p$ and $\beta \in \mathbb{R}$ such that

$$G \cdot \nabla \Phi - \operatorname{div} G - \theta V \leq \beta.$$

(A4) There are constants $\gamma > 0$ and $\lambda_2 \geq 0$ such that

$$|\nabla V| \leq \gamma V^{\frac{3}{2}} + \lambda_2.$$

(A5) There is a constant $\xi > 0$ such that

$$\Big| G - \frac{p-2}{p} \nabla \Phi \Big| \leq \xi |x|.$$

We mention that under the assumptions (A1) for all $\tau > 0$, (A2), (A3) for some constants $\theta \in \mathbb{R}$, $\beta_1 \in \mathbb{R}$ and (A4) Sobajima–Yokota established in [12, Theorem 1.1] that the operator $A_{\Phi,G,V}$ with domain

$$W_V^{2,p}(\mathbb{R}^N, d\mu) = \big\{ u \in W_\mu^{2,p}(\mathbb{R}^N) : Vu \in L_\mu^p(\mathbb{R}^N) \big\}$$

generates an analytic semigroup on $L_\mu^p(\mathbb{R}^N)$ for $1 < p < \infty$ if

$$\frac{\theta}{p} + (p-1)\gamma \Big(\frac{\kappa}{p} + \frac{\gamma}{4} \Big) < 1.$$

The paper is structured as follows. In Section 2, we prove an L^p-weighted Hardy inequality. Besides, we use them to study the accretivity and the dispersivety of $A_{\Phi,G,V,c}$. In Section 3, we state and prove the main generation results.

2. Hardy inequality

Our main aim of this section is to extend the result of [7, Theorem 2.1] to the whole space $L_\mu^p(\mathbb{R}^N)$ for $1 < p < \infty$.

Theorem 2.1. *Assume $N \geq 3$ and* (A1) *hold. Then, for any $u \in C_c^\infty(\mathbb{R}^N)$, one has*

$$\gamma_N^\star \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} d\mu \leq (4+\sigma) \int_{\mathbb{R}^N} |u|^{p-2} |\nabla u|^2 d\mu + c_\sigma \int_{\mathbb{R}^N} |u|^p d\mu$$

if $p \geq 2$ and

$$\gamma_N^\star \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} d\mu \leq (4+\sigma) \lim_{\delta \to 0} \int_{\mathbb{R}^N} (|u|^2+\delta)^{\frac{p-2}{2}} |\nabla u|^2 d\mu + c_\sigma \int_{\mathbb{R}^N} |u|^p d\mu$$

if $1 < p < 2$, for any $\sigma > 0$ with a corresponding constant $c_\sigma > 0$, where $\gamma_N^\star = \left(\frac{N-2}{p}\right)^2$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$. Take $\delta > 0$ if $1 < p < 2$ and $\delta = 0$ if $p \geq 2$. Hence, we have

$$(|u|^2+\delta)^{\frac{p}{4}}(x) \exp\Big(-\frac{\Phi(x)}{2}\Big) = -\int_1^\infty \frac{d}{dt}\Big((|u|^2+\delta)^{\frac{p}{4}}(tx)\exp(-\frac{\Phi(tx)}{2})\Big) dt.$$

Thus, by a change of variables, it follows that

$$\begin{aligned}
&\Big\| \frac{(|u|^2+\delta)^{\frac{p}{4}}}{|x|} \Big\|_{L^2_\mu} \\
&\leq \Big(\int_1^\infty t^{-\frac{N}{2}} dt\Big) \Big\| \frac{p}{2} \nabla |u| |u| (|u|^2+\delta)^{\frac{p-4}{4}} - \frac{1}{2}(|u|^2+\delta)^{\frac{p}{4}} \nabla \Phi \Big\|_{L^2_\mu} \\
&\leq \Big(\frac{p}{N-2}\Big)^2 \Big\| \nabla |u| |u| (|u|^2+\delta)^{\frac{p-4}{4}} - \frac{1}{p}(|u|^2+\delta)^{\frac{p}{4}} \nabla \Phi \Big\|_{L^2_\mu}.
\end{aligned}$$

Moreover, by using the Höder, Young and Jensen inequalities, we infer that

$$\begin{aligned}
&\Big(\frac{N-2}{p}\Big)^2 \int_{\mathbb{R}^N} \frac{(|u|^2+\delta)^{\frac{p}{2}}}{|x|^2} d\mu \\
&\leq \int_{\mathbb{R}^N} |\nabla |u||^2 |u|^2 (|u|^2+\delta)^{\frac{p-4}{2}} d\mu + \frac{1}{p^2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 (|u|^2+\delta)^{\frac{p}{2}} d\mu \\
&\quad + \frac{2}{p} \int_{\mathbb{R}^N} \nabla \Phi \cdot \nabla |u| |u| (|u|^2+\delta)^{\frac{p-2}{2}} d\mu \\
&\leq \int_{\mathbb{R}^N} |\nabla |u||^2 |u|^2 (|u|^2+\delta)^{\frac{p-4}{2}} d\mu + \frac{1}{p^2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 (|u|^2+\delta)^{\frac{p}{2}} d\mu \\
&\quad + \frac{2}{p} \Big(\int_{\mathbb{R}^N} |u|^2 |\nabla |u||^2 (|u|^2+\delta)^{\frac{p-4}{2}} d\mu \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |\nabla \Phi|^2 (|u|^2+\delta)^{\frac{p}{2}} d\mu \Big)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{p^2} + \frac{\eta}{p}\right) \int_{\mathbb{R}^N} |\nabla \Phi|^2 (|u|^2+\delta)^{\frac{p}{2}} d\mu \\
&\quad + \left(1 + \frac{1}{\eta p}\right) \int_{\mathbb{R}^N} |u|^2 |\nabla |u||^2 (|u|^2+\delta)^{\frac{p-4}{2}} d\mu.
\end{aligned} \tag{2.1}$$

Furthermore, combining integration by parts, (A1) and Young inequalities, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\Phi|^2(|u|^2+\delta)^{\frac{p}{2}}\,d\mu \\
&= \int_{\mathbb{R}^N} \Delta\Phi(|u|^2+\delta)^{\frac{p}{2}}\,d\mu + p\int_{\mathbb{R}^N} \nabla\Phi\cdot\nabla|u||u|(|u|^2+\delta)^{\frac{p-2}{2}}\,d\mu \\
&\le \left(N\tau+\frac{1}{2}\right)\int_{\mathbb{R}^N} |\nabla\Phi|^2(|u|^2+\delta)^{\frac{p}{2}}\,d\mu + NC_\tau\int_{\mathbb{R}^N}(|u|^2+\delta)^{\frac{p}{2}}\,d\mu \\
&\quad + \frac{p^2}{2}\int_{\mathbb{R}^N} |u|^2|\nabla|u||^2(|u|^2+\delta)^{\frac{p-4}{2}}\,d\mu.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\Phi|^2(|u|^2+\delta)^{\frac{p}{2}}\,d\mu \\
&\le \frac{2NC_\tau}{1-2N\tau}\int_{\mathbb{R}^N}(|u|^2+\delta)^{\frac{p}{2}}\,d\mu + \frac{p^2}{1-2N\tau}\int_{\mathbb{R}^N}|u|^2|\nabla|u||^2(|u|^2+\delta)^{\frac{p-4}{2}}\,d\mu
\end{aligned}$$

for every $\tau\in(0,\frac{1}{2N})$. Hence, collecting all the terms and using the identity $|\nabla u|\ge|\nabla|u||$, we conclude that

$$\begin{aligned}
&\Big(\frac{N-2}{p}\Big)^2\int_{\mathbb{R}^N}\frac{(|u|^2+\delta)^{\frac{p}{2}}}{|x|^2}\,d\mu \\
&\le \Big[\left(\frac{1}{p^2}+\frac{\eta}{p}\right)\frac{p^2}{1-2N\tau}+1+\frac{1}{\eta p}\Big]\int_{\mathbb{R}^N}|\nabla u|^2(|u|^2+\delta)^{\frac{p-2}{2}}\,d\mu \\
&\quad + \left(\frac{1}{p^2}+\frac{\eta}{p}\right)\frac{2NC_\tau}{1-2N\tau}\int_{\mathbb{R}^N}(|u|^2+\delta)^{\frac{p}{2}}\,d\mu \\
&\quad - \delta\Big[\left(\frac{1}{p^2}+\frac{\eta}{p}\right)\frac{p^2}{1-2N\tau}+1+\frac{1}{\eta p}\Big]\int_{\mathbb{R}^N}|\nabla|u||^2(|u|^2+\delta)^{\frac{p-4}{2}}\,d\mu.
\end{aligned}$$

So, taking the minimum with respect to η, that is, choosing $\eta=\frac{1}{p}$, τ small, and letting δ go to zeros, we get (2.1). The case where $p\ge 2$ can be handled similarly. □

3. Dissipativity and dispersivity of $A_{\Phi,G,V,c}$

As an application of Theorem 2.1, we establish firstly the dissipativity of the operator $A_{\Phi,G,V,c}$.

Proposition 3.1. *Assume that* (A1) *and* (A3) *hold. Then, the operator* $A_{\Phi,G,V,c}-\gamma_2$ *with domain* $C_c^\infty(\mathbb{R}^N)$ *is dissipative in* $L^p_\mu(\mathbb{R}^N)$ *if and only if* $c\le\gamma_0$, *where* $\gamma_0=\frac{(N-2)^2(p-1)}{4(4+\sigma)}$ *and* $\gamma_2=\frac{\beta}{p}+\frac{c_\sigma(p-1)}{4+\sigma}$, $\sigma>0$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$. Take $\delta > 0$ if $1 < p < 2$ and $\delta = 0$ if $p \geq 2$. Then, by using the identity $\mathrm{Re}(\overline{u}\nabla u) = |u|\nabla|u|$ and integration by parts, it follows that

$$\begin{aligned}
&\mathcal{R}e\langle A_{\Phi,G,V}u, u(|u|^2+\delta)^{\frac{p-2}{2}}\rangle_{L^p_\mu} \\
= &-\int_{\mathbb{R}^N} |\nabla u|^2(|u|^2+\delta)^{\frac{p-2}{2}} d\mu \\
&-(p-2)\int_{\mathbb{R}^N} |\mathrm{Re}(\overline{u}\nabla u)|^2(|u|^2+\delta)^{\frac{p-4}{2}} d\mu \\
&-\frac{1}{p}\int_{\mathbb{R}^N} \Big(\mathrm{div}\, G - G\cdot\nabla\Phi\Big)(|u|^2+\delta)^{\frac{p}{2}} d\mu \\
&-\int_{\mathbb{R}^N} V(|u|^2+\delta)^{\frac{p}{2}} d\mu + \delta\int_{\mathbb{R}^N} V(|u|^2+\delta)^{\frac{p-2}{2}} d\mu.
\end{aligned}$$

So, using the identity $|\nabla u|^2 \geq |\nabla|u||^2$, we obtain

$$\begin{aligned}
&\mathcal{R}e\langle A_{\Phi,G,V}u, u|u|^{p-2}\rangle_{L^p_\mu} \\
\leq &-(p-1)\int_{\mathbb{R}^N} |\nabla|u||^2|u|^{p-2} d\mu - \frac{1}{p}\int_{\mathbb{R}^N} \Big(\mathrm{div}\, G - G\cdot\nabla\Phi\Big)|u|^p d\mu \\
&-\int_{\mathbb{R}^N} V|u|^p d\mu
\end{aligned}$$

if $p \geq 2$ and

$$\begin{aligned}
&\mathcal{R}e\langle A_{\Phi,G,V}u, u|u|^{p-2}\rangle_{L^p_\mu} \\
\leq &-(p-1)\lim_{\delta\to 0^+}\int_{\mathbb{R}^N} |\nabla|u||^2(|u|^2+\delta)^{\frac{p-2}{2}} d\mu \\
&-\frac{1}{p}\int_{\mathbb{R}^N} \Big(\mathrm{div}\, G - G\cdot\nabla\Phi\Big)|u|^p d\mu - \int_{\mathbb{R}^N} V|u|^p d\mu
\end{aligned}$$

if $1 < p < 2$. Using now Theorem 2.1 and (A3), we infer, in both cases, that

$$\begin{aligned}
&\mathcal{R}e\langle A_{\Phi,G,V}u, u|u|^{p-2}\rangle_{L^p_\mu} \\
\leq &-\frac{(N-2)^2(p-1)}{4(4+\sigma)}\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} d\mu + \Big(\frac{\beta}{p} + \frac{c_\sigma(p-1)}{4+\sigma}\Big)\int_{\mathbb{R}^N} |u|^p d\mu.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\mathcal{R}e\langle A_{\Phi,G,V}u + c|x|^{-2}u, u|u|^{p-2}\rangle_{L^p_\mu} \\
\leq &\Big(c - \frac{(N-2)^2(p-1)}{4(4+\sigma)}\Big)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} d\mu + \Big(\frac{\beta}{p} + \frac{c_\sigma(p-1)}{4+\sigma}\Big)\int_{\mathbb{R}^N} |u|^p d\mu.
\end{aligned}$$

Thus, it follows that

$$\mathcal{R}e\langle A_{\Phi,G,V,c}u - \gamma_2 u, u|u|^{p-2}\rangle_{L^p_\mu} \leq 0$$

if and only if $c \leq \gamma_0$ so the proof is now complete. □

Now, we present sufficient conditions for the dispersivity of $A_{\Phi,G,V,c}$.

Proposition 3.2. *Suppose that* (A1) *and* (A3) *are verified. Then, the operator* $A_{\Phi,G,V,c}-\gamma_2$ *with domain* $C_c^\infty(\mathbb{R}^N)$ *is dispersive in* $L^p_\mu(\mathbb{R}^N)$ *if and only if* $c\le\gamma_0$.

Proof. Let $u\in C_c^\infty(\mathbb{R}^N)$ be real-valued and fix $\delta>0$. By straightforward computation we deduce that

$$\begin{aligned}\langle A_{\Phi,G}u,u_+(u_+^2+\delta)^{\frac{p-2}{2}}\rangle_{L^p_\mu} = &-\int_{\mathbb{R}^N}(u_+^2+\delta)^{\frac{p-2}{2}}|\nabla u_+|^2d\mu\\ &-(p-2)\int_{\mathbb{R}^N}(u_+^2+\delta)^{\frac{p-4}{2}}u_+^2|\nabla u_+|^2d\mu\\ &-\frac{1}{p}\int_{\mathbb{R}^N}(u_+^2+\delta)^{\frac{p}{2}}\Big(\operatorname{div}G-G\cdot\nabla\Phi\Big)d\mu\\ &-\int_{\mathbb{R}^N}V(u_+^2+\delta)^{\frac{p}{2}}d\mu+\delta\int_{\mathbb{R}^N}V(u_+^2+\delta)^{\frac{p-2}{2}}d\mu.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\langle A_{\Phi,G}u,(u_+)^{p-1}\rangle_{L^p_\mu}\\ &\le(1-p)\int_{\mathbb{R}^N}u_+^{p-2}|\nabla u_+|^2d\mu-\frac{1}{p}\int_{\mathbb{R}^N}\Big(\operatorname{div}G-G\cdot\nabla\Phi\Big)u_+^pd\mu-\int_{\mathbb{R}^N}Vu_+^pd\mu\end{aligned}$$

if $p\ge2$ and

$$\begin{aligned}&\langle A_{\Phi,G}u,(u_+)^{p-1}\rangle_{L^p_\mu}\\ &\le(1-p)\lim_{\delta\to0}\int_{\mathbb{R}^N}(u_+^2+\delta)^{\frac{p-2}{2}}|\nabla u_+|^2d\mu-\frac{1}{p}\int_{\mathbb{R}^N}\Big(\operatorname{div}G-G\cdot\nabla\Phi\Big)u_+^pd\mu\\ &\quad-\int_{\mathbb{R}^N}Vu_+^pd\mu\end{aligned}$$

if $1<p<2$. Whence, by applying Theorem 2.1 where u is replaced by u_+ and (A3), we get the thesis. □

4. Main result

In this section, we present and prove our main results of this paper. First, we deal with the case when $2p(4+\sigma)\ge N$.

Theorem 4.1. *Let* $1<p<\infty$, $N\ge3$ *and* $\sigma>0$ *such that* $2p(4+\sigma)\ge N$. *Suppose that* (A1)-(A5) *are verified and*

$$\frac{\theta}{p}+(p-1)\gamma\Big(\frac{\kappa}{p}+\frac{\gamma}{4}\Big)<1.$$

Then, for every $c<\alpha_0$, $A_{\Phi,G,V}+c|x|^{-2}$ *endowed with domain* $W^{2,p}_\mu(\mathbb{R}^N)\cap D(|x|^{-2})$ *generates a quasi-contractive analytic semigroup in* $L^p_\mu(\mathbb{R}^N)$. *Furthermore, the closure of* $\big(A_{\Phi,G,V}+\alpha_0|x|^{-2},W^{2,p}_\mu(\mathbb{R}^N)\cap D(|x|^{-2})\big)$ *generates a quasi-contractive semigroup in* $L^p_\mu(\mathbb{R}^N)$.

Proof. As the main consequence of Theorem 1.1 together with Proposition 3.1, we have

$$-A_{\Phi,G,V} + \gamma_2 - c|x|^{-2}$$

with domain $W^{2,p}_\mu(\mathbb{R}^N) \cap D(|x|^{-2})$ is m-accretive if $c < \alpha_0$ and

$$-A_{\Phi,G,V} + \gamma_2 - \alpha_0|x|^{-2}$$

is essentially m-accretive.

Furthermore, thanks to [12, Theorem 1.1], the semigroup generated by $A_{\Phi,G,V}$ is analytic if

$$\frac{\theta}{p} + (p-1)\gamma\Big(\frac{\kappa}{p} + \frac{\gamma}{4}\Big) < 1.$$

Whence, under this condition, $A_{\Phi,G,V}$ is sectorial and therefore there exists γ_p such that

$$|\mathcal{I}m\langle A_{\Phi,G,V}u, |u|^{p-2}u\rangle_{L^p_\mu}| \leq \gamma_p\ \mathcal{R}e\langle A_{\Phi,G,V}u, |u|^{p-2}u\rangle_{L^p_\mu}$$

for every $u \in W^{2,p}(\mathbb{R}^N, d\mu)$. Replacing $A_{\Phi,G,V}$ by $A_{\Phi,G,V} - \gamma_2 + c|x|^{-2}$ where $c \leq \alpha_0$, the above estimate continues to hold for all $u \in W^{2,p}_\mu(\mathbb{R}^N) \cap D(|x|^{-2})$. This means that $A_{\Phi,G,V} + c|x|^{-2}$ is sectorial and whence by virtue of [3, Theorem 1.54], we infer that $A_{\Phi,G,V} + c|x|^{-2}$ generates an analytic semigroup in $L^p_\mu(\mathbb{R}^N)$. ☐

Next, we treat the case when $2p(4+\sigma) \leq N$. In this connection, in order to apply Theorem 1.2, we will need the following result.

Proposition 4.2. *Set* $U_\epsilon = \frac{1}{|x|^2+\epsilon}$. *Assume that* (A1)-(A6) *hold. Then, for every* $u \in C^\infty_c(\mathbb{R}^N)$, *one has*

$$\begin{aligned}&\mathcal{R}e\langle -A_{\Phi,G,V}u + \gamma_2 u, \|U_\epsilon u\|^{2-p}|U_\epsilon u|^{p-2}U_\epsilon u\rangle_{L^p_\mu}\\ &\geq \alpha_0\|U_\epsilon u\|^2_{L^p_\mu} - \alpha_1\|U_\epsilon u\|_{L^p_\mu}\|u\|_{L^p_\mu},\end{aligned}$$

where

$$\alpha_0 = \frac{(p-1)}{p^2}\Big(\frac{N}{4+\sigma} - 2p\Big)N, \qquad \alpha_1 = \frac{2\xi(p-1)}{p}.$$

Proof. Let $u \in C^\infty_c(\mathbb{R}^N)$ and set $u_\delta = ((R|u|)^2+\delta)^{\frac{1}{2}}$ where $R^p = U_\epsilon^{p-1}$. In the computations below, we have to take $\delta > 0$ in the case $1 < p < 2$, whereas we only take $\delta = 0$ to deal with the case $p \geq 2$. We have

$$\begin{aligned}&\langle -A_{\Phi,G,V}u, |U_\epsilon u|^{p-2}U_\epsilon u\rangle_{L^p_\mu}\\ &= \lim_{\delta\to 0}\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\overline{u}(-\Delta u + \nabla\Phi\cdot\nabla u - G\cdot\nabla u + Vu)d\mu.\end{aligned}$$

Integration by parts gives

$$\begin{aligned}&\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\overline{u}(-\Delta u + \nabla\Phi\cdot\nabla u)d\mu\\ &= \int_{\mathbb{R}^N} u_\delta^{p-2}R(\overline{u}\nabla u)\cdot\nabla R d\mu + \int_{\mathbb{R}^N} u_\delta^{p-2}|\nabla(Ru)|^2 d\mu\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} u_\delta^{p-2} u \nabla R \cdot \nabla(R\overline{u}) d\mu \\
&+ (p-2) \int_{\mathbb{R}^N} u_\delta^{p-4} R^2 |u| R(\overline{u}\nabla u) \cdot \nabla(R|u|) d\mu.
\end{aligned}
$$

Since $\mathrm{Re}(\overline{u}\nabla u) = |u|\nabla|u|$, taking the real parts in the identity above we see that

$$
\begin{aligned}
&- \mathcal{R}e \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \overline{u}(\Delta u - \nabla\Phi\cdot\nabla u) d\mu \\
= &\int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(Ru)|^2 d\mu - \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 |\nabla R|^2 d\mu \\
&+ (p-2) \underbrace{\int_{\mathbb{R}^N} u_\delta^{p-4} R^2 |u|^2 R \nabla|u| \cdot \nabla(R|u|) d\mu}_{=I}.
\end{aligned}
$$

Now, we rearrange the last integral in the following way

$$
\begin{aligned}
I = &\int_{\mathbb{R}^N} u_\delta^{p-4} R^2 |u|^2 |\nabla(R|u|)|^2 d\mu - \int_{\mathbb{R}^N} u_\delta^{p-4} R^2 |u|^2 |u| \nabla(R|u|)\cdot\nabla R d\mu \\
= &\int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(R|u|)|^2 d\mu - \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 |\nabla R|^2 d\mu \\
&- \int_{\mathbb{R}^N} u_\delta^{p-2} R|u| \nabla|u| \cdot \nabla R d\mu - \delta \int_{\mathbb{R}^N} u_\delta^{p-4} R \nabla(R|u|)\cdot\nabla|u| d\mu.
\end{aligned}
$$

On the other hand, an integration by parts implies

$$
\begin{aligned}
&- \mathcal{R}e \int_{\mathbb{R}^N} G\cdot(\overline{u}\nabla u) R^2 u_\delta^{p-2} d\mu + \int_{\mathbb{R}^N} V|u|^2 R^2 u_\delta^{p-2} d\mu \\
= &\frac{1}{p} \int_{\mathbb{R}^N} \Big(\operatorname{div} G - G\cdot\nabla\Phi\Big) u_\delta^p d\mu + \int_{\mathbb{R}^N} |u|^2 R(G\cdot\nabla R) u_\delta^{p-2} d\mu \\
&+ \int_{\mathbb{R}^N} V u_\delta^p d\mu - \delta \int_{\mathbb{R}^N} V u_\delta^{p-2} d\mu.
\end{aligned}
$$

Collecting all the terms gives

$$
\begin{aligned}
&\int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \overline{u}(-A_{\phi,G,V} u) d\mu \\
\geq &(p-1) \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(R|u|)|^2 d\mu - (p-1) \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 |\nabla R|^2 d\mu \\
&- (p-2) \int_{\mathbb{R}^N} u_\delta^{p-2} R|u| \nabla|u| \cdot \nabla R d\mu - (p-2)\delta \int_{\mathbb{R}^N} u_\delta^{p-4} R \nabla(R|u|)\cdot\nabla|u| d\mu \\
&+ \frac{1}{p} \int_{\mathbb{R}^N} \Big(\operatorname{div} G - G\cdot\nabla\Phi + \theta V\Big) u_\delta^p d\mu + \int_{\mathbb{R}^N} |u|^2 R(G\cdot\nabla R) u_\delta^{p-2} d\mu \\
&+ \left(1 - \frac{\theta}{p}\right) \int_{\mathbb{R}^N} V u_\delta^p d\mu - \delta \int_{\mathbb{R}^N} V u_\delta^{p-2} d\mu,
\end{aligned}
$$

where we have used the inequality $|\nabla(Ru)| \geq |\nabla(R|u|)|$. Letting $\delta \to 0^+$ and recalling the definition of $R^p = U_\epsilon^{p-1}$, we infer that

$$
\begin{aligned}
&\mathcal{R}e\langle -A_{\Phi,G,V}u, |U_\epsilon u|^{p-2}U_\epsilon u\rangle_{L^p_\mu} \\
&\geq (p-1)\int_{\mathbb{R}^N} |u|^{p-2}R^{p-2}|\nabla(R|u|)|^2 d\mu - (p-1)\int_{\mathbb{R}^N} |u|^p R^{p-2}|\nabla R|^2 d\mu \\
&\quad + \frac{4p(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N} |u|^p|x|^2 U_\epsilon^{p+1} d\mu - \frac{2N(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N} |u|^p U_\epsilon^p d\mu \\
&\quad - \frac{p-2}{p^2}\int_{\mathbb{R}^N} \nabla\Phi\cdot\nabla R^p|u|^p d\mu + \frac{1}{p}\int_{\mathbb{R}^N}\Big(\operatorname{div} G - G\cdot\nabla\Phi + \theta V\Big)R^p|u|^p d\mu \\
&\quad + \int_{\mathbb{R}^N} |u|^p R(G\cdot\nabla R)R^{p-2} d\mu.
\end{aligned}
$$

Thus, combining Theorem 1.2, the inequalities $|\nabla U_\epsilon| \leq 2U_\epsilon^{\frac{3}{2}}$, the assumptions (A3) and the Young inequality, we deduce that

$$
\begin{aligned}
&\mathcal{R}e\langle -A_{\Phi,G,V}u, |U_\epsilon u|^{p-2}U_\epsilon u\rangle_{L^p_\mu} \\
&\geq \frac{(p-1)(N-2)^2}{p^2(4+\sigma)}\int_{\mathbb{R}^N} U_\epsilon^{p-1}\frac{|u|^p}{|x|^2} d\mu - \frac{C_\sigma(p-1)}{4+\sigma}\int_{\mathbb{R}^N} U_\epsilon^{p-1}|u|^p d\mu \\
&\quad - 4\frac{(p-1)^3}{p^2}\int_{\mathbb{R}^N} |x|^2 U_\epsilon^{p+1}|u|^p d\mu + \frac{4p(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N} |u|^p|x|^2 U_\epsilon^{p+1} d\mu \\
&\quad - \frac{2N(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N} |u|^p U_\epsilon^p d\mu - \frac{\beta}{p}\int_{\mathbb{R}^N} U_\epsilon^{p-1}|u|^p d\mu \\
&\quad - \frac{p-2}{p^2}\int_{\mathbb{R}^N} \nabla\Phi\cdot\nabla U_\epsilon^{p-1}|u|^p d\mu + \frac{1}{p}\int_{\mathbb{R}^N} G\cdot\nabla U_\epsilon^{p-1}|u|^p d\mu.
\end{aligned}
$$

Thus, by means of the inequality $|x|^2 U_\epsilon \leq 1$ and (A5), we obtain

$$
\begin{aligned}
&\mathcal{R}e\langle -A_{\Phi,G,V}u, |U_\epsilon u|^{p-2}U_\epsilon u\rangle_{L^p_\mu} \\
&\geq \frac{(p-1)(N-2)^2}{p^2(4+\sigma)}\int_{\mathbb{R}^N} U_\epsilon^p|u|^p d\mu - \frac{C_\sigma(p-1)}{4+\sigma}\int_{\mathbb{R}^N} U_\epsilon^{p-1}|u|^p d\mu \\
&\quad - 4\frac{(p-1)^3}{p^2}\int_{\mathbb{R}^N} U_\epsilon^p|u|^p d\mu + \frac{4p(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N} |u|^p|x|^2 U_\epsilon^{p+1} d\mu \\
&\quad - \frac{2N(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N} |u|^p U_\epsilon^p d\mu - \frac{\beta}{p}\int_{\mathbb{R}^N} U_\epsilon^{p-1}|u|^p d\mu \\
&\quad - \frac{2\xi(p-1)}{p}\int_{\mathbb{R}^N} U_\epsilon^{p-1}|u|^p d\mu.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\mathcal{R}e\langle -A_{\Phi,G,V}u + \gamma_2 u, |U_\epsilon u|^{p-2}U_\epsilon u\rangle_{L^p_\mu} \\
&\geq \alpha_0\int_{\mathbb{R}^N} U_\epsilon^p|u|^p d\mu - \alpha_1\int_{\mathbb{R}^N} U_\epsilon^{p-1}|u|^p d\mu.
\end{aligned}
$$

This completes the proof. □

We now come to state and establish the second main result of this paper.

Theorem 4.3. *Let* $1 < p < \infty$, $N \geq 3$ *and* $\sigma > 0$ *such that* $2p(4+\sigma) < N$ *and set* $\nu = \min\{\alpha_0, \gamma_0\}$, *where* $\gamma_0 = \frac{(p-1)(N-2)^2}{p^2(4+\sigma)}$ *and* $\alpha_0 = \frac{(p-1)}{p^2}\Big(\frac{N}{4+\sigma} - 2p\Big)N$. *Assume that* (A1)-(A5) *hold. Then, for every* $c < \nu$, $A_{\Phi,G,V} + c|x|^{-2}$ *endowed with domain* $W^{2,p}(\mathbb{R}^N, d\mu)$ *generates a quasi-contractive positive semigroup in* $L^p_\mu(\mathbb{R}^N)$ *and* $C_c^\infty(\mathbb{R}^N)$ *is a core for such an operator. Moreover, the closure of* $\big(A_{\Phi,G,V} + \nu|x|^{-2}, W^{2,p}(\mathbb{R}^N, d\mu)\big)$ *generates a quasi-contractive semigroup in* $L^p_\mu(\mathbb{R}^N)$.

Proof. Our purpose is to apply Theorem 1.2. Indeed, set $A = -A_{\Phi,G,V} + \gamma_2$ with $D(A) = W^{2,p}_\mu(\mathbb{R}^N)$ and let B be the multiplicative operator by $|x|^{-2}$ endowed with the maximal domain $D(|x|^{-2}) = \big\{u \in L^p_\mu(\mathbb{R}^N) : |x|^{-2}u \in L^p_\mu(\mathbb{R}^N)\big\}$ in $L^p_\mu(\mathbb{R}^N)$. We mention that the Yosida approximation B_ϵ of B is the multiplicative operator by $U_\epsilon = (|x|^2 + \epsilon)^{-1}$. Both A and B are m-accretive in $L^p_\mu(\mathbb{R}^N)$. Set $D = C_c^\infty(\mathbb{R}^N)$. Then, Proposition 4.2 yields (i) with $k_1 = \alpha_0$, $d = 0$ and $a = \alpha_1$. The second assumption (ii) is obviously satisfied. Moreover, (iii) holds with $\gamma_0 = k_2$ thanks to Proposition 3.1. As a consequence of Theorem 1.2, we infer that for every $c > -\nu$, $-A_{\Phi,G,V} + \gamma_2 + c|x|^{-2}$ with domain $W^{2,p}_\mu(\mathbb{R}^N)$ is m-accretive in $L^p_\mu(\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R}^N)$ is a core for $-A_{\Phi,G,V} + \gamma_2 + c|x|^{-2}$. In addition, $-A_{\Phi,G,V} + \gamma_2 - \nu|x|^{-2}$ is essentially m-accretive. The generation results follow then by Lumer Phillips Theorem [2, Theorem 3.15]. Lastly, the positivity of the generated semigroups follows by virtue of Proposition 3.2, which implies that $A_{\Phi,G,V} - \gamma_2 - c|x|^{-2}$ is dispersive for every $c \geq -\nu$. The dispersivity is equivalent to the positivity of the resolvent, which is equivalent to the positivity of the semigroup, we complete so the proof of our results. □

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