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THE CLASS OF p-DEMICOMPACT OPERATORS ON LATTICE NORMED SPACES

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Abstract. In the present paper, we introduce a new class of operators called p-demicompact operators between two lattice normed spaces X and Y. We study the basic properties of this class. Precisely, we give some conditions under which a p-bounded operator be p-demicompact. Also, a sufficient condition is given, under which each p-demicompact operator has a modulus which is p-demicompact. Further, we put in place some properties of this class of operators on lattice normed spaces.

1. Introduction

In 1936, lattice-normed spaces were first defined by L. Kantorovich in [18]. After that, the theory of lattice-normed spaces was studied and then well-developed by S. Kutateladze, and A. Kusraev. Many results from ergodic theory, probability theory have been extended to lattice-normed vector lattices (see for instance [9, 15, 16]). It should be noticed that the theory of lattice-normed spaces was always studied under the condition of decomposability of lattice norm in [7, 10, 22, 23]. In this paper, we develop a general approach to lattice-normed vector lattices without requiring decomposability of lattice norm. We recall that a vector lattice X equipped with a norm $\|\cdot\|$ is said to be a normed lattice if $|x| \leq |y|$ in X implies $\|x\| \leq \|y\|$. If a normed lattice is norm complete, then it is called a Banach lattice.

Recently, based on the theory of Banach lattice and the class of demicompactness, H. Benkhaled et al. in [6] introduced the notion of order weakly demicompact operator. Note that, the class of demicompactness was used by W. V. Petryshyn in [25, 26] to construct and investigate the structure of fixed point sets for nonlinear operators acting on Hilbert and Banach spaces. Let us recall from [25] that an operator $T : D \subset X \longrightarrow X$ is said to be demicompact if every bounded sequence $(x_n)_n$ in D such that $(x_n - Tx_n)_n$ converges strongly, has a convergent subsequence. Further, several results focused on this

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class which contains compact operators [8, 19, 20, 25] and its important role in spectral theory (see [11, 13, 17]). Some other analyzes were related with the class of demicompact linear operators (see for instance [2, 12, 14, 21, 27]).

Motivated by demicompactness and related results on bounded linear operators acting on lattice spaces, we introduce in this paper the notion of p-demicompact operator. Then, we use the framework of the theory of lattice normed spaces to provide a systematic approach to the demicompactness criteria and generalize some results regarding the characterization of p-compact operators between lattice normed spaces, which was introduced by A. Aydın et al. in [4]. These operators generalize several known classes of operators such as compact, weakly compact and order weakly compact. Let us recall that, a linear operator T between two lattices normed spaces is said to be p-compact if, for any p-bounded net $(x_\alpha)_\alpha$, the net $(Tx_\alpha)_\alpha$ has a p-convergent subnet.

In what follows, we present some notations and recall some basic definitions that will be used in the sequel. Let $\leq$ be an order relation on a real vector space X. Then X is called an ordered vector space, if it satisfies the following conditions:

(i) $x \leq y$ implies $x + z \leq y + z$ for all $z \in X$.
(ii) $x \leq y$ implies $\lambda x \leq \lambda y$ for all $\lambda \in \mathbb{R}_+$.

For an ordered vector space X we let $X_+ := \{x \in X : x \geq 0\}$. The subset X_+ is called the positive cone of X. For each x and y in an ordered vector space X we let $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$. If $x \in X_+$ and $x \neq 0$, then we write $x > 0$. A net $(x_\alpha)_\alpha$ in a vector lattice X is order convergent (or o-convergent for short) to $x \in X$, if there exists another net $(y_\beta)_\beta$ satisfying $y_\beta \downarrow 0$ and for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. In this case we write $x_\alpha \xrightarrow{o} x$. A vector $e > 0$ is called a strong unit in vector lattice E if, for every $x \in E$, there exists a positive number λ, depending on x, such that $|x| \leq \lambda e$.

Let X be a vector space, E be a vector lattice and $p : X \to E_+$ be a vector norm, i.e.,

(i) $p(x) = 0 \Longleftrightarrow x = 0$.
(ii) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R},\ x \in X$.
(iii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Then the triple (X, p, E) is called a lattice-normed-space abbreviated as LNS. The lattice norm p in an LNS (X, p, E) is said to be decomposable if for all $x \in X$ and $e_1, e_2 \in E_+$ it follows from $p(x) = e_1 + e_2$, that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$, $k = 1, 2$. If X is a vector lattice and the vector norm p is monotone ($|x| \leq |y| \Longrightarrow p(x) \leq p(y)$), then the triple (X, p, E) is called a lattice-normed vector lattice abbreviated as LNVL. We abbreviate the convergence $p(x_\alpha - x) \xrightarrow{o} 0$ as $x_\alpha \xrightarrow{p} x$ and say that x_α p-converges to x.

In an LNS (X, p, E) a subset A of X is called p-bounded if there exists $e \in E$ such that $p(a) \leq e$ for all $a \in A$. An LNVL (X, p, E) is called op-continuous if $x_\alpha \xrightarrow{o} 0$ implies that $p(x_\alpha) \xrightarrow{o} 0$. Consider LNSs (X, p, E) and (Y, m, F).

A linear operator $T : X \to Y$ is said to be dominated if there is a positive operator $S : E \to F$ satisfying

$$m(T(x)) \leq S(p(x)) \text{ for all } x \in X.$$

In this case, S is called a dominant for T. The set of all dominated operators from X to Y is denoted by $M(X, Y)$.

The sets $\mathcal{L}(X, Y)$ and $L^{\sim}(E, F)$ denote, respectively, the space of all linear operators between vector spaces X and Y, and the ordered vector spaces of all order bounded operators from E into F. Recall that $T \in \mathcal{L}(X; Y)$, where X and Y are normed spaces, is called a Dunford-Pettis operator if $x_n \xrightarrow{w} 0$ in X implies that $Tx_n \xrightarrow{\|\cdot\|} 0$ in Y, here $\xrightarrow{w}$ denotes the weak convergence. For an operator $T \in \mathcal{L}(X; Y)$, the range of T is denoted by $\mathcal{R}(T)$.

A normed lattice $(X, \|\cdot\|)$ is called order continuous if a net $x_\alpha \downarrow 0$ in X implies $\|x\| \downarrow 0$ or equivalently $x_\alpha \xrightarrow{o} 0$ in X implies $\|x_\alpha\| \to 0$. A normed lattice $(X, \|\cdot\|)$ is called σ-order continuous if a sequence $x_n \downarrow 0$ in X implies $\|x_n\| \downarrow 0$ or equivalently $x_n \xrightarrow{o} 0$ in X implies $\|x_n\| \to 0$. Every order continuous normed lattice is σ-order continuous. A normed lattice $(X, \|\cdot\|)$ is called a KB-space if for $0 \leq x_\alpha \uparrow$ and $\sup_\alpha \|x_\alpha\| < \infty$ we get that the net (x_α) is norm convergent. A positive vector $a \neq 0$ in a vector lattice X is called atom if, for any $x \in [0, a]$, there is $\lambda \in \mathbb{R}$ such that $x = \lambda a$.

Let a be an atom in a vector lattice X. The principal band B_a generated by a is a projection band, and $B_a = I_a = \text{span}\{a\} = \{\lambda a, \lambda \in \mathbb{R}\}$, where I_a is the ideal generated by a. A vector lattice X is called atomic if the band generated by its atoms is X. If a vector lattice X is atomic, then for any $x > 0$, there is an atom a such that $a \leq x$. A Banach lattice E is said to be an AM-space if $x \wedge y = 0$ in E implies $\|x \vee y\| = \max\{\|x\|; \|y\|\}$. It is know that, in an AM-space with strong unit, every norm bounded set is order bounded. For more details on lattice spaces, the reader can see ([1, 3, 5, 7, 22, 24]).

An outline of this paper is as follows: In Section 2, we introduce a new class of operators, called p-demicompact (Definition 2.1). Then, we use our new class to generalize some results regarding the characterizations of the operators p-compact. Furthermore, we give relations between demicompact operator on acting mixed norm and p-demicompact operators. Note also that under a sufficient condition, we show that each p-demicompact operator has a modulus which is p-demicompact (see Theorem 2.5). We end this paper by studying the relationship between polynomially p-demicompact and p-demicompact operators (see Theorem 2.6).

2. Main results

Definition 2.1. Let X, Y be two LNSs and $T \in \mathcal{L}(X, Y)$ such that $\mathcal{R}(T) \subset X$. T is called p-demicompact if, for every p-bounded net $(x_\alpha)_\alpha$ in X such that $x_\alpha - Tx_\alpha \xrightarrow{p} y$, there exists a p-convergent subnet $(x_{\alpha\beta})_\beta$.

In the next theorem, we show that if a net of p-demicompact dominated operators p-convergent to a dominated operator, then it is also p-demicompact.

Theorem 2.1. *Let (X,p,E) be a decomposable LNS and (Y,q,F) LNS such that F is order complete. Let $(T_m)_m$ be a sequence in $M(X,Y)$ such that $\mathcal{R}(T_m)$ and $\mathcal{R}(T)$ are subsets of X. If each T_m is p-demicompact with $T_m \xrightarrow{p} T$ in $M(X,Y)$, then T is a p-demicompact operator.*

Proof. Let x_α be a p-bounded net in X such that $x_\alpha - Tx_\alpha \xrightarrow{p} y$. Since x_α is a p-bounded net in X, there is $e \in E_+$ such that $p(x_\alpha) \leq e$ for all α. Now, we can write

$$x_\alpha - T_m x_\alpha = x_\alpha - T_m x_\alpha + Tx_\alpha - Tx_\alpha,$$

therefore, we get

$$\begin{aligned} q(x_\alpha - T_m x_\alpha) &= q(x_\alpha - T_m x_\alpha + Tx_\alpha - Tx_\alpha) \\ &\leq q(x_\alpha - Tx_\alpha) + q(T_m x_\alpha - Tx_\alpha). \end{aligned}$$

Since $T_m \in M(X,Y)$ for all $m \in \mathbb{N}$, we have

$$q(T_m x_\alpha - Tx_\alpha) \leq |T_m - T|(p(x_\alpha)) \leq |T_m - T|(e).$$

Since $T_m \xrightarrow{p} T$ in $M(X,Y)$, by Theorem VIII.2.3 [28] it follows that $|T_m - T|(e) \xrightarrow{o} 0$ in F as $n \to \infty$. On the other hand, by hypothesis, we have $x_\alpha - Tx_\alpha \xrightarrow{p} y$, this implies that $q(x_\alpha - Tx_\alpha) \xrightarrow{o} y$. Hence, we deduce that $q(x_\alpha - T_m x_\alpha) \xrightarrow{o} y$. Thus, $x_\alpha - T_m x_\alpha \xrightarrow{p} y$. Now, using the fact that T_m is p-demicompact, we infer that there exists a p-convergent subnet $(x_{\alpha\beta})_\beta$ of $(x_\alpha)_\alpha$. Consequently, T is p-demicompact. ☐

In following two propositions, we have relations between demicompact operators on acting mixed norm and p-demicompact operators.

Proposition 2.1. *Let (X,p,E) be an LNS, where $(E,\|\cdot\|_E)$ is a normed lattice and (Y,q,F) be an LNS, where $(F,\|\cdot\|_F)$ is a Banach lattice. Let $T:(X,p,\|\cdot\|_E) \to (Y,q,\|\cdot\|_F)$ such that $\mathcal{R}(T) \subset X$. If T is demicompact, then $T:(X,p,E) \to (Y,q,F)$ is p-demicompact.*

Proof. Let $(x_\alpha)_\alpha$ be a p-bounded net in X such that $x_\alpha - Tx_\alpha \xrightarrow{p} y$. Since x_α is a p-bounded net in X, there is $e \in E$ such that $p(x_\alpha) \leq e$ for all α. So, $\|p(x_\alpha)\|_E \leq \|e\|_E < \infty$. Hence, x_α is norm bounded in $(X,p,\|\cdot\|_E)$. This allows us to get $q(x_\alpha - Tx_\alpha) \xrightarrow{o} y$ or q-$\|x_\alpha - Tx_\alpha\|_F \to y$. Since T is demicompact, there exists a subnet $x_{\alpha\beta}$ such that q-$\|x_{\alpha\beta} - x\|_F \to 0$ or $\|q(x_{\alpha\beta} - x)\|_F \to 0$. Since $(F,\|\cdot\|_F)$ is a Banach lattice, by Theorem VII.2.1 [28] there is a further subnet $x_{\alpha\beta_k}$ such that $q(x_{\alpha\beta_k} - x) \xrightarrow{o} 0$. Therefore, $x_{\alpha\beta_k} \xrightarrow{p} x$. Consequently, T is a p-demicompact operator. ☐

Proposition 2.2. *Let (X,p,E) be an LNS, where $(E,\|\cdot\|_E)$ is an AM-space with a strong unit. Let (Y,q,F) be an LNS, where $(F,\|\cdot\|_F)$ is an order* *continuous normed lattice. If $T : (X, p, E) \to (Y, q, F)$ such that $\mathcal{R}(T) \subset X$ is p-demicompact, then $T : (X, p, \|\cdot\|_E) \to (Y, q, \|\cdot\|_F)$ is demicompact.*

Proof. Let $(x_\alpha)_\alpha$ be a normed bounded net in $(X, p, \|\cdot\|_E)$ that is $\|p(x_\alpha)\|_E \leq k < \infty$ for all α such that $\|x_\alpha - Tx_\alpha\|_F \to y$. Since $(E, \|\cdot\|_E)$ is an AM-space with a strong unit, $p(x_\alpha)$ is order bounded in E. Thus x_α is a p-bounded net in (X, p, E). Thus, we have $x_\alpha - Tx_\alpha \xrightarrow{p} y$. Now, from the p-demicompactness of T, it follows that there exists a subnet $x_{\alpha\beta}$ such that $x_{\alpha\beta} \xrightarrow{p} x$, then $q(x_{\alpha\beta} - x) \xrightarrow{o} 0$ in F. Since $(F, \|\cdot\|_F)$ is order continuous, $\|q(x_{\alpha\beta} - x)\|_F \to 0$. Hence, $\|x_{\alpha\beta}\|_E \to x$. Consequently, T is a demicompact operator. □

It is known that a finite rank operator is demicompact. Similarly, we have the following result.

Proposition 2.3. *Let (X, p, E) and (Y, q, F) be LNSs. Let $T : (X, p, E) \to (Y, q, F)$ such that $\mathcal{R}(T) \subset X$. If T is a p-bounded finite rank operator, then T is p-demicompact.*

Proof. Without loss of generality, we may suppose that T is given by $Tx = f(x)y_0$ for some p-bounded functional $f : (X, p, E) \to (\mathbb{R}, |\cdot|, \mathbb{R})$ and $y_0 \in Y$. Let x_α be a p-bounded net in X such that $x_\alpha - Tx_\alpha \xrightarrow{p} y$. Since x_α is a p-bounded net in X, $f(x_\alpha)$ is bounded in $\mathbb{R}$. So there is a subnet $x_{\alpha\beta}$ such that $f(x_{\alpha\beta}) \to \lambda$ for some $\lambda \in \mathbb{R}$. Now, we have

$$\begin{aligned} q(x_{\alpha\beta} - \lambda y_0) &= q(x_{\alpha\beta} - Tx_{\alpha\beta} + Tx_{\alpha\beta} - \lambda y_0) \\ &\leq q(x_{\alpha\beta} - Tx_{\alpha\beta}) + q(Tx_{\alpha\beta} - \lambda y_0). \end{aligned}$$

We have

$$\begin{aligned} q(Tx_{\alpha\beta} - \lambda y_0) &= q(f(x_{\alpha\beta})y_0 - \lambda y_0) \\ &= |f(x_{\alpha\beta}) - \lambda|\, q(y_0) \xrightarrow{o} 0. \end{aligned}$$

Further, by hypothesis, we have $x_\alpha - Tx_\alpha \xrightarrow{p} y$, this implies that $q(x_\alpha - Tx_\alpha) \xrightarrow{o} y$. Hence, we deduce that $q(x_{\alpha\beta} - \lambda y_0) \xrightarrow{o} y$. Consequently, $x_{\alpha\beta} - \lambda y_0 \xrightarrow{p} y$. Thus, T is p-demicompact. □

Remark 2.1 ([4]). If X is an atomic KB-space, then every order bounded net has an order convergent subnet.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a normed space. Then $x_n \xrightarrow{\|\cdot\|} x$ if and only if for any subsequence $(x_{n_k})_k$, there is a further subsequence $(x_{n_{k_j}})_j$ such that $x_{n_{k_j}} \xrightarrow{\|\cdot\|} x$.*

The following proposition gives information about when an order bounded operator is p-demicompact.

Proposition 2.4. *Let X be a vector lattice and (Y, q, F) be an op-continuous LNVL such that Y is an atomic KB-space. If $T \in \mathcal{L}^{\sim}(X, Y)$ such that $\mathcal{R}(T) \subset X$, then $T : (X, |\cdot|, X) \to (Y, q, F)$ is p-demicompact.*

Proof. Let $(x_\alpha)_\alpha$ be a p-bounded net in $(X, |\cdot|, X)$ such that $x_\alpha - Tx_\alpha \xrightarrow{p} y$. Since x_α be a p-bounded net in $(X, |\cdot|, X)$, x_α is order bounded in X. The fact that T is order bounded, allows us to get $(Tx_\alpha)_\alpha$ is order bounded in Y, which is an atomic KB-space, so from Remark 2.1 there are a subnet $x_{\alpha\beta}$ and $z \in Y$ such that $Tx_{\alpha\beta} \xrightarrow{o} z$. Since (Y, q, F) is op-continuous, $q(Tx_{\alpha\beta} - z) \xrightarrow{o} 0$. By hypothesis, we have $x_\alpha - Tx_\alpha \xrightarrow{p} y$, which implies that $q(x_\alpha - Tx_\alpha - y) \xrightarrow{o} 0$. Now, we can write $x_{\alpha\beta} = x_{\alpha\beta} - Tx_{\alpha\beta} + Tx_{\alpha\beta}$. Thus

$$\begin{aligned} q(x_{\alpha\beta} - (y + z)) &= q(x_{\alpha\beta} - Tx_{\alpha\beta} - y + Tx_{\alpha\beta} - z) \\ &\leq q(x_{\alpha\beta} - Tx_{\alpha\beta} - y) + q(Tx_{\alpha\beta} - z) \xrightarrow{o} 0. \end{aligned}$$

Hence, we deduce that $q(x_{\alpha\beta} - (y + z)) \xrightarrow{o} 0$. Thus, $x_{\alpha\beta} \xrightarrow{p} y + z$. Consequently, T is p-demicompact. □

In the next proposition, under some conditions, we see that p-bounded operator is p-demicompact.

Proposition 2.5. *Let (X, p, E) and $(Y, |\cdot|, Y)$ be two LNVLs such that Y is an atomic KB-space. If $T : (X, p, E) \to (Y, |\cdot|, Y)$ such that $\mathcal{R}(T) \subset X$ is p-bounded, then T is p-demicompact.*

Proof. Let $(x_\alpha)_\alpha$ be a p-bounded net in $(X, |\cdot|, X)$ such that $x_\alpha - Tx_\alpha \xrightarrow{p} y$. From the fact that T is p-bounded, it follows that Tx_α is order bounded in Y. Since Y is an atomic KB-space, by Remark 2.1, there is a subnet $x_{\alpha\beta}$ and $z \in Y$ such that $Tx_{\alpha\beta} \xrightarrow{o} z$. Now, we have $x_{\alpha\beta} = x_{\alpha\beta} - Tx_{\alpha\beta} + Tx_{\alpha\beta}$, which implies that

$$\begin{aligned} |x_{\alpha\beta} - (y + z)| &= |x_{\alpha\beta} - Tx_{\alpha\beta} - y + Tx_{\alpha\beta} - z| \\ &\leq |x_{\alpha\beta} - Tx_{\alpha\beta} - y| + |Tx_{\alpha\beta} - z| \xrightarrow{o} 0. \end{aligned}$$

Hence, $|x_{\alpha\beta}| \xrightarrow{o} y + z$. Thus, $x_{\alpha\beta} \xrightarrow{p} y + z$. Consequently, T is p-demicompact. □

In the next examples, we see that we can not omit the atomicity in Propositions 2.4 and 2.5.

Example 2.1. We consider the identity operator

$$I : (L_1[0; 1]; |\cdot|; L_1[0; 1]) \to (L_1[0; 1]; |\cdot|; L_1[0; 1]).$$

The sequence of Rademacher functions, that is the function $r_n : [0; 1] \to \mathbb{R}$ defined by

$$r_n(t) = \operatorname{sgn} \sin(2^n \pi t) \quad \text{for } t \in [0; 1],$$

is order bounded by 1 and has no order convergent subsequence. Indeed, let r_{nk} be a subsequence of r_n such that

$$r_{nk} \to f.$$

Then

$$r_{nk}(x) \to f(x) \text{ a.e. for each } x \in [0;1].$$

But, for each $x \in [0;1]$, there are infinitely many n's such that $r_{nk}(x) = 1$ and infinitely many n's such that $r_{nk}(x) = -1$. So, I is not p-demicompact.

Example 2.2. The identity operator $I : (l_1; |\cdot|; l_1) \to (l_1; |\cdot|; l_1)$ satisfies the conditions of Proposition 2.4, where $X = l_1$, $(Y; m; F) = (l_1; |\cdot|; l_1)$, $Y = l_1$, which is an atomic KB-space, because l_1 has no copy of c_0 (note that $x = \left(\frac{1}{n}\right) \in c_0$ but $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, so $\left(\frac{1}{n}\right) \notin l_1$), and also $(l_1; |\cdot|; l_1)$ is op-continuous, so I is p-demicompact. This shows that the identity operator on an infinite dimensional space can be p-demicompact.

Theorem 2.2. *Let (X, p, E) be an LNS with $(E, \|\cdot\|_E)$ be an order continuous Banach lattice. $T : (X, p, E) \to (X, p, E)$ is p-demicompact if and only if T is order weakly demicompact.*

Proof. Assume that T is p-demicompact. Let x_α be an order bounded net in E such that $x_\alpha \xrightarrow{w} 0$ and $\|x_\alpha - Tx_\alpha\|_E \to 0$. We have $\|x_\alpha - Tx_\alpha\|_E \to 0$, since $(E, \|\cdot\|_E)$ is a Banach lattice, it follows that there is a subnet x_{α_β} such that $x_{\alpha_\beta} - Tx_{\alpha_\beta} \xrightarrow{o} 0$ in E. From Theorem VII.2.1 [28], we get $x_{\alpha_\beta} - Tx_{\alpha_\beta} \xrightarrow{p} 0$ in (X, p, E). Now, taking into account that T is p-demicompact, there exists a subnet $x_{\alpha_{\beta_k}}$ such that $x_{\alpha_{\beta_k}} \xrightarrow{p} 0$. Now, we can write $Tx_{\alpha_{\beta_k}} = Tx_{\alpha_{\beta_k}} - x_{\alpha_{\beta_k}} + x_{\alpha_{\beta_k}}$. We get

$$\begin{aligned} p(Tx_{\alpha_{\beta_k}}) &= p(Tx_{\alpha_{\beta_k}} - x_{\alpha_{\beta_k}} + x_{\alpha_{\beta_k}}) \\ &\le p(Tx_{\alpha_{\beta_k}} - x_{\alpha_{\beta_k}}) + p(x_{\alpha_{\beta_k}}) \to 0. \end{aligned}$$

Which implies that $p(Tx_{\alpha_{\beta_k}}) \to 0$. So, $Tx_{\alpha_{\beta_k}} \xrightarrow{o} 0$. Thus, since $(E, \|\cdot\|_E)$ is order continuous, we obtain that $\|Tx_{\alpha_{\beta_k}}\|_E \to 0$. Hence, from Lemma 2.1, we get $\|Tx_\alpha\|_E \to 0$. Further, we have $x_\alpha = x_\alpha - Tx_\alpha + Tx_\alpha$. Thus,

$$\begin{aligned} \|x_\alpha\|_E &= \|x_\alpha - Tx_\alpha + Tx_\alpha\|_E \\ &\le \|x_\alpha - Tx_\alpha\|_E + \|Tx_\alpha\|_E \to 0. \end{aligned}$$

Hence, we deduce that $\|x_\alpha\|_E \to 0$. Consequently, T is order weakly demicompact. To prove the converse. Assume that T is order weakly demicompact. Let x_α be a p-bounded net in E such that $x_\alpha \xrightarrow{w} 0$ and $x_\alpha - Tx_\alpha \xrightarrow{p} 0$ in (X, p, E). This implies that $x_\alpha - Tx_\alpha \xrightarrow{o} 0$ in E. Since $(E, \|\cdot\|_E)$ is an order continuous Banach lattice, $\|x_\alpha - Tx_\alpha\|_E \to 0$. Now, from the fact that T is order weakly demicompact, we get $\|x_\alpha\|_E \to 0$. Therefore, since $(E, \|\cdot\|_E)$ is a Banach lattice, there is a subnet x_{α_β} such that $x_{\alpha_\beta} \xrightarrow{o} 0$ in E and so $x_{\alpha_\beta} \xrightarrow{p} 0$ in (X, p, E). □

Theorem 2.3. *Let (X, p, E) be an LNS with $(E, \|\cdot\|_E)$ is a Banach lattice. Let $T : (X, p, E) \to (X, p, E)$ such that T is Dunford-Pettis. If T is p-demicompact, then T is order weakly demicompact.*

Proof. Let $(x_\alpha)_\alpha$ be an order bounded net in E such that $x_\alpha \xrightarrow{w} 0$ and $\|x_\alpha - Tx_\alpha\|_E \to 0$. We have $\|x_\alpha - Tx_\alpha\|_E \to 0$, since $(E, \|\cdot\|_E)$ is a Banach lattice, there is a subnet x_{α_β} such that $x_{\alpha_\beta} - Tx_{\alpha_\beta} \xrightarrow{o} 0$ in E. So, we get $x_{\alpha_\beta} - Tx_{\alpha_\beta} \xrightarrow{p} 0$ in (X, p, E). Now, since T is p-demicompact, there exists a subnet $x_{\alpha_{\beta_k}}$ such that $x_{\alpha_{\beta_k}} \xrightarrow{p} 0$. Since $x_\alpha \xrightarrow{w} 0$ and T is Dunford-Pettis, we obtain $\|Tx_\alpha\|_E \to 0$, so we have $\|Tx_{\alpha_{\beta_k}}\|_E \to 0$. Now, we can write $x_{\alpha_{\beta_k}} = x_{\alpha_{\beta_k}} - Tx_{\alpha_{\beta_k}} + Tx_{\alpha_{\beta_k}}$. Thus

$$\begin{aligned}\|x_{\alpha_{\beta_k}}\|_E &= \|x_{\alpha_{\beta_k}} - Tx_{\alpha_{\beta_k}} + Tx_{\alpha_{\beta_k}}\|_E \\ &\leq \|x_{\alpha_{\beta_k}} - Tx_{\alpha_{\beta_k}}\|_E + \|Tx_{\alpha_{\beta_k}}\|_E \to 0.\end{aligned}$$

Hence, we deduce that $\|x_{\alpha_{\beta_k}}\|_E \to 0$. Now, applying Lemma 2.1, we infer that $\|x_\alpha\|_E \to 0$. Consequently, T is order weakly demicompact. □

Theorem 2.4. *Let (X, p, E) be an LNS with $(E, \|\cdot\|_E)$ is order continuous. Let $T : (X, p, E) \to (X, p, E)$. If T is order weakly demicompact, then T is p-demicompact.*

Proof. Let $(x_\alpha)_\alpha$ be a p-bounded net, then order bounded in E such that $x_\alpha \xrightarrow{w} 0$ and $x_\alpha - Tx_\alpha \xrightarrow{p} 0$. We have $x_\alpha - Tx_\alpha \xrightarrow{p} 0$, then $x_\alpha - Tx_\alpha \xrightarrow{o} 0$ in E. Since $(E, \|\cdot\|_E)$ is order continuous, we get $\|x_\alpha - Tx_\alpha\|_E \to 0$. Now, from the fact that T is order weakly demicompact, we obtain that $\|x_\alpha\|_E \to 0$. Thus, using Lemma 2.1, there is further subnet such that $\|x_{\alpha_{\beta_k}}\|_E \to 0$. Hence, we deduce that $x_{\alpha_{\beta_k}} \xrightarrow{\mathrm{p}} 0$. Consequently, T is p-demicompact. □

A linear mapping $T : E \to F$ between two vector lattices is called disjointness preserving if $|T(x)| \wedge |T(y)| = 0$ for all $x, y \in E$ satisfying $|x| \wedge |y| = 0$. Now, we give a sufficient condition under which each p-demicompact operator has a modulus which is p-demicompact.

Theorem 2.5. *Let (X, p, E) and (Y, q, F) be LNSs. Let $T : (X, p, E) \to (Y, q, F)$ such that $\mathcal{R}(T) \subset X$ be p-demicompact. Then the modulus of T is p-demicompact if T is a p-bounded disjointness preserving operator.*

Proof. T is a p-bounded disjointness preserving operator, so it is an order disjointness preserving operator, then a theorem of Meyer Nieberg ([24], Theorem 3.1.4) implies that $|T|$ exists and that $|T|(x) = |T(x)|$ for all $x \in E^+$. Now, let x_α be a p-bounded net in E such that $x_\alpha - |T|x_\alpha \xrightarrow{p} y$. This implies that $x_\alpha - |Tx_\alpha| \xrightarrow{p} y$. Hence, $q(x_\alpha - |Tx_\alpha|) \xrightarrow{o} y$.

$$\begin{aligned}q(x_\alpha - Tx_\alpha) &= q(|x_\alpha - Tx_\alpha|) \\ &\leq q(x_\alpha - |T|x_\alpha) \xrightarrow{o} y.\end{aligned}$$

Thus, we obtain that $q(x_\alpha - Tx_\alpha) \xrightarrow{o} y$. Hence, $x_\alpha - Tx_\alpha \xrightarrow{p} y$. Since T is p-demicompact, it follows that there is a p-convergent subnet $(x_{\alpha_\beta})_\beta$ of $(x_\alpha)_\alpha$. Consequently, $|T|$ is a p-demicompact operator. □

Theorem 2.6. *Let (X,p,E) be an LNS and $T:(X,p,E)\to(X,p,E)$. Then, T^m is p-demicompact for some $m\geq 1$ if and only if T is p-demicompact.*

Proof. We assume that the assumption holds and let $(x_\alpha)_\alpha$ be a p-bounded net in E such that $x_\alpha - Tx_\alpha \xrightarrow{p} y$. Now, we have the following equality

$$I - T^m = \sum_{j=0}^{m-1} T^j(I-T).$$

Thus, we obtain that

$$(I-T^m)x_\alpha = \sum_{j=0}^{m-1} T^j(I-T)x_\alpha.$$

From hypothesis, we have that $x_\alpha - Tx_\alpha \xrightarrow{p} y$, and this implies that $(I - T^m)x_\alpha \xrightarrow{p} z$. Now, using the fact that T^m is p-demicompact, we infer that there exists a p-convergent subnet $(x_{\alpha_\beta})_\beta$ of $(x_\alpha)_\alpha$. Consequently, we deduce that T is a p-demicompact operator. The converse is similarly. □

Theorem 2.7. *Let (X,p,E) be an LNS and $T:(X,p,E)\to(X,p,E)$ be a p-demicompact operator. If $S:(X,p,E)\to(X,p,E)$ is p-compact, then $T+S$ is a p-demicompact operator.*

Proof. Let $(x_\alpha)_\alpha$ be a p-bounded net in E such that $x_\alpha - (T+S)x_\alpha \xrightarrow{p} y$. Using the fact that S is p-compact, it follows that there is a subnet x_{α_β} such that $Sx_{\alpha_\beta} \xrightarrow{p} z$. Thus, from hypothesis, we obtain that $x_{\alpha_\beta} - Tx_{\alpha_\beta} \xrightarrow{p} y+z$. Now, since T is p-demicompact, we infer that there is a subnet $x_{\alpha_{\beta_k}}$ of x_{α_β} p-convergent. Consequently, $T+S$ is a p-demicompact operator. □

Remark 2.2. (i) Every p-compact operator $T:(X,p,E)\to(X,p,E)$ is p-demicompact. Indeed, let x_α be a p-bounded net in E such that $x_\alpha - Tx_\alpha \xrightarrow{p} y$. Since T is p-compact, it follows that there exists a subnet x_{α_β} such that $Tx_{\alpha_\beta} \xrightarrow{p} z$. Thus, we get $x_{\alpha_\beta} - Tx_{\alpha_\beta} \xrightarrow{p} y$, which implies that $x_{\alpha_\beta} \xrightarrow{p} y+z$. Consequently, T is p-demicompact.

(ii) Let $(X,\|\cdot\|_X,\mathbb{R})$ and $(Y,\|\cdot\|_Y,\mathbb{R})$ be normed spaces. Then

$$T:(X,\|\cdot\|_X,\mathbb{R})\to(Y,\|\cdot\|_Y,\mathbb{R})$$

such that $\mathcal{R}(T)\subset X$ is p-demicompact if and only if $T:X\to Y$ is demicompact.

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