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NONNIL-S-COHERENT RINGS

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Abstract. Let R be a commutative ring with identity. If the nilpotent radical $Nil(R)$ of R is a divided prime ideal, then R is called a ϕ-ring. Let R be a ϕ-ring and S be a multiplicative subset of R. In this paper, we introduce and study the class of nonnil-S-coherent rings, i.e., the rings in which all finitely generated nonnil ideals are S-finitely presented. Also, we define the concept of ϕ-S-coherent rings. Among other results, we investigate the S-version of Chase's result and Chase Theorem characterization of nonnil-coherent rings. We next study the possible transfer of the nonnil-S-coherent ring property in the amalgamated algebra along an ideal and the trivial ring extension.

1. Introduction

Throughout this paper, it is assumed that all rings are commutative with non-zero identity and all modules are unitary. If R is a ring, then we denote by $Nil(R)$ and $Z(R)$ the ideal of all nilpotent elements and the set of all zero-divisors of R, respectively. A nonempty subset S of R is said to be a multiplicative subset if $1 \in S$, $0 \notin S$ and for each $a, b \in S$ we have $ab \in S$. A prime ideal P of R is called divided prime if it is comparable to every ideal of R. Set $H = \{R \,|\, R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. If $R \in H$, then R is called a ϕ-ring. Let R be a ϕ-ring with a total quotient ring T. As in [4], we define $\phi : T \to K := R_{Nil(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ for each $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from T into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = \frac{x}{1}$ for every $x \in R$.

Let R be a ring and M be an R-module. Set

$$\phi - tor(M) = \{x \in M \mid sx = 0 \text{ for some } s \in R \setminus Nil(R)\}.$$

If $\phi - tor(M) = M$, then M is called a ϕ-torsion module, and if $\phi - tor(M) = 0$, then M is called a ϕ-torsion free module. Recall from [22] that an R-module F

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is said to be ϕ-flat if for every R-monomorphism $f : A \to B$ with Coker$f$ a ϕ-torsion R-module, we have $1_F \otimes_R f : F \otimes_R A \to F \otimes_R B$ is an R-monomorphism; equivalently, $\mathrm{Tor}_1^R(F, M) = 0$ for every ϕ-torsion R-module M. The suitable background on ϕ-flat modules is [15, 17, 21, 22]. In [3], Khoualdia and Benhissi introduced two versions of coherent rings that are in the class H. A ϕ-ring R is called nonnil-coherent if each finitely generated nonnil ideal of R is finitely presented, and R is said to be ϕ-coherent if $\phi(R)$ is a nonnil-coherent ring. Among other things, they proved the Chase Theorem for nonnil-coherent rings using ϕ-flat modules. Next, the authors of [18] showed that any nonnil-coherent ring is ϕ-coherent, and they gave an example to show that the converse does not hold (see [18, Example 1.5]).

In [1], Anderson and Dumitrescu introduced the notion of S-Noetherian rings as a generalization of Noetherian rings. Let R be a ring, S be a multiplicative set of R, and M be an R-module. We say that M is S-finite if there exist a finitely generated sub-module F of M and $s \in S$ such that $sM \subseteq F$. Also, we say that M is S-Noetherian if each submodule of M is S-finite. A ring R is said to be S-Noetherian if it is S-Noetherian as an R-module (i.e., if each ideal of R is S-finite). In 2018, D. Bennis and M. El Hajoui [6] introduced S-finitely presented modules and S-coherent rings, which are S-versions of finitely presented modules and coherent rings, and they proved that a ring R is an S-coherent ring, if and only if, $(I : a)$ is an S-finite ideal of R for every finitely generated ideal I of R and $a \in R$, if and only if, $(0 : a)$ is an S-finite ideal of R for every $a \in R$ and the intersection of two finitely generated ideals of R is an S-finite ideal of R (cf. [6, Theorem 3.8]). After that, the authors of [19] investigate the open Question. (How to give an S-version of flatness that characterizes S-coherent rings similarly to the classical case?), and they proved that a ring R is an S-coherent ring, if and only if any product of flat R-modules is S-flat, if and only if, any product of R is S-flat (cf. [19, Theorem 4.4.]), where an R-module M is called S-flat if $S^{-1}M$ is a flat $S^{-1}R$-module. In [16], Kwon and Lim introduced the notion of nonnil-S-Noetherian rings as a generalization of both nonnil-Noetherian rings and S-Noetherian rings. Let R be a ring and S be a multiplicative set of R. Then R is said to be a nonnil-S-Noetherian ring if each nonnil ideal of R is S-finite.

Let A and B be two rings, J an ideal of B and $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\},$$

which is called the amalgamation of A and B along J with respect to f. The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra: pullbacks and trivial ring extensions. See for instance [7, 8, 11].

The main purpose of this paper is to integrate the concepts of nonnil-coherent rings and S-coherent rings. Then we construct a new class of rings that contains the class of nonnil-coherent rings. Let R be a ϕ-ring and S be a multiplicative subset of R. We define R to be a nonnil-S-coherent if every finitely generated nonnil ideal of R is S-finitely presented, and R is said to be a ϕ-S-coherent ring if $\phi(R)$ is a nonnil-$\phi(S)$-coherent ring. Note that if S consists of units of R, then the concept of S-finitely presented modules is the same as that of finitely presented module; so if S consists of units of R, then the notion of nonnil-S-coherent (resp., ϕ-S-coherent) rings coincides with that of nonnil-coherent (resp., ϕ-coherent) rings. Furthermore, if R is a domain, then the concepts of nonnil-S-coherent and ϕ-S-coherent rings are precisely the same as that of S-coherent domains. Clearly, if $S_1 \subseteq S_2$ are multiplicative subsets, then any nonnil-S_1-coherent ring (resp., ϕ-S_1-coherent) is nonnil-S_2-coherent (resp., ϕ-S-coherent); and if S^* is the saturation of S in R, then R is a nonnil-S-coherent (resp., ϕ-S-coherent) ring if and only if R is a nonnil-S^*-coherent (resp., ϕ-S^*-coherent) ring.

2. On nonnil-S-coherent rings

Let R be a ring, S be a multiplicative set of R, and M be an R-module. M is said to be S-finitely presented, if there exists an exact sequence of R-modules $0 \to K \to F \to M \to 0$, where K is S-finite and F is a finitely generated free R-module. A ring R is S-coherent if every finitely generated ideal of R is S-finitely presented.

The following theorem gives a characterization of nonnil-S-coherent rings.

Theorem 2.1. *Let R be a ϕ-ring and S be a multiplicative subset of R. Then the following assertions are equivalent*:

(1) *R is nonnil-S-coherent,*
(2) *$(I : a)$ is an S-finite ideal of R for any non-nilpotent element $a \in R$ and any finitely generated ideal I of R,*
(3) *$(0 : a)$ is an S-finite ideal for any non-nilpotent element $a \in R$, and the intersection of two finitely generated nonnil ideals of R is an S-finite nonnil ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let I be a finitely generated ideal of R and b a non-nilpotent element in R. So $J = I + Rb$ is a finitely generated nonnil ideal of R, and so it is S-finitely presented. Thus, there exists an exact sequence $0 \to K \to R^{n+1} \to J \to 0$, where K is S-finite. By [12, Lemma 2.3.1], there exists a surjective homomorphism $g : K \to (I : b)$, which shows that $(I : b)$ is S-finite.

(2) $\Rightarrow$ (1) Let I be a finitely generated nonnil ideal of R generated by $\{a_1, \ldots, a_n\}$, where each a_i is non-nilpotent. We will show that I is S-finitely presented by induction on n. The case $n = 1$ follows from the exact sequence $0 \to (0 : a_1) \to R \to Ra_1 \to 0$. For $n > 1$; let $L = \langle a_1, \ldots, a_{n-1} \rangle$ and consider the exact sequence:

$$0 \to (L : a_n) \to R \to (Ra_n + L)/L \to 0.$$

Then $(Ra_n + L)/L = I/L$ is S-finitely presented by (2). Consider the exact sequence $0 \to L \to I \to I/L \to 0$. Since L and I/L are S-finitely presented, I is also S-finitely presented by [6, Theorem 2.5(2)].

(1) $\Rightarrow$ (3) Let a be a non-nilpotent element in R. Then Ra is S-finitely presented. Consider the exact sequence:

$$0 \to (0 : a) \to R \to aR \to 0.$$

So the ideal $(0 : a)$ is S-finite by [6, Proposition 2.4]. Now, let I and J be two finitely generated nonnil ideals of R. Since $Nil(R) \subseteq I \cap J \subseteq I + J$, it follows that $I + J$ is a finitely generated nonnil ideal, and so it is S-finitely presented. Consider the following exact sequence:

$$0 \to I \cap J \to I \oplus J \to I + J \to 0.$$

Since $I + J$ is S-finitely presented and $I \oplus J$ is finitely generated, we get that $I \cap J$ is S-finite by [6, Theorem 2.5(5)].

(3) $\Rightarrow$ (1) Let I be a finitely generated nonnil ideal of R. Write $I = Rx_1 + \cdots + Rx_n$ with all $a_i \in R \setminus Nil(R)$, and we will prove the result by induction on n. For $n = 1$, we have $I = Rx_1$. Since x_1 is non-nilpotent, we get $(0, x_1)$ is S-finite. Hence I is S-finitely presented. For $n > 1$, set $J = Rx_1 + \cdots + Rx_{n-1}$ is a finitely generated nonnil ideal of R. Then we have the following exact sequence:

$$0 \to J \cap Rx_n \to J \oplus Rx_n \to I \to 0.$$

Note that $J \oplus Rx_n$ is S-finitely presented by [6, Theorem 2.5(2)]. On the other hand, since $J \cap Rx_n$ is the intersection of two finitely generated nonnil ideals of R, $J \cap Rx_n$ is S-finitely generated. So I is S-finitely presented by [6, Theorem 2.5(5)]. □

Let R be a ring and P be a prime ideal of R. Then $R \setminus P$ is a multiplicative subset of R. We define an R-module M to be P-finitely presented if M is an $(R \setminus P)$-finitely presented module. The next result gives a local characterization of finitely presented modules.

Proposition 2.2. *Let R be a ring and M be a finitely generated R-module. Then the following conditions are equivalent*:

1. *M is finitely presented,*
2. *M is P-finitely presented for every prime ideal P of R,*
3. *M is Q-finitely presented for every maximal ideal Q of R.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are straightforward.

(3) $\Rightarrow$ (1) Assume that M is Q-finitely presented for all maximal ideals Q of R. Consider the following exact sequence:

$$0 \to A = \mathrm{Ker}(f) \to R^n \xrightarrow{f} M \to 0.$$

Then A is Q-finite for every maximal ideal Q of R by [6, Proposition 2.4]. So for each maximal ideal Q of R, there exist an element $s_Q \in R \setminus m$ and a finitely generated sub-module F_Q of A such that $s_Q A \subseteq F_Q$. Let $S = \{s_m \mid m \text{ is a maximal ideal of } R\}$. Since S generates R, there exist finite elements $s_{Q_1}, \ldots, s_{Q_n}$ of S such that

$$A = (s_{Q_1}R + \cdots + s_{Q_n}R)A \subseteq F_{Q_1} + \cdots + F_{Q_n} \subseteq A,$$

which means that $A = F_{Q_1} + \cdots + F_{Q_n}$. So A is finitely generated. Therefore M is finitely presented. □

Let P be a prime ideal of R. We say R is P-coherent (resp., nonnil-P-coherent) provided R is $(R \setminus P)$-coherent (resp., nonnil-$(R \setminus P)$-coherent).

Corollary 2.3. *Let R be a ring. Then the following conditions are equivalent*:

(1) *R is a coherent ring,*
(2) *R is a P-coherent ring for all prime ideals P of R,*
(3) *R is a Q-coherent ring for all maximal ideals Q of R.*

Corollary 2.4. *Let R be a ϕ-ring. Then the following conditions are equivalent*:

(1) *R is a nonnil-coherent ring,*
(2) *R is a nonnil-P-coherent ring for all prime ideals P of R,*
(3) *R is a nonnil-Q-coherent ring for all maximal ideals Q of R.*

The following result gives us a criterion to an S-finitely presented nonnil ideal.

Theorem 2.5. *Let R be a ϕ-ring, S be a multiplicative subset of R and I be an S-finitely presented nonnil ideal of R. Then $I/Nil(R)$ is a T-finitely presented nonzero ideal of $R/Nil(R)$ with $T = S + Nil(R)$.*

Proof. Let $0 \to \mathrm{Ker}(\pi_1) \xrightarrow{i_1} R^n \xrightarrow{\pi_1} I \to 0$ be an exact sequence of R-modules with $\mathrm{Ker}\,\pi_1$ S-finite. Since I is a finitely generated nonnil ideal of R, $I/Nil(R)$ is a finitely generated non-zero ideal of $R/Nil(R)$. Then there is an epimorphism $\pi_2 : (R/Nil(R))^n \to I/Nil(R)$. Therefore,

$$0 \to \mathrm{Ker}(\pi_2) \xrightarrow{i_2} (R/Nil(R))^n \xrightarrow{\pi_2} I/Nil(R) \to 0$$

is an exact sequence of $(R/Nil(R))$-module. We can take as an exact sequence of R-modules with $(R/Nil(R))^n$ a finitely generated R-module. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \xrightarrow{i_1} & \mathrm{Ker}\,\pi_1 & \longrightarrow & R^n & \xrightarrow{\pi_1} & I & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \xrightarrow{i_2} & \mathrm{Ker}\,\pi_2 & \longrightarrow & (R/Nil(R))^n & \xrightarrow{\pi_2} & I/Nil(R) & \longrightarrow & 0
\end{array}$$

We have $\pi_1(Nil(R)^n) = Nil(R)$. Indeed, let $m' \in Nil(R)^n$. Thus $\pi_2 \circ \beta(m') = 0 = \gamma \circ \pi_1(m')$. Then $\pi_1(m') \in \mathrm{Ker}\gamma = Nil(R)$. Conversely, let $m \in Nil(R)$, and let $a \in I \setminus Nil(R)$. Then $Nil(R) \subseteq Ra$. Hence $m = ar$ with

$r \in Nil(R)$ since $Nil(R)$ is a prime ideal of R. Therefore $m = ar = r\pi_1(x) = \pi_1(rx) \in \pi_1(Nil(R)^n)$.

Our aim is to show that α is an epimorphism, therefore, $\alpha(\mathrm{Ker}(\pi_1)) = \beta(\mathrm{Ker}(\pi_2)) = \mathrm{Ker}(\pi_2)$. For this, let $k \in \mathrm{Ker}\,\pi_1$. Then $\gamma \circ \pi_1(k) = 0 = \pi_2 \circ \beta(k)$. Therefore, $\beta(k) \in \mathrm{Ker}(\pi_2)$. Conversely, let $k_2 = \beta(k) \in \mathrm{Ker}(\pi_2)$. Then $\pi_2 \circ \beta(k) = 0 = \gamma \circ \pi_1(k)$. Therefore $\pi_1(k) \in \mathrm{Ker}\,\gamma = Nil(R) = \pi_1(Nil(R)^n)$. So there is $j \in Nil(R)^n$ such that $\pi_1(k) = \pi_1(j)$. Then $k - j \in \mathrm{Ker}(\pi_1)$. Hence $\beta(k) = \beta(k-j) + \beta(j) = \beta(k-j) \in \beta(\mathrm{Ker}\,\pi_1)$. Then α is an epimorphism, consequently $\mathrm{Ker}(\pi_2)$ is S-finite. Hence, $I/Nil(R)$ is a T-finitely presented nonzero ideal of $R/Nil(R)$. □

In light of Theorem 2.5, we give a new characterization of nonnil-S-coherent rings using the integral domain $R/Nil(R)$.

Theorem 2.6. *Let R be a ϕ-ring and S be a multiplicative subset of R. Then R is a nonnil-S-coherent ring if and only if $R/Nil(R)$ is a T-coherent domain with $T = S + Nil(R)$ and $(0 : r)$ is an S-finite ideal for every non-nilpotent element $r \in R$.*

Proof. Assume that R is a nonnil-S-coherent ring. Then $(0 : r)$ is an S-finite ideal for every non-nilpotent element $r \in R$ by Theorem 2.1. Now, let J be a finitely generated ideal of $R/Nil(R)$. So $J = I/Nil(R)$ for some finitely generated nonnil ideal I of R. Since R is nonnil-S-coherent, we conclude that I is S-finitely presented. Hence $I/Nil(R)$ is a T-finitely presented nonzero ideal of $R/Nil(R)$ according to Theorem 2.5. Therefore $R/Nil(R)$ is a T-coherent domain.

Conversely, let I and J be two finitely generated nonnil ideals of R. Then $I/Nil(R)$ and $J/Nil(R)$ are finitely generated non-zero ideals of $R/Nil(R)$. Thus $(I \cap J)/Nil(R) = I/Nil(R) \cap J/Nil(R)$ is T-finite, therefore there exist $s \in S$ and a finitely generated nonnil ideal K of R such that $s(I \cap J)/Nil(R) \subseteq K/Nil(R) \subseteq I \cap J/Nil(R)$. Hence $s(I \cap J) \subseteq K \subseteq I \cap J$, so $I \cap J$ is S-finite. Whence R is nonnil-S-coherent according to Theorem 2.1. □

Example 2.7. Let R be a nonnil-S-Noetherian ring such that $(0 : r)$ is an S-finite ideal of R for every non-nilpotent element $r \in R$. Then R is a nonnil-S-coherent ring.

Corollary 2.8. *Let R be a ϕ-strong ring and S be a multiplicative subset of R. Then R is a nonnil-S-coherent ring if and only if $R/Nil(R)$ is a T-coherent domain with $T = S + Nil(R)$.*

Corollary 2.9. *Let R be a ϕ-ring and S be a multiplicative subset of R. Then R is a ϕ-S-coherent ring if and only if $\phi(R)/Nil(\phi(R))$ is a S'-coherent domain, with $S' = \phi(S) + Nil(\phi(R))$.*

Proof. Note that R is a ϕ-S-coherent ring if and only if $\phi(R)$ is a nonnil-$\phi(S)$-coherent ring. Since $\phi(R)$ is a ϕ-ring with $Nil(\phi(R)) = Z(\phi(R))$, and according to Corollary 2.8, we have the result. □

Recall from [5, Lemma 1.1] that $R/Nil(R) \cong \phi(R)/Nil(\phi(R))$ for every ϕ-ring R. Then we have the following corollary as a direct consequence of this result and Theorem 2.6.

Corollary 2.10. *Let R be a ϕ-ring such that $(0 : r)$ is an S-finite ideal for every non-nilpotent element $r \in R$, and S be a multiplicative subset of R. Then the following statements are equivalent*:

(1) *R is a nonnil-S-coherent ring,*
(2) *$R/Nil(R)$ is an $\overline{S}$-coherent domain with $\overline{S} = S + Nil(R)$,*
(3) *$\phi(R)/Nil(\phi(R))$ is an S'-coherent domain with $S' = \phi(S) + Nil(\phi(R))$,*
(4) *$\phi(R)$ is a nonnil-$\phi(S)$-coherent ring.*

Let R be a ring, M be an R-module. Then $R \propto M$, the set of pairs (r, m) with component-by-component addition and multiplication defined by: $(r, m)(b, f) = (rb, rf + bm)$, is a unitary commutative ring, called the trivial extension (or idealization) of R by M. For a suitable background on the commutative trivial ring extensions, see [2, 13, 14]. Now we will give an example of a nonnil-S-coherent ring that is neither nonnil-coherent nor S-coherent.

Example 2.11. Let D be a domain that is not a field, Q its quotient field and $E = \bigoplus_{i=1}^{\infty} Q/D$. Let $R = D \propto E$ be the trivial extension construction and $S = S_0 \propto 0$ with $S_0 = D \setminus Nil(R)$. Then R is a nonnil-S-coherent ring which is neither nonnil-coherent nor S-coherent.

Proof. We have R is a ϕ-ring which is not nonnil-coherent by [18, Example 1.5]. Note that $R/Nil(R) \cong D$, and thus R is ϕ-S-coherent by Corollary 2.9. Let (d, e) be a non-nilpotent element of R. Then it is easy to verify that $(d, 0)(0 : (d, e)) = 0$. Hence $(0 : (d, e))$ is S-finite. Consequently R is a nonnil-S-coherent ring. But R is not a S-coherent ring; indeed, let $x \in E$ and so $(0 : (0, x)) = Ann_R(x) \propto E$ with $Ann_R(x) := (0 : x)$. Assume that $Ann_R(x) \propto E$ is S-finite. Then there exist $(r_1, e_1), \ldots, (r_n, e_n) \in Ann_R(x) \propto E$ such that

$$(d, 0)Ann_R(x) \propto E \subseteq F = \langle (r_1, e_1), \ldots, (r_n, e_n)\rangle \subseteq Ann_R(x) \propto E.$$

Since $dE = E$, we get $E = \langle e_1, \ldots, e_n\rangle$, which is contradiction. So $(0 : (0, x))$ is not S-finite. Therefore, R is not S-coherent. □

Recall from [3, Theorem 2.4] that a ϕ-ring R is nonnil-coherent, if and only if, any direct product of ϕ-flat R-modules is ϕ-flat, if and only if, any product of R is a ϕ-flat R-module. Now, we aim to give an S-version of flatness that characterizes nonnil-S-coherent rings similarly to the classical case. For this, we well start by the following definition from [19]. Let M and N be R-modules and set $\tau_S(M) \doteqdot \{x \in M \mid sx = 0 \text{ for some } s \in S\}$. Then $\tau_S(M)$ is called the total S-torsion submodule of M; if $\tau_S(M) = 0$, then M is called an S-torsion-free module, and if $\tau_S(M) = M$, then M is called an S-torsion module. An R-homomorphism $f : M \to N$ is an S-monomorphism (resp., an S-epimorphism, an S-isomorphism) if the induced $S^{-1}R$-homomorphism $S^{-1}f : S^{-1}M \to S^{-1}N$ is a monomorphism (resp., an epimorphism, an isomorphism). A sequence $0 \to M \to N \to L \to 0$ is S-exact if the induced sequence $0 \to S^{-1}M \to S^{-1}N \to S^{-1}L \to 0$ is exact.

Definition 2.12. Let M be an R-module. Then M is said to be ϕ-S-flat if for any finitely generated nonnil ideal I of R, the natural homomorphism $I \otimes_R M \to R \otimes_R M$ is an S-monomorphism.

Obviously, every ϕ-flat module is ϕ-S-flat. However, the converse does not hold; indeed, let R be a domain which is not a field and S be the set of nonzero elements in R. Then every R-module is ϕ-S-flat. Since R is not a ϕ-Von Neumann regular, there exists some ϕ-S-flat module which is not ϕ-flat by [22, Theorem 4.1]. Clearly, if $S_1 \subseteq S_2$ are multiplicative subsets, then any ϕ-S_1-flat module is ϕ-S_2-flat; and if S^* is the saturation of S in R, then an R-module M is ϕ-S-flat if and only it is ϕ-S^*-flat.

Now, we give a characterization of ϕ-S-flat modules.

Proposition 2.13. *Let M be an R-module. Then the following assertions are equivalent*:

(1) *M is ϕ-S-flat,*
(2) *for any finitely generated nonnil ideal I of R, $\psi : I \otimes_R M \to IM$ is an S-isomorphism,*
(3) *$S^{-1}M$ is a ϕ-flat $S^{-1}R$-module.*

Proof. (1) $\iff$ (2) Let I be a finitely generated nonnil ideal of R. Consider the following commutative diagram:

$$\begin{array}{ccccc} & & I \otimes_R M & \xrightarrow{f} & R \otimes_R M \\ & & \psi \downarrow & & \cong \downarrow \\ 0 & \longrightarrow & IM & \longrightarrow & M \end{array}$$

We have M is ϕ-S-flat, if and only if, f is an S-monomorphism, if and only if, ψ is an S-monomorphism.

(1) $\Rightarrow$ (3) Let $J = S^{-1}I$ be a finitely generated nonnil ideal of $S^{-1}R$, where I is a finitely generated nonnil ideal of R. Since M is ϕ-S-flat, the natural homomorphism $I \otimes_R M \to R \otimes_R M$ is an S-monomorphism. By localizing at S, the natural homomorphism:

$$S^{-1}I \otimes_{S^{-1}R} S^{-1}M \cong S^{-1}(I \otimes_R M) \to S^{-1}(R \otimes_R M) \cong S^{-1}M$$

is an $S^{-1}R$-monomorphism. Thus $S^{-1}I$ is a ϕ-flat $S^{-1}R$-module.

(3) $\Rightarrow$ (1) Let I be a finitely generated nonnil ideal of R. Then $S^{-1}I$ is a finitely generated nonnil ideal of $S^{-1}R$. Since $S^{-1}M$ is a ϕ-flat $S^{-1}R$-module, the natural homomorphism $S^{-1}(I \bigotimes_R M) \to S^{-1}(R \bigotimes_R M)$ is an $S^{-1}R$-monomorphism. Hence M is a ϕ-S-flat module. □

Note that if there exists $s \in S \cap Nil(R)$, then there exists a positive integer n such that $0 = s^n \in S$, a contradiction. Hence we always have $S \cap Nil(R) = \emptyset$. Consequently every S-torsion R-module is ϕ-torsion and every ϕ-torsion free R-module is S-torsion free.

In [3, Theorem 2.4], Khoualdia and Benhissi proved that a ϕ-ring R is nonnil-coherent if and only if any product of R is ϕ-flat if and only if any product of ϕ-flat R-modules is ϕ-flat. Now we extend this to the S-version and obtain the promised result.

Theorem 2.14 (Nonnil-S-version of Chase Theorem)**.** *Let R be a ϕ-ring. The following statements are equivalent*:

(1) *R is a nonnil-S-coherent ring,*
(2) *any product of ϕ-flat R-modules is ϕ-S-flat,*
(3) *any product of R is ϕ-S-flat.*

Proof. (1) $\Rightarrow$ (2) Let R be a nonnil-S-coherent ring, $\{F_i\}_{i\in I}$ be a family of ϕ-S-flat R-modules and J be a finitely generated nonnil ideal of R. The following exact sequence $0 \to A = ker(\pi_J) \to R^n \xrightarrow{\pi_J} J \to 0$ shows that A is S-finite. Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} A\bigotimes\prod_{i\in I}F_i & \longrightarrow & R^n\bigotimes\prod_{i\in I}F_i & \longrightarrow & J\bigotimes\prod_{i\in I}F_i & & \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\alpha} & & \\ \prod_{i\in I}(A\bigotimes F_i) & \longrightarrow & \prod_{i\in I}(R^n\bigotimes F_i) & \longrightarrow & \prod_{i\in I}(J\bigotimes F_i) & \longrightarrow & 0 \end{array}$$

To prove that α is an S-monomorphism, we only need to show that β is an S-epimorphism. Since A is S-finite, there exists a finitely generated submodule of A such that $sA \subseteq K \subseteq A$ for some $s \in S$. The natural commutative diagram:

$$\begin{array}{ccc} K\bigotimes_R\prod_{i\in I}F_i & \longrightarrow & A\bigotimes_R\prod_{i\in I}F_i \\ \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\beta} \\ \prod_{i\in I}(K\bigotimes_R F_i) & \longrightarrow & \prod_{i\in I}(A\bigotimes_R F_i) \end{array}$$

induces the following commutative diagram by localizing at S.

$$\begin{array}{ccc} S^{-1}K\bigotimes_{S^{-1}R}S^{-1}(\prod_{i\in I}F_i) & \xrightarrow{\cong} & S^{-1}A\bigotimes_{S^{-1}R}S^{-1}(\prod_{i\in I}F_i) \\ \downarrow{\scriptstyle S^{-1}\nu} & & \downarrow{\scriptstyle S^{-1}\beta} \\ S^{-1}\left(\prod_{i\in I}(K\bigotimes F_i)\right) & \xrightarrow{f} & S^{-1}\left(\prod_{i\in I}(A\bigotimes F_i)\right) \end{array}$$

Since K is finitely generated, ν is an epimorphism, so $S^{-1}\nu$ is also. On the other hand, for any $a_i \in A$, $q_i \in F_i (i \in I)$ and $t \in S$, we obtain:

$$\frac{(a_i\bigotimes_R q_i)_{i\in I}}{t} = \frac{s(a_i\bigotimes_R q_i)_{i\in I}}{st} = \frac{(sa_i\bigotimes_R q_i)_{i\in I}}{st} \in S^{-1}\left(\prod_{i\in I}(K\bigotimes F_i)\right).$$

Thus f is an epimorphism. Now consider the following exact sequence:

$$0 \to K \to A \to A/K \to 0.$$

Since $sA/K = 0$, A/K is a ϕ-torsion R-module, hence:

$$0 \to K \bigotimes F_i \to A \bigotimes F_i \to A/K \bigotimes F_i \to 0$$

is an exact sequence by [22, Theorem 3.2]. Then f is a monomorphism, so it is an isomorphism. Thus $S^{-1}\beta$ is an epimorphism, and then β is an S-epimorphism. Hence $\prod_{i\in I} F_i$ is ϕ-S-flat by Proposition 2.13.

(2) $\Rightarrow$ (3) This is straightforward.

(3) $\Rightarrow$ (1) Let J be a nonnil ideal of R, the following exact sequence $0 \to J \to R \to R/J \to 0$ induces the following commutative diagram:

$$\begin{array}{ccccccccc} & & J\bigotimes\prod R & \xrightarrow{f} & R\bigotimes\prod R & \longrightarrow & R/J\bigotimes\prod R & \longrightarrow & 0 \\ & & \beta\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow & & \\ 0 & \longrightarrow & \prod(J\bigotimes R) & \longrightarrow & \prod(R\bigotimes R) & \longrightarrow & \prod(R/J\bigotimes R) & \longrightarrow & 0 \end{array}$$

Since $\prod R$ is a ϕ-S-flat module, we get that f is an S-monomorphism. Thus $ker(f) = ker(\beta)$ is S-torsion, and consequently β is an S-monomorphism. Now consider the following exact sequence:

$$0 \to A = ker(f) \to R^n \xrightarrow{i} J \to 0.$$

Then there is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & A\bigotimes\prod R & \xrightarrow{f} & R^n\bigotimes\prod R & \longrightarrow & J\bigotimes\prod R & \longrightarrow & 0 \\ & & \alpha\big\downarrow & & \cong\big\downarrow & & \beta\big\downarrow & & \\ 0 & \longrightarrow & \prod(A\bigotimes R) & \longrightarrow & \prod(R^n\bigotimes R) & \longrightarrow & \prod(J\bigotimes R) & \longrightarrow & 0 \end{array}$$

Since β is an S-monomorphism, α is an S-epimorphism. Hence A is S-finite by [19, Lemma 4.1]. Therefore, I is S-finitely presented. $\square$

3. Nonnil-S-coherent properties on some ring constructions

Now, we study the transfer of nonnil-coherent rings in the trivial ring extensions and in the amalgamation algebra along an ideal. From [10, Corollary 2.4], the trivial ring extension $R \propto M$ is a ϕ-ring if and only if R is a ϕ-ring and $sM = M$ for all $s \in R \setminus Nil(R)$.

Let M be an R-module and $r \in R$. Set $(0 :_M r) := \{m \in M \mid rm = 0\}$. From [9], $(0 :_M r)$ is a submodule of M such that $(0 : r)M \subset (0 :_M r)$, and so $(0 : r) \propto (0 :_M r)$ is an ideal of $R \propto M$. The following theorem characterizes when a trivial ring extension is a nonnil-S-coherent ring.

Theorem 3.1. *Let A be a ϕ-ring, and M be an A-module such that $aM = M$ for every $a \in A \setminus Nil(A)$. Let S be a multiplicative subset of $R = A \propto E$. Set S_0 as the projection of S on A. Then the following statements are equivalent*:

(1) *R is a nonnil-S-coherent ring,*
(2) *A is a nonnil-S_0-coherent ring and $(0:r) \propto (0:_M r)$ is an S-finite ideal of R for each $r \in A \setminus Nil(A)$,*
(3) *A is a nonnil-S_0-coherent ring and $R(r,0)$ is S-finitely presented for all $r \in A \setminus Nil(A)$.*

Proof. (1) $\Rightarrow$ (2) Assume that R is a nonnil-S-coherent ring. Let I and J be finitely generated nonnil ideals of A. It is easy to see that if $I = \langle a_1, \dots, a_n \rangle$, then $I \propto M = \langle (a_1, 0), \dots, (a_n, 0) \rangle$. Hence $I \propto M$ and $J \propto M$ are finitely generated nonnil ideals of R. Since R is a nonnil-S-coherent ring, $(I \propto M) \cap (J \propto M) = (I \cap J) \propto M$ is S-finite by Theorem 2.1. So there exist $(s, e) \in S$ and $(a_1, m_1), \dots, (a_n, m_n) \in (I \cap J) \propto M$ such that:

$$(s, e)(I \cap J) \propto M \subseteq (a_1, m_1)R + \cdots + (a_n, m_n)R.$$

In particular, $s(I \cap J) \subseteq a_1A + \cdots + a_nA$. Therefore, $I \cap J$ is S_0-finite. Let $r \in A \setminus Nil(A)$. Then, $((0,0) : (r,0)) = (0 : r) \propto (0 :_M r)$ is S-finite by Theorem 2.1, and so $(0 : r)$ is S_0-finite. Therefore, A is a nonnil-S_0-coherent ring by Theorem 2.1.

(2) $\Rightarrow$ (1) Assume that A is a nonnil-S_0-coherent ring and $(0 : r) \propto (0 :_M r)$ is an S-finite ideal of R for each $r \in A \setminus Nil(A)$. Let $I \propto M$ and $J \propto M$ be finitely generated nonnil ideals of R. Then, I and J are finitely generated nonnil ideals of A. Since A is a nonnil-S-coherent ring, $I \cap J$ is an S_0-finite ideal of A, and so there exist $s \in S_0$ and $a_1, \dots, a_n \in I \cap J$ such that $s(I \cap J) \subseteq a_1A + \cdots + a_nA$. Since $s \in S_0$, there exists $u \in M$ such that $(s, u) \in S$, and consequently $(s, u)(I \cap J) \subseteq (a_1, 0)R + \cdots + (a_n, 0)R$. Thus $(I \propto M) \cap (J \propto M) = (I \cap J) \propto M$ is an S-finite ideal of R. Let $(r, u) \in R \setminus Nil(R)$. Then, $((0,0) : (r,u)) = (0 : r) \propto (0 :_M r)$ is S-finite by hypothesis. Therefore, R is a nonnil-S-coherent ring by Theorem 2.1.

(2) $\Leftrightarrow$ (3) Let $r \in A \setminus Nil(A)$. Then, the following sequence $0 \to ((0,0) : (r,0)) \to R \to R(r,0) \to 0$ is exact. Therefore, by [6, Proposition 2.4] $(0 : r) \propto (0 : Mr)$ is S-finite if and only if $R(r, 0)$ is S-finitely presented. ☐

Corollary 3.2. *Let $R = A \propto M$ be a ϕ-ring such that $Z(A) = Nil(A)$ and S be a multiplicative subset of R. Set S_0 as the projection of S on A. Then R is a nonnil-S-coherent ring if and only if A is a nonnil-S_0-coherent ring and $(0 :_M r)$ is S_0-finite A-submodule of M for every $r \in A \setminus Nil(A)$.*

Corollary 3.3. *Let $R = A \propto M$ be a ϕ-ring such that $Z(A) = Nil(A)$, and let S be a multiplicative subset of R, set S_0 as the projection of S on A and let M be an S_0-Noetherian A-module. Then R is a nonnil-S-coherent ring if and only if A is a nonnil-S_0-coherent ring.*

For a ring R and an R-module M, set $Z_R(M) := \{r \in R \mid rm = 0 \text{ for some nonzero } m \in M\}$.

Corollary 3.4. *Let $R = A \propto M$ be a ϕ-ring such that $Z(A) = Nil(A) = Z_A(M)$ and S be a multiplicative subset of R. Set S_0 as the projection of S on*

A. Then R is a nonnil-S-coherent ring if and only if A is a nonnil-S_0-coherent ring.

Example 3.5. Let D be an integral with quotient K. Then $R = D \propto K$ is a nonnil-S-coherent ring with $S = (D \setminus \{0\}) \propto K$.

Let A and B be two rings, J a nonzero ideal of B, and $f : A \to B$ be a ring homomorphism. Set $R := A \bowtie^f J$ and $N(J) := Nil(B) \cap J$. Recall from [10, Theorem 2.1] that (1) If J is a nonnil ideal of B, then R is a ϕ-ring if and only if $f^{-1}(J) = 0$, A is an integral domain, and $N(J)$ is a divided prime ideal of $f(A) + J$, (2) If $J \subseteq Nil(B)$, then R is a ϕ-ring if and only if A is a ϕ-ring, and for each $i, j \in J$ and each $a \in A \setminus Nil(A)$, there exist $x \in Nil(A)$ and $k \in J$ such that $xa = 0$ and $j = kf(a) + i(f(x) + k)$. Moreover, let $\iota : A \to A \bowtie^f J$ be the natural embedding defined by $a \to (a, f(a))$ for each $a \in A$, and S be a multiplicative subset of A. Then $S' := \{(s, f(s)) \mid s \in S\}$ and $f(S)$ are multiplicative subsets of $A \bowtie^f J$ and B, respectively.

Now, we study the transfer of being ϕ-S-coherent rings in the amalgamation algebra along an ideal.

Theorem 3.6. *Let A and B be two rings and $f : A \to B$ be a ring homomorphism. Let J be a nonnil ideal of B and S be a multiplicative subset of A. Define $\bar{f} : A \to B/N(J)$ by $\bar{f}(a) = f(a) + N(J)$ for any $a \in A$. Assume that $A \bowtie^f J$ is a ϕ-ring. Then the following statements are equivalent:*

(1) *$A \bowtie^f J$ is a ϕ-S'-coherent ring,*
(2) *$A \bowtie^{\bar{f}} \frac{J}{N(J)}$ is an $\bar{S}'$-coherent domain with $\bar{S}' := \{(s, \bar{f}(s)) \mid s \in S\}$,*
(3) *$\bar{f}(A) + J/N(J)$ is an $\bar{f}(S)$-coherent domain.*

Proof. (1) $\Rightarrow$ (2) Assume that $A \bowtie^f J$ is a ϕ-S'-coherent ring. Since $A \bowtie^f J$ is a ϕ-ring, it follows that A is an integral domain by [10, Theorem 2.1(1)], and so $Nil\left(A \bowtie^f J\right) = 0 \times N(J)$. As $A \bowtie^f J$ is a ϕ-S'-coherent ring, $\frac{A \bowtie^f J}{0 \times N(J)}$ is an $\bar{S}'$-coherent domain. Therefore, $A \bowtie^{\bar{f}} \frac{J}{N(J)}$ is an $\bar{S}'$-coherent domain.

(2) $\Rightarrow$ (1) This follows directly from [20, Remark 2.6].

(2) $\Rightarrow$ (3) Assume that $A \bowtie^{\bar{f}} J/N(J)$ is a coherent domain. Then according to [10, Theorem 2.1(1)], we conclude that $f^{-1}(J) = \bar{f}^{-1}(J/N(J)) = 0$, and so by [7, Proposition 5.2] $\bar{f}(A) + J/N(J)$ is an integral domain. From [7, Proposition 5.1], $\bar{f}(A) + J/N(J) \cong A \bowtie^{\bar{f}} J/N(J)$, as desired.

(3) $\Rightarrow$ (2) By [10, Theorem 2.1(1)], we have $\bar{f}^{-1}(J/N(J)) = 0$ and from [7, Proposition 5.1], we obtain $\bar{f}(A) + J/N(J) \cong A \bowtie^{\bar{f}} J/N(J)$, which is an $\bar{S}'$-coherent domain, as desired. □

Corollary 3.7 investigates the transfer of being a nonnil-S-coherent ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nonnil ideal J.

Corollary 3.7. *Let A and B be two rings and $f : A \to B$ be a ring homomorphism. Let J be a nonnil ideal of B. Define $\bar{f} : A \to B/N(J)$ by*

$\bar{f}(a) = f(a) + N(J)$ *for any* $a \in A$. *Assume that* $A \bowtie^f J$ *is a* ϕ*-ring. Then* $A \bowtie^f J$ *is a nonnil-*S'*-coherent ring if and only if* $\bar{f}(A) + J/N(J)$ *is an* $\bar{S}'$*-coherent domain and* $\left(A \bowtie^f J\right)(r, f(r)+j)$ *is an* S'*-finitely presented ideal for any non-nilpotent element* $(r, f(r)+j)$ *of* $A \bowtie^f J$.

Proof. This follows immediately from Theorem 2.6 and Theorem 3.6. □

Theorem 3.8 studies the transfer of being a ϕ-S-coherent ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nil ideal J.

Theorem 3.8. *Let* A *and* B *be two rings and* $f : A \to B$ *be a ring homomorphism. Let* J *be a nil ideal of* B *and* S *be a multiplicative subset of* A. *Assume that* $A \bowtie^f J$ *is a* ϕ*-ring. Then,* $A \bowtie^f J$ *is a* ϕ*-*S'*-coherent ring if and only if* A *is a* ϕ*-*S*-coherent ring.*

Proof. Since $J \subseteq Nil(B)$, we have $N(J)=J$, and so $Nil\left(A \bowtie^f J\right)=Nil(A) \bowtie^f J$. Since $A \bowtie^f J$ is a ϕ-S'-coherent ring, $\frac{A \bowtie^f J}{Nil(A) \bowtie^f J} \cong \frac{A}{Nil(A)}$ is an $(S+Nil(A))$-coherent domain. Thus A is a ϕ-S-coherent ring. Conversely, since A is a ϕ-S-coherent ring, $\frac{A}{Nil(A)} \cong \frac{A \bowtie^f J}{Nil(A) \bowtie^f J}$ is an $(S' + Nil(A \bowtie^f J))$-coherent domain. Whence $A \bowtie^f J$ is a ϕ-S'-coherent ring. □

Corollary 3.9 studies the transfer of being a nonnil-coherent ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nil ideal J.

Corollary 3.9. *Let* A *and* B *be two rings and* $f : A \to B$ *be a ring homomorphism. Let* J *be a nil ideal of* B *and* S *be a multiplicative subset of* A. *Assume that* $A \bowtie^f J$ *is a* ϕ*-ring. Then the following are equivalent*:

(1) $A \bowtie^f J$ *is a nonnil-*S'*-coherent ring,*
(2) A *is a* ϕ*-*S*-coherent ring and* $\left(A \bowtie^f J\right)(r, f(r)+j)$ *is an* S'*-finitely presented ideal for any non-nilpotent element* $(r, f(r)+j)$ *of* $A \bowtie^f J$.

Proof. This follows immediately from Theorem 2.6 and Theorem 3.9. □

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