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RESULTS OF 3-DERIVATIONS AND COMMUTATIVITY FOR PRIME RINGS WITH INVOLUTION INVOLVING SYMMETRIC AND SKEW SYMMETRIC COMPONENTS

Hanane Aharssi, Kamal Charrabi, and Abdellah Mamouni

Abstract. This article examines the connection between 3-derivations and the commutativity of a prime ring R with an involution $*$ that fulfills particular algebraic identities for symmetric and skew symmetric elements. In practice, certain well-known problems, such as the Herstein problem, have been studied in the setting of three derivations in involuted rings.

1. Introduction

R will be used to refer to an associative ring with the center $Z(R)$ throughout this piece. The commutator $xy - yx$ is denoted as $[x, y]$ for any $x, y \in R$, while the anti-commutator $xy + yx$ is written as $x \circ y$. If $aRb = 0$ indicates either $a = 0$ or $b = 0$, then R is prime. As is common knowledge, a derivation is an additive mapping d: $R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. On the other hand, if d is a derivation, a generalized derivation of R with an attached derivation d is referred to as an additive mapping $F : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Numerous articles have been written about the connection between certain unique types of maps and the commutativity of a ring R in the literature (see [2–4]). Posner's theorem, which states that a prime ring becomes commutative if it admits a nonzero centralizing derivation, is one of the most significant findings in this area.

A map's commutativity is preserved if all instances of $[\psi(x), \psi(y)] = 0$ for all instances of $[x, y] = 0$ for all $x, y \in R$. Matrix theory, operator theory, and ring theory have all explored the idea of commutativity preserving mapping (for references, see [5, 13]). If $[\psi(x), \psi(y)] = [x, y]$ for all $x, y \in S$, then a map $\psi : R \to R$ is said to be strong commutativity preserving (SCP) on a subgroup of R. In this regard, Bell and Daif [3] looked into the commutativity of rings that allow derivations that are SCP on nonzero right ideals. In fact,

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they demonstrated that $I \subseteq Z(R)$ in the case where a semiprime ring R allows a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R. Additionally, Ali and Huang demonstrated that if R is a 2-torsion free semiprime ring and d is a derivation of R satisfying $[d(x), d(y)] + [x, y] = 0$ for all x, y in nonzero ideal I of R, then R contains a nonzero central ideal. They specifically demonstrated that R is commutative if the ideal is $I = R$.

These findings have been expanded upon and investigated in the context of bands with involution as time goes on. An involution is an additive map $* : R \to R$ satisfying $(x^*)^* = x$ for all $x \in R$. In a ring with the involution $(R, *)$, a member x is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. $H(R)$ will stand for the collections of all hermitian elements in R, while $S(R)$ will stand for the set of skew-hermitian elements. If $Z(R) \subseteq H(R)$, the involution is said to be of the first kind, otherwise, it is said to be of the second kind.

Many writers have recently examined the commutativity of prime rings accepting pair of derivations meeting specific algebraic identities (see, for example, [1,7,9,10]). In [9] Lanski has demonstrated that if R is a noncommutative prime ring and d, g are two derivations of R into itself with g is a nonzero derivation, then $[d(x), g(x)] = 0$ holds for every $x \in R$. Then $d = \lambda g$, where λ is a component of $C(R)$. Abbassi et al. in [10], recently showed that if R is a prime ring with involution $*$ of the second kind such that $char(R) \neq 2$ and d_1, d_2 are nonzero derivations of R such that $[d_1(x); d_2(x^*)] = 0$ for all $x \in R$. Then R is commutative. Afterwards, El Mir et al. claimed in [7] that as long as R is a ring, P is a prime ideal, and R has d_1, d_2 derivations, then $d_1(x)d_2(y) - [x, y] \in P$ for every $x, y \in R$ reveals that R/P is a commutative integral domain.

In order to progress this field of study, we look into the commutativity conditions for rings with involution enabling three derivations that meet specific algebraic identities as well as a few additional identities.

2. Preliminary results

Lemma 2.1 ([14]). *Let R be a semiprime ring. Suppose that the relation $axb+bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case $(a+c)xb = 0$ is satisfied for all $x \in R$.*

We begin by establishing the following facts, which will be used frequently.

Fact 1. Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $H(R) \subset Z(R)$ or $S(R) \subset Z(R)$, then R is commutative.

Fact 2. Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If d is a nonzero derivation of R such that $d(h) \in Z(R)$ for all $h \in H(R)$ or $d(k) \in Z(R)$ for all $k \in S(R)$, then R is commutative.

3. Main theorems

Proposition 3.1. *Let R be a prime ring. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(x), d_2(y)] + [d_3(x), y] = 0$ for all $x, y \in R$, then R is commutative.*

Remark 3.2. In the above proposition, we must suppose that ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Proof. Suppose that R is noncommutative, by the given assumption, we have

$$[d_1(x), d_2(y)] + [d_3(x), y] = 0 \text{ for all } x, y \in R. \tag{3.1}$$

Substituting yr for y in (3.1), we obtain

$$\begin{aligned} &d_2(y)[d_1(x), r] + [d_1(x), y]d_2(r) + ([d_1(x), d_2(y)] + [d_3(x), y])r \\ &\qquad + y([d_1(x), d_2(r)] + [d_3(x), r]) = 0 \text{ for all } r, x, y \in R. \end{aligned} \tag{3.2}$$

By invoking equation (3.1), the last equation yields

$$d_2(y)[d_1(x), r] + [d_1(x), y]d_2(r) = 0 \text{ for all } r, x, y \in R. \tag{3.3}$$

We replace r by rt in (3.3), we get

$$\begin{aligned} &d_2(y)r[d_1(x), y] + [d_1(x), y]rd_2(t) + (d_2(y)[d_1(x), r] \\ &\qquad + [d_1(x), y]d_2(r))t = 0 \text{ for all } r, t, x, y \in R. \end{aligned} \tag{3.4}$$

By equation (3.3) together with the last equation, it follows that

$$d_2(y)r[d_1(x), t] + [d_1(x), y]rd_2(t) = 0 \text{ for all } r, t, x, y \in R. \tag{3.5}$$

Taking $y = t$, we obtain

$$d_2(y)r[d_1(x), y] + [d_1(x), y]rd_2(y) = 0 \text{ for all } r, x, y \in R. \tag{3.6}$$

Applying Lemma 2.1, we obtain from the above relation

$$d_2(y)r[d_1(x), y] = 0 \text{ for all } r, x, y \in R. \tag{3.7}$$

Since R is prime, the last equation implies that $d_2 = 0$ or $[d_1(x), y] = 0$ for all $x, y \in R$. Our supposition forces $d_2(R) = \{0\}$, then (3.1) is reduced to $[d_3(x), y] = 0$ for all $x, y \in R$. In accordance with Posner's Theorem ([12, Lemma 3]), R is commutative, this is in conflict with our presumption. Therefore, we have $d_3 = 0$, which is a contradiction. Thus R is commutative. $\square$

Theorem 3.3. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(h), d_2(h')] + [d_3(h), h'] = 0$ for all $h, h' \in H(R)$, then R is commutative.*

Remark 3.4. In the above theorem, we must suppose that ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Proof. Take into account R is noncommutative, with the stated assumption, we have

$$[d_1(h), d_2(h')] + [d_3(h), h'] = 0 \text{ for all } h, h' \in H(R). \tag{3.8}$$

Replacing h' by hh_0, $h_0 \in H(R) \cap Z(R)$ in (3.8), we find $[d_1(h), h]d_2(h_0) = 0$ for all $h \in H(R)$ and $h_0 \in H(R) \cap Z(R)$. The result of R being prime is that $[d_1(h), h] = 0$ for all $h \in H(R)$ or $d_2(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. The first case R is commutative in light of [6, Theorem 2.5], which goes against our selection of R. Thus we must have $d_2(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$, hence $d_2(Z(R)) = \{0\}$. Substituting hh_0 for h in (3.8), where $h_0 \in Z(R) \cap H(R)$, we obtain

$$[h, d_2(h')]d_1(h_0) + [h, h']d_3(h_0) = 0 \quad \text{for all } h, h' \in H(R). \tag{3.9}$$

Taking $h = h'$ in (3.9), we arrive at $[d_2(h), h]d_1(h_0) = 0$. Primeness of R leads to $[d_2(h), h] = 0$ for all $h \in H(R)$ or $d_1(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. When viewed in the context of [6, Theorem 2.5], the first instance R is commutative, which is at odds with the R we chose. Thus we must have $d_1(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$, then $d_1(Z(R)) = \{0\}$. So equation (3.9) gives $[h, h']d_3(h_0) = 0$, again using primeness of R and reasoning as above, we find $d_3(Z(R)) = \{0\}$. Replacing h by kk_0 in (3.8), where $k \in S(R)$ and $k_0 \in Z(R) \cap S(R)$, we get $([d_1(k), d_2(h')] + [d_3(k), h'])k_0 = 0$. Since R is prime and $Z(R) \cap S(R) \neq \{0\}$, we find

$$[d_1(k), d_2(h')] + [d_3(k), h'] = 0 \quad \text{for all } h' \in H(R) \text{ and } k' \in S(R). \tag{3.10}$$

Given that R is 2-torsion free, every $x \in R$ can be expressed as $2x = h + k$ with $h \in H(R)$ and $k \in S(R)$. We have

$$\begin{aligned} &2([d_1(x), d_2(h')] + [d_3(x), h']) \\ &= [d_1(2x), d_2(h')] + [d_3(2x), h'] \\ &= [d_1(h + k), d_2(h')] + [d_3(h + k), h'] \\ &= [d_1(h), d_2(h')] + [d_3(h), h'] + [d_1(k), d_2(h')] + [d_3(k), h']. \end{aligned} \tag{3.11}$$

Using (3.8) and (3.10) gives that $[d_1(x), d_2(h')] + [d_3(x), h'] = 0$ for all $x \in R$ and $h' \in H(R)$. Taking kk_0 for h' in the last expression, where $k \in S(R)$ and $k_0 \in Z(R) \cap S(R)$, we get $[d_1(x), d_2(k)] + [d_3(x), k] = 0$. Combining the last two expressions, we finally get $[d_1(x), d_2(y)] + [d_3(x), y] = 0$ for all $x, y \in R$, then R is commutative in light of Proposition 3.1, a contradiction. Thus R is commutative. □

Theorem 3.5. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(k), d_2(k')] + [d_3(k), k'] = 0$ for all $k, k' \in S(R)$, then R is commutative.*

Remark 3.6. In the above theorem, we must suppose that ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Proof. According to the presumption that R is noncommutative, we have

$$[d_1(k), d_2(k')] + [d_3(k), k'] = 0 \quad \text{for all } k, k' \in S(R). \tag{3.12}$$

Replacing k' by kh_0, $h_0 \in H(R) \cap Z(R)$ in (3.12), we find $[d_1(k), k]d_2(h_0) = 0$ for all $k \in S(R)$ and $k_0 \in H(R) \cap Z(R)$. Primeness of R leads to $[d_1(k), k] = 0$ for all $k \in S(R)$ or $d_2(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. The first case R is commutative in light of [6, Theorem 2.6]. This runs counter to what we assume. Due to this, we must only have $d_2(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$, hence $d_2(Z(R)) = \{0\}$. Substituting kh_0 for k in (3.12), where $h_0 \in Z(R) \cap H(R)$, we obtain

$$[k, d_2(k')]d_1(h_0) + [k, k']d_3(h_0) = 0 \quad \text{for all } k, k' \in S(R). \tag{3.13}$$

Taking $k = k'$ in (3.13), we arrive at $[d_2(k), k]d_1(h_0) = 0$. Primacy of R results in $[d_2(k), k] = 0$ for all $k \in S(R)$ or $d_1(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. With reference to [6, Theorem 2.6], the first case R is commutative, a contradiction. Thus we must have $d_1(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$, then $d_1(Z(R)) = \{0\}$. So equation (3.13) gives $[k, k']d_3(h_0) = 0$, using the primeness of R and the same logic as before, we discover $d_3(Z(R)) = \{0\}$. Replacing k by hk_0 in (3.12), where $h \in H(R)$ and $k_0 \in Z(R) \cap S(R)$, we get $([d_1(h), d_2(k')] + [d_3(h), k'])k_0 = 0$. Since R is prime and $Z(R) \cap S(R) \neq \{0\}$, we find

$$[d_1(h), d_2(k')] + [d_3(h), k'] = 0 \quad \text{for all } h \in H(R) \text{ and } k' \in S(R). \tag{3.14}$$

With regard to the fact that R is 2-torsion free, each $x \in R$ can be stated as $2x = h + k$ with $h \in H(R)$ and $k' \in S(R)$. Following the same procedures as in Theorem 3.3 and using (3.12) and (3.14), we find $[d_1(x), d_2(k')] + [d_3(x), k'] = 0$ for all $x \in R$ and $k' \in S(R)$. Taking hk_0 for k' in the last expression, where $h \in H(R)$ and $k_0 \in Z(R) \cap S(R)$, we get $[d_1(x), d_2(h)] + [d_3(x), h] = 0$. Combining the last two expressions, we finally get $[d_1(x), d_2(y)] + [d_3(x), y] = 0$ for all $x, y \in R$. Then applying Proposition 3.1, we achieve the desired outcome. □

Following the same arguments and under the same conditions as in Theorem 3.3 and Theorem 3.5, we can readily obtain the following result.

Theorem 3.7. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(h), d_2(k)] + [d_3(h), k] = 0$ for all $h \in H(R)$ and $k \in S(R)$, then R is commutative.*

The following conclusions are drawn as uses of the aforementioned findings.

Corollary 3.8. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(x), d_2(y)] + [d_3(x), y] = 0$ for all $x, y \in R$, then R is commutative.*

Remark 3.9. In Corollary 3.8, we must have ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Corollary 3.10. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind and d is a nonzero derivation of R. If $[d(x), d(y)] + [d(x), y] = 0$ for all $x, y \in R$, then R is commutative.*

Additionally, for rings with involution, if we take $d_1 = d_2$ and $d_3 = 0$, we obtain a variation of Herstein's finding.

Corollary 3.11 ([8]). *Let R be a 2-torsion free prime ring with involution $*$ of the second kind and d is a nonzero derivation of R. If $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative.*

Theorem 3.12. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $d_1(h) \circ d_2(h) + d_3(h) \circ h \in Z(R)$ for all $h \in H(R)$, then R is commutative.*

Remark 3.13. In the above theorem, we must suppose that ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Proof of Theorem 3.12. Consider that

$$d_1(h) \circ d_2(h) + d_3(h) \circ h \in Z(R) \quad \text{for all } h \in H(R). \tag{3.15}$$

Following linearization, we arrive at

$$\begin{aligned} &d_1(h) \circ d_2(h') + d_1(h') \circ d_2(h) + d_3(h) \circ h' \\ &\qquad + d_3(h') \circ h \in Z(R) \quad \text{for all } h, h' \in H(R). \end{aligned} \tag{3.16}$$

Replacing h' by h_0, where $h_0 \in Z(R) \cap H(R) \backslash \{0\}$, and using the last equation, we obtain

$$\begin{aligned} &d_1(h)d_2(h_0) + d_2(h)d_1(h_0) + d_3(h)h_0 \\ &\qquad + d_3(h_0)h \in Z(R) \quad \text{for all } h \in H(R). \end{aligned} \tag{3.17}$$

Substituting h_0^2 for h' in (3.16), where $h_0 \in Z(R) \cap H(R) \backslash \{0\}$, and using the fact that R is 2-torsion free, we have

$$\begin{aligned} &2h_0 d_1(h)d_2(h_0) + 2h_0 d_2(h)d_1(h_0) + d_3(h)h_0^2 \\ &\qquad + 2h_0 d_3(h_0)h \in Z(R) \text{ for all } h \in H(R). \end{aligned} \tag{3.18}$$

Hence

$$\begin{aligned} &h_0(d_1(h)d_2(h_0) + d_2(h)d_1(h_0) + d_3(h)h_0 + d_3(h_0)h) \\ &\qquad + h_0(d_1(h)d_2(h_0) + d_2(h)d_1(h_0) + d_3(h_0)h) \in Z(R) \\ &\qquad \text{for all } h \in H(R). \end{aligned} \tag{3.19}$$

Invoking (3.17), (3.19) yields

$$h_0 \left(d_1(h)d_2(h_0) + d_2(h)d_1(h_0) + d_3(h_0)h\right) \in Z(R) \quad \text{for all } h \in H(R). \tag{3.20}$$

Since $h_0 \neq 0$ and R is prime, we arrive at

$$[d_1(h)d_2(h_0) + d_2(h)d_1(h_0) + d_3(h_0)h, y] = 0 \quad \text{for all } y \in R. \tag{3.21}$$

From (3.17), we get $[d_1(h)d_2(h_0) + d_2(h)d_1(h_0) + d_3(h_0)h, y] = -[d_3(h)h_0, y]$ for all $y \in R$. Therefore the last expression becomes $[d_3(h), y]h_0 = 0$ for all $y \in R$. R is prime and $h_0 \neq 0$ implies $d_3(h) \in Z(R)$ for all $h \in H(R)$, and by Fact 2 R is commutative. □

Theorem 3.14. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $d_1(k) \circ d_2(k) + d_3(k) \circ k \in Z(R)$ for all $k \in S(R)$, then R is commutative.*

Remark 3.15. In the above theorem, we must suppose that ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Proof. Assume that R is not commutative. By hypothesis we have

$$d_1(k) \circ d_2(k) + d_3(k) \circ k \in Z(R) \quad \text{for all } k \in S(R). \tag{3.22}$$

Linearizing the above equation, we get

$$\begin{aligned} d_1(k) \circ d_2(k') + d_1(k') \circ d_2(k) + d_3(k) \circ k' \\ + d_3(k') \circ k \in Z(R) \quad \text{for all } k, k' \in S(R). \end{aligned} \tag{3.23}$$

Replacing k' by $k'h_0$ in (3.23), where $h_0 \in Z(R) \cap H(R) \backslash \{0\}$ and making use equation (3.23), we arrive at

$$\begin{aligned} (d_1(k) \circ k')d_2(h_0) + (k' \circ d_2(k))d_1(h_0) \\ + (k \circ k')d_3(h_0) \in Z(R) \quad \text{for all } k, k' \in S(R). \end{aligned} \tag{3.24}$$

Taking $k' \in Z(R) \cap S(R) \backslash \{0\}$ and using the fact that R is 2-torsion free prime, we obtain

$$d_1(k)d_2(h_0) + d_2(k)d_1(h_0) + kd_3(h_0) \in Z(R) \quad \text{for all } k \in S(R). \tag{3.25}$$

Replacing k by kh_0, where $h_0 \in Z(R) \cap H(R) \backslash \{0\}$, we find

$$\begin{aligned} (d_1(k)d_2(h_0) + d_2(k)d_1(h_0) + kd_3(h_0))h_0 \\ + 2kd_1(h_0)d_2(h_0) \in Z(R) \quad \text{for all } k \in S(R). \end{aligned} \tag{3.26}$$

Making use equation (3.25) together with 2-torsion freeness of R, we get

$$kd_1(h_0)d_2(h_0) \in Z(R)$$

for all $k \in S(R)$. Primeness of R leads to $k \in Z(R)$ for all $k \in S(R)$ or $d_1(h_0)d_2(h_0) = 0$ for $h_0 \in Z(R) \cap H(R)$. Contrary to what we had assumed, in the first instance, R is commutative by Fact 1. So in view of primeness of R, we can see $d_1(h_0) = 0$ or $d_2(h_0) = 0$ for $h_0 \in Z(R) \cap H(R)$. If $d_1(h_0) = 0$ for $h_0 \in Z(R) \cap H(R)$, then $d_1(Z(R)) = \{0\}$. Taking $k \in Z(R) \cap S(R) \backslash \{0\}$ in (3.24), we find that $kk'd_3(h_0) \in Z(R)$ for all $k' \in S(R)$. Again using primeness of R and our supposition, we must have $d_3(h_0) = 0$ for $h_0 \in Z(R) \cap H(R)$, thus $d_3(Z(R)) = \{0\}$. Therefore, equation (3.25) becomes $d_1(k)d_2(h_0) \in Z(R)$ for all $k \in S(R)$ and $h_0 \in Z(R) \cap H(R)$. Since R is prime, we arrive at $d_1(k) \in Z(R)$ for all $k \in S(R)$ or $d_2(h_0) = 0$ for $h_0 \in Z(R) \cap H(R)$. The first case together with Fact 2 forces R to be commutative which contradicts our supposition. Thus $d_2(h_0) = 0$ for $h_0 \in Z(R) \cap H(R)$, hence $d_2(Z(R)) = \{0\}$. Taking $k' \in Z(R) \backslash \{0\}$ in (3.23), we find $d_3(k)k' \in Z(R)$ for all $k \in S(R)$. Primeness of R and the fact that $k' \neq 0$ leads to $d_3(k) \in Z(R)$ for all $k \in S(R)$, therefore, in view of Fact 2, which runs counter to our assumption, R is commutative. Hence R is commutative. □

Using the same conditions, the following result can be reached by using the same justifications as in Theorem 3.14.

Theorem 3.16. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $d_1(h) \circ d_2(k) + d_3(h) \circ k \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$, then R is commutative.*

As a result of applying the aforementioned results, we derive the following corollaries.

Corollary 3.17. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $d_1(x) \circ d_2(x^*) + d_3(x) \circ x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Corollary 3.18. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $d_1(x) \circ d_2(y) + d_3(x) \circ y \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Remark 3.19. In Corollaries 3.17 and 3.18, we must have ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Corollary 3.20. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a nonzero derivation of R satisfying $d(x) \circ d(x^*) + d(x) \circ x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Corollary 3.21. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a nonzero derivation of R satisfying $d(x) \circ d(y) + d(x) \circ y \in Z(R)$ for all $x, y \in R$, then R is commutative.*

The subsequent corollaries become clear if we select $d_3 = 0$. The next corollary is a generalization of [11, Theorem 3.5].

Corollary 3.22. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1 and d_2 are a nonzero derivations of R satisfying $d_1(x) \circ d_2(x^*) \in Z(R)$ for all $x \in R$, then R is commutative.*

Corollary 3.23. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1 and d_2 are a nonzero derivations of R satisfying $d_1(x) \circ d_2(y) \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Corollary 3.24 ([11], Theorem 3.5)**.** *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a nonzero derivation of R such that $d(x) \circ d(x^*) \in Z(R)$ for all $x \in R$, then R is commutative.*

Corollary 3.25 ([11], Corollary 3.6)**.** *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a nonzero derivation of R such that $d(x) \circ d(y) \in Z(R)$ for all $x, y \in R$, then R is commutative.*

We can come to the following result if either $d_1 = 0$ or $d_2 = 0$.

Corollary 3.26 ([11], Theorem 3.7)**.** *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a nonzero derivation of R such that $d(x) \circ x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Theorem 3.27. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(h), d_2(h)] + d_3(h) \circ h \in Z(R)$ for all $h \in H(R)$, then R is commutative.*

Remark 3.28. In the above theorem, we must suppose that ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Proof. Suppose that

$$[d_1(h), d_2(h)] + d_3(h) \circ h \in Z(R) \text{ for all } h \in H(R). \tag{3.27}$$

Linearizing, we obtain

$$\begin{aligned} &[d_1(h), d_2(h')] + [d_1(h'), d_2(h)] + d_3(h) \circ h' \\ &\qquad + d_3(h') \circ h \in Z(R) \text{ for all } h, h' \in H(R). \end{aligned} \tag{3.28}$$

Replacing h' by h_0, where $h_0 \in Z(R) \cap H(R) \backslash \{0\}$, and using the last equation, we obtain

$$d_3(h)h_0 + d_3(h_0)h \in Z(R) \text{ for all } h \in H(R). \tag{3.29}$$

Substituting h_0^2 for h' in (3.28), where $h_0 \in Z(R) \cap H(R) \backslash \{0\}$, and using the fact that R is 2-torsion free, we have

$$d_3(h)h_0^2 + 2h_0 d_3(h_0)h \in Z(R) \text{ for all } h \in H(R). \tag{3.30}$$

Hence

$$h_0(d_3(h)h_0 + d_3(h_0)h) + h_0 d_3(h_0)h \in Z(R) \text{ for all } h \in H(R). \tag{3.31}$$

Invoking (3.29), (3.31) yields

$$h_0 d_3(h_0)h \in Z(R) \text{ for all } h \in H(R). \tag{3.32}$$

Since $h_0 \neq 0$ and R is prime, we arrive at

$$[d_3(h_0)h, y] = 0 \text{ for all } y \in R. \tag{3.33}$$

From (3.29), we get $[d_3(h_0)h, y] = -[d_3(h)h_0, y]$ for all $y \in R$. Therefore, by using (3.33), the last expression becomes $[d_3(h), y]h_0 = 0$ for all $y \in R$. R is prime and $h_0 \neq 0$ implies $d_3(h) \in Z(R)$ for all $h \in H(R)$, and by Fact 2 R is commutative. ☐

Theorem 3.29. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(k), d_2(k)] + d_3(k) \circ k \in Z(R)$ for all $k \in S(R)$, then R is commutative.*

Remark 3.30. In the above theorem, we must suppose that ($d_1 \neq 0$ and $d_2 \neq 0$) or $d_3 \neq 0$.

Proof. By the given assumption, we have

$$[d_1(k), d_2(k)] + d_3(k) \circ k \in Z(R) \text{ for all } k \in S(R). \tag{3.34}$$

Linearizing, we obtain

$$\begin{aligned}(3.35)\qquad &[d_1(k), d_2(k')] + [d_1(k'), d_2(k)] + d_3(k) \circ k' \\ &\qquad + d_3(k') \circ k \in Z(R) \quad \text{for all } k, k' \in S(R).\end{aligned}$$

Replacing k' by k_0, where $k_0 \in Z(R) \cap S(R) \backslash \{0\}$, and using the last equation, we obtain

$$(3.36)\qquad d_3(k)k_0 + kd_3(k_0) \in Z(R) \quad \text{for all } k \in S(R).$$

Replacing k by kh_0, where $h_0 \in Z(R) \cap H(R) \backslash \{0\}$, the last equation yields

$$(3.37)\qquad d_3(k)k_0h_0 + kd_3(h_0)k_0 + kd_3(k_0)h_0 \in Z(R) \quad \text{for all } k \in S(R).$$

Invoking (3.36) and (3.37), we get

$$(3.38)\qquad kd_3(h_0)k_0 \in Z(R) \quad \text{for all } k \in S(R).$$

Hence

$$(3.39)\qquad [k, r]Rd_3(h_0)k_0 = 0 \quad \text{for all } k \in S(R) \text{ and } r \in R.$$

Primeness of R and the fact that $k_0 \neq 0$ leads to $k \in Z(R)$ for all $k \in S(R)$ or $d_3(h_0) = 0$, where $h_0 \in Z(R) \cap H(R) \backslash \{0\}$. In light of Fact 1, the first case forces R to be commutative. Otherwise if $d_3(h_0) = 0$ for all $h_0 \in Z(R) \cap H(R) \backslash \{0\}$, we obtain $d_3(Z(R)) = \{0\}$. Therefore equation (3.36) becomes $d_3(k)k_0 \in Z(R)$ for all $k \in Z(R)$. Since R is prime and $k_0 \neq 0$, we finally have $d_3(k) \in Z(R)$. Thus R is commutative in view of Fact 2. □

With the same justifications and conditions as in Theorem 3.29, we can quickly arrive at the conclusion shown below.

Theorem 3.31. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(h), d_2(k)] + d_3(h) \circ k \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$, then R is commutative.*

As a result of the preceding discoveries, the following corollaries are acquired.

Corollary 3.32. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R satisfying $[d_1(x), d_2(x^*)] + d_3(x) \circ x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Corollary 3.33. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1, d_2 and d_3 are derivations of R, then R is commutative if and only if $[d_1(x), d_2(y)] + d_3(x) \circ y \in Z(R)$ for all $x, y \in R$.*

Remark 3.34. In Corollaries 3.32 and 3.33, we must have $d_1 \neq 0$ and $d_2 \neq 0$ or $d_3 \neq 0$.

Corollary 3.35. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a nonzero derivation of R satisfying $[d(x), d(x^*)] + d(x) \circ x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Corollary 3.36. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a nonzero derivation of R satisfying $[d(x), d(y)] + d(x) \circ y \in Z(R)$ for all $x, y \in R$, then R is commutative.*

The following corollaries are found if we select $d_3 = 0$. The first corollary is an improved version of [11, Theorem 3.1].

Corollary 3.37. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1 and d_2 are a nonzero derivations of R satisfying $[d_1(x), d_2(x^*)] \in Z(R)$ for all $x \in R$, then R is commutative.*

The next result is an improved version of [11, Corollary 3.3].

Corollary 3.38. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d_1 and d_2 are a nonzero derivations of R satisfying $[d_1(x), d_2(y)] \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Corollary 3.39 ([11], Theorem 3.1)**.** *Let R be a 2-torsion free prime ring with involution $*$ of the second kind and d is a nonzero derivation of R. If $[d(x), d(x^*)] \in Z(R)$ for all $x \in R$, then R is commutative.*

Corollary 3.40 ([11], Corollary 3.3)**.** *Let R be a 2-torsion free prime ring with involution $*$ of the second kind and d is a nonzero derivation of R. If $[d(x), d(y)] \in Z(R)$ for all $x, y \in R$, then R is commutative.*

4. Examples

In this part, we go over a few illustrations that demonstrate how, in some circumstances, our findings do not hold. We start by demonstrating through the following instances that the condition "$*$" is of the second kind is required.

Example 4.1. Let us consider $R = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in Z \right\}$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. We have $Z(R) = \{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in Z \}$, therefore, it is simple to verify that R is a prime ring and $*$ is an involution of the first kind. Moreover, we set $d_1 \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -b \\ c & 0 \end{smallmatrix}\right)$, $d_2 = d_1$ and $d_3 = 0$. Thus d_1, d_2 and d_3 satisfying the conditions of Theorem 3.3, Theorem 3.12, and Theorem 3.27, but R is not commutative. Consequently, the hypothesis of second kind involution is essential.

In the following illustration, we show that the "primeness hypothesis" of R in our work is not a purely theoretical construct.

Example 4.2. Let R with involution $*$ and d_1, d_2, and d_3 be as in Example 4.1, and $\mathbb{C}$ be the field of complex numbers. If we set $R_1 = R \times \mathbb{C}$, then R_1 is a semi-prime ring provided with the involution of the second kind $\tau : R_1 \to R_1$, where $\tau(r, s) = (r^*, \bar{s})$ for all $(r, s) \in R \times \mathbb{C}$. Consider the derivation $D_1 : R_1 \to R_1$ defined as $D_1(x, s) = (d_1(x), 0)$, the derivation $D_2 : R_1 \to R_1$ defined as $D_2(x, s) = (d_2(x), 0)$ and the derivation $D_3 : R_1 \to R_1$ defined as $D_3(x, s) = (0, 0)$. Furthermore D_1, D_2 and D_3 satisfies the conditions of Theorem 3.3, but R_1 is not commutative.

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Hanane Aharssi
Department of Mathematics
Moulay Ismaïl University
Meknes, Morocco
Email address: `aharssi.hanane@gmail.com`

Kamal Charrabi
Department of Mathematics
Moulay Ismaïl University
Meknes, Morocco
Email address: `kamal95charrabi@gmail.com`

Abdellah Mamouni
Department of Mathematics
Moulay Ismaïl University
Meknes, Morocco
Email address: a.mamouni.fste@gmail.com