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3-HOM-LIE SUPERBIALGEBRAS AND 3-HOM-LIE CLASSICAL YANG-BAXTER EQUATIONS

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Abstract. 3-Lie algebras are in close relationships with many fields. In this paper we are concerned with the study of 3-Hom-Lie super algebras, the concepts of 3-Hom-Lie coalgebras and how they make a 3-Hom-Lie superbialgebras, we study the structures of such categories of algebras and the relationships between each others. We study a super twisted 3-ary version of the Yang-Baxter equation, called the super 3-Lie classical Hom-Yang-Baxter equation (3-Lie CHYBE), which is a general form of 3-Lie classical Yang-Baxter equation and prove that the superbialgebras induced by the solutions of the super 3-Lie CHYBE induce the coboundary local cocycle 3-Hom-Lie superbialgebras.

1. Introduction

3-Lie algebras [16] are a generalisation of Lie algebras which have an important relationship with different fields in mathematics and mathematical physics. For example, the structure of 3-Lie algebras is applied to the study of supersymmetry and gauge symmetry transformations of the world-volume theory. The structure of 3-Hom-Lie superalgebras is more complicated than that of Lie superalgebras (see [7, 13] for more details about Lie superalgebras). We need to excavate more constructions of 3-Hom-Lie superalgebras, and more relationships with algebraic systems related to 3-Hom-Lie supercoalgebras, such as 3-Hom-Lie superbialgebras.

V. G. Drinfel'd introduced Lie bialgebras in [8], which is the algebraic structure corresponding to a Poisson-Lie group and the classical structure of a quantized universal enveloping algebra, with a suitable compatibility condition between the Lie bracket $[\cdot,\cdot]$ and the Lie comultiplication Δ in the following form:

$$\Delta([x,y]) = [\Delta(x), y] + [x, \Delta(y)]. \tag{1.1}$$

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As demonstrated in [4,9,19] this compatibility condition can be concisely stated as the condition that the Lie comultiplication Δ is a derivation with respect to the Lie algebra structure on $\wedge^2 L$. Note that, in fact,

$$[x, \Delta(y)] = (ad_x \otimes 1 + 1 \otimes ad_x)\Delta(y),$$

where $ad : L \to gl(L)$ is the adjoint representation.

On the other hand, it is interesting to consider the bialgebra structures of 3-Lie algebras as we can see in [6, 12]. Giving a 3-Hom-Lie superalgebra $(L, [\cdot, \cdot, \cdot], \alpha)$, a 3-Hom-Lie supercoalgebra (L, Δ, α) such that (L^*, Δ^*, α) is also a 3-Hom-Lie superalgebra, the most important part for a bialgebra theory is the compatibility conditions (see [5, 12, 18] for the classic case). It is quite common for an algebraic system to have multiple bialgebra structures that differ only by their compatibility conditions:

$$\Delta([x, y, z]) = [\Delta(x), y, z] + [x, \Delta(y), z] + [x, y, \Delta(z)]. \tag{1.2}$$

The compatibility condition for n-Lie algebras see [1].

The coboundary Lie bialgebra [2, 14] associates to a solution of the classical Yang-Baxter equation, and it has been playing an important role in mathematics and physics. On the other hand 3-Hom-Lie supercoalgebras and 3-Lie superbialgebras are important concepts in 3-Hom-Lie superalgebras. The classical Yang-Baxter equation (shorthand for CYBE) is a 1-1 correspondence between triangular Lie bialgebras and solutions of the CYBE [18]. Motivated by recent work on Hom-Lie bialgebras and the Hom-Yang-Baxter equation [5], 3-Lie algebra and 3-Lie classical Yang-Baxter equation [2], CHYBE has far-reaching mathematical significance and is closely related to many topics in mathematical physics, including Hamiltonian structures, Kac-Moody algebras, Poisson-Lie groups, quantum groups, Hopf algebras, and Lie bialgebras. Motivated by this, we define the 3-Hom Lie supercoalgebra and the 3-Hom-Lie superbialgebra, and study the structure of them and the relationships with 3-Hom-Lie superalgebras. We introduce 3-Hom-Lie superbialgebras whose compatibility conditions between the multiplication and co-multiplication are given by local cocycle condition, and is called the local cocycle 3-Hom-Lie superbialgebra. We study a twisted 3-ary Hom super version of the classical Yang-Baxter equation, called the 3-Lie super classical Hom-Yang-Baxter equation (shorthand for 3-Lie super CHYBE), which is a general form of 3-Lie classical Yang-Baxter equation studied in [18], and use the solutions of 3-Lie super CHYBE to induce the coboundary local cocycle 3-Hom-Lie superbialgebras.

The paper is organized as follow. In Section 2, we provide some basics about the 3-Hom-Lie supercoalgebra and define 3-Hom-Lie superbialgebra with the derivation compatibility, the local cocycle 3-Hom-Liesuper bialgebra with cocycle compatibility, and the 3-Lie super CHBYE. In Section 3, we define the coboundary local cocycle 3-Hom-Lie superbialgebra and prove that it can be constructed by a multiplicative 3-Hom-Lie superalgebra and a solution of the 3-Lie super CHBYE, expending the notion and results obtained in [18] to superalgebras.

2. 3-Hom-Lie superalgebras and 3-Hom-Lie superbialgebras

Definition 2.1. A quadruple $(L, [\cdot,\cdot,\cdot], \alpha)$, where L is a $\mathbb{Z}_2$-gradded vector space, α is a linear map of L, and $[\cdot,\cdot,\cdot] : L^{\otimes 3} \to L$ is a 3-linear map, is called a 3-Hom-Lie superalgebra if the following conditions are satisfied:

(a) $[x_1, x_2, x_3] = -(-1)^{|x_1||x_2|}[x_2, x_1, x_3]$,

(b) $[\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]]$
$= [[x_1, x_2, x_3], \alpha(x_4), \alpha(x_5)] + (-1)^{|x_3|(|x_1|+|x_2|)}[\alpha(x_3), [x_1, x_2, x_4], \alpha(x_5)]$
$+ (-1)^{(|x_3|+|x_4|)(|x_1|+|x_2|)}[\alpha(x_3), \alpha(x_4), [x_1, x_2, x_5]]$
for all $x_1, x_2, x_3, x_4, x_5 \in L$.

The fundamental identity could be rewritten with the operator:

$$ad_{x_1,x_2} : L \to L,\ ad_{x_1,x_2}x = [x_1, x_2, x] \tag{2.1}$$

in the form as:

$$\begin{aligned} ad_{\alpha(x_1),\alpha(x_2)}[x_3, x_4, x_5] = {} & [ad_{x_1,x_2}x_3, \alpha(x_4), \alpha(x_5)] \\ & + (-1)^{|x_3|(|x_1|+|x_2|)}[\alpha(x_3), ad_{x_1,x_2}x_4, \alpha(x_5)] \\ & + (-1)^{(|x_3|+|x_4|)(|x_1|+|x_2|)}[\alpha(x_3), \alpha(x_4), ad_{x_1,x_2}x_5]. \end{aligned} \tag{2.2}$$

Remark 2.2. Let $(A, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie superalgebra where the skew-symmetric linear map $[\cdot,\cdot,\cdot] : \otimes^3 A \to A$ with $\alpha : A \to A$ that $\alpha([\cdot,\cdot,\cdot]) = [\cdot,\cdot,\cdot] \circ \alpha^{\otimes^3}$. Then we get the following equation:

$$\begin{aligned} 0 = {} & [[x_1, x_2, x_3], \alpha(x_4), \alpha(x_5)] - (-1)^{|x_4||x_3|}[[x_1, x_2, x_4], \alpha(x_3), \alpha(x_5)] \\ & + (-1)^{|x_2|(|x_3|+|x_4|)}[[x_1, x_3, x_4], \alpha(x_2), \alpha(x_5)] \\ & - (-1)^{|x_1|(|x_2|+|x_3|+|x_4|)}[[x_2, x_3, x_4], \alpha(x_1), \alpha(x_5)]. \end{aligned}$$

Proof. If $(A, [\cdot,\cdot,\cdot], \alpha)$ is a 3-Hom-Lie superalgebra, then applying Eq. (2.2) to the last term of Eq. (2.2), we have

$$\begin{aligned} & [\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]] \\ = {} & [[x_1, x_2, x_3], \alpha(x_4), \alpha(x_5)] + (-1)^{|x_3|(|x_1|+|x_2|)}[\alpha(x_3), [x_1, x_2, x_4], \alpha(x_5)] \\ & + (-1)^{(|x_3|+|x_4|)(|x_1|+|x_2|)}[[x_3, x_4, x_1], \alpha(x_2), \alpha(x_5)] \\ & + (-1)^{(-1)^{|x_2|(|x_3|+|x_4|)}}[\alpha(x_1), [x_3, x_4, x_2], \alpha(x_5)] \\ & + [\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]], \end{aligned}$$

proving the remark. ☐

The 3-Lie superalgebra structure on the $\mathbb{Z}_2$-vector space $(L \oplus V)$ is given by:

$$\overline{\rho}(x_1 + a_1, x_2 + a_2, x_3 + a_3) = \rho_1(x_1, x_2)a_3 + (-1)^{|x_1|(|x_2|+|x_3|)}\rho_1(x_2, x_3)a_1$$

$$+ (-1)^{|x_3|(|x_2|+|x_1|)}\rho_1(x_3, x_1)a_2,$$

where

$$\overline{\rho} : \wedge^3(L \oplus V) \longrightarrow (L \oplus V).$$

Proposition 2.3. *Let $(L, [\cdot,\cdot,\cdot], \alpha_1)$ be a 3-Hom-Lie superalgebra. A graded representation of L is a 4-tuple (V, ρ, α_2), where V is a linear space, $\alpha_2 \in gl(V)$ is a commuting linear map and $\rho : \wedge^2 L \to End(V)$ is a linear map such that, for all $x_1, x_2, x_3, x_4 \in L$, we have*:

$$\rho(\alpha_1(x_1), \alpha_1(x_2)) \circ \alpha_2 = \alpha_2 \circ \rho(x_1, x_2), \tag{2.3}$$

$$\begin{aligned} &\rho(\alpha_1(x_1), \alpha_1(x_2))\rho(x_3, x_4) \\ &= (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|}\rho(\alpha_1(x_3), \alpha_1(x_4))\rho(x_1, x_2) \\ &\quad + \rho([x_1, x_2, x_3], \alpha_1(x_4))\alpha_2 \\ &\quad - (-1)^{|x_3||x_4|}\rho([x_1, x_2, x_4], \alpha_1(x_3))\alpha_2, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\rho(\alpha_1(x_1), [x_2, x_3, x_4])\alpha_2 \\ &= (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|}\rho(\alpha_1(x_3), \alpha_1(x_4))\rho(x_1, x_2) \\ &\quad - (-1)^{|x_1|(|x_4|+|x_2|)}\rho(\alpha_1(x_1), \alpha_1(x_3))\rho(x_4, x_2) \\ &\quad + (-1)^{|x_1|(|x_2|+|x_3|)}\rho(\alpha_1(x_2), \alpha_1(x_3))\rho(x_1, x_4). \end{aligned} \tag{2.5}$$

Proof. The quadruple (V, ρ, α_1) is a representation if and only if

$$\begin{aligned} &[x_1 + a_1, x_2 + a_2, x_3 + a_3] \\ &= [x_1, x_2, x_3] + \rho_1(x_1, x_2)a_3 + (-1)^{|x_1|(|x_2|+|x_3|)}\rho_1(x_2, x_3)a_1 \\ &\quad + (-1)^{|x_3|(|x_2|+|x_1|)}\rho_1(x_3, x_1)a_2. \end{aligned}$$

By straightforward computations it is easy to prove

$$\begin{aligned} 0 = &[(\alpha_1 \oplus \alpha_2)(x_1 + a_1), (\alpha_1 \oplus \alpha_2)(x_2 + a_2), [x_3 + a_3, x_4 + a_4, x_5 + a_5]] \\ &- [[x_1 + a_1, x_2 + a_2, x_3 + a_3](\alpha_1 \oplus \alpha_2)(x_4 + a_4), (\alpha_1 \oplus \alpha_2)(x_5 + a_5)] \\ &- (-1)^{|x_3|(|x_1|+|x_2|)}[(\alpha_1 \oplus \alpha_2)(x_3 + a_3)[x_1 + a_1, x_2 + a_2, x_4 + a_4] \\ &\times (\alpha_1 \oplus \alpha_2)(x_5 + a_5)] - (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}[(\alpha_1 \oplus \alpha_2)(x_3 + a_3), \\ &(\alpha_1 \oplus \alpha_2)(x_4 + a_4)[x_1 + a_1, x_2 + a_2, x_5 + a_5]]. \end{aligned}$$

□

Remark 2.4. With the above notations, (V, ρ, α_1) is called a representation of L, or (V, ρ, α_1) is an L-module.

Example 2.5. Let L be a 3-Hom-Lie superalgebra. The linear map ad : $L \otimes L \to \mathfrak{gl}(L)$ with $x_1, x_2 \to ad_{x_1,x_2}$ for any $x_1, x_2 \in L$ defines a representation (L, ad, α) which is called the adjoint representation of L, where ad_{x_1,x_2} is given by Eq. (2.1). The dual representation (L^*, ad^*, α^*) of the adjoint representation (L, ad, α) of a 3-Hom-Lie superalgebra L is called the coadjoint representation.

Example 2.6. Let V be a 4-dimensional vector space with the basis $\{e_1, e_2, e_3, e_4\}$. Define the following brackets:

$$[e_1, e_2, e_3] = -e_4; \quad [e_1, e_2, e_4] = e_3; \quad [e_1, e_3, e_4] = -e_2; \quad [e_2, e_3, e_4] = e_1.$$

With this bracket, $(V, [\cdot,\cdot,\cdot], \alpha)$ is a 3-Hom-Lie superalgebra. Let α be a linear map of V defined by:

$$\alpha(e_1) = -e_2; \quad \alpha(e_2) = -e_1; \quad \alpha(e_3) = -e_4; \quad \alpha(e_4) = -e_3.$$

Definition 2.7. Let (V, ρ, α) be a representation of a 3-Hom-Lie superalgebra L. Define $\rho^* : \otimes^2 \to gl(V^*)$ by

$$\langle \rho^*(x_1, x_2)a, v\rangle = -(-1)^{(|x_1|+|x_2|)|a|}\langle a, \rho(x_1, x_2)v\rangle$$

$\forall a \in V^*, x_1, x_2 \in L, v \in V$.

Definition 2.8. Let $(L, [\cdot,\cdot], \alpha)$ be a 3-Hom-Lie superalgebra, L^* be the dual space of L. Then we get the dual mapping $\Delta : L^* \to L^* \otimes L^* \otimes L^*$ of $[\cdot,\cdot]$, satisfying for every $x, y, z \in L$ and $\xi, \eta, \zeta \in L^*$,

$$\langle \Delta(\xi), x \otimes y \otimes z\rangle = \langle \xi, [x, y, z]\rangle, \tag{2.6}$$
$$\langle \xi \otimes \eta \otimes \zeta, x \otimes y \otimes z\rangle = \langle \xi, x\rangle\langle \eta, y\rangle\langle \zeta, z\rangle,$$

where $\langle,\rangle$ is the natural nondegenerate symmetric bilinear form on the vector space $L \oplus L^*$ is defined by $\langle \xi,\, x\rangle = \xi(x)$, $\xi \in L^*$, $x \in L$.

Definition 2.9. Given a representation (ρ, V, α), denote by $C^p_{\alpha,A}(L, V) = C^p_{\alpha,A}(L, V)_{\bar{0}} \oplus C^p_{\alpha,A}(L, V)_{\bar{1}}$ the set of p-cochains where:

$$C^p_{\alpha,A}(L, V) := \{\text{linear maps } f : \underbrace{L \otimes L \otimes \cdots \otimes L}_{n} \to V,\ \alpha \circ f = f \circ \alpha^{\otimes^n}\}.$$

The coboundary operators associated to the module are given in ([3, 10, 11, 15, 17, 20]). For $n \geq 1$, the coboundary operator $\delta : C^n_{\alpha,A}(L, V) \to C^{n+2}_{\alpha,A}(L, V)$ is defined as follows:

Define the $\mathbb{K}$-linear maps $\delta : C^{n-1}(L, M) \to C^n(L, M)$ given by

$$\begin{aligned}
&\delta f(x_1, \dots, x_{2n+1})\\
= &-(-1)^{(|f|+|x_1|+|x_2|+\cdots+|x_{2n-2}|)(|x_{2n-1}|+|x_{2n+1}|)+|x_{2n+1}||x_{2n}|}\\
&\rho(\alpha^n(x_{2n-1}), \alpha^n(x_{2n+1}))f(x_1, \dots, x_{2n-2}, x_{2n})\\
&+(-1)^{(|f|+|x_1|+|x_2|+\cdots+|x_{2n-1}|)(|x_{2n}|+|x_{2n+1}|)}\\
&\rho(\alpha^n(x_{2n}), \alpha^n(x_{2n+1}))f(x_1, \dots, x_{2n-1})\\
&+\sum_{k=1}^{n}(-1)^{k+n+(|f|+|x_1|+|x_2|+\cdots+|x_{2k-2}|)(|x_{2k-1}|+|x_{2k}|)}\\
&\rho(\alpha^n(x_{2k-1}), \alpha^n(x_{2k}))f(x_1, \dots, \widehat{x}_{2k-1}, \widehat{x}_{2k}, \dots, x_{2n+1})\\
&+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{k+(|x_{2k+1}|+\cdots+|x_{j-1}|)(|x_{2k-1}|+|x_{2k}|)}
\end{aligned}$$

$$f(\alpha(x_1), \dots, \widehat{x}_{2k-1}, \widehat{x}_{2k}, \dots, [x_{2k-1}, x_{2k}, x_j], \dots, \alpha(x_{2n+1})).$$

Let L be a 3-Hom-Lie superalgebra and (ρ, V, α) be the representation of L. A linear map $f : L \to V$ is called a 1-cocycle on L associated to (ρ, V, α) if it satisfies:

$$\begin{aligned} f([x,y,z]) = {} & (-1)^{|f|(|x|+|y|)}\rho(\alpha(x),\alpha(y))f(z) \\ & + (-1)^{(|f|+|x|)(|y|+|z|)}\rho(\alpha(y),\alpha(z))f(x) \\ & - (-1)^{|z||y|+|f|(|z|+|x|)}\rho(\alpha(x),\alpha(z))f(y). \end{aligned} \tag{2.7}$$

Definition 2.10. A 3-Hom-Lie supercoalgebra is a triple (L, Δ, α) consisting of a linear space L, a bilinear map $\Delta : L \to L \otimes L \otimes L$ and a linear map $\alpha : L \to L$ satisfying

(2.8) $Im(\Delta) \subset L \otimes L \otimes L$,

(2.9) $(1-\omega_1-\omega_2-\omega_3)(\alpha \otimes \alpha \otimes \Delta)\Delta(x)=0$ (Hom-super-co-Jacobi Identity),

where $\forall x, x_1, x_2, x_3, x_4, x_5 \in L$,

$$1,\ \omega_i : L \otimes L \otimes L \otimes L \otimes L \to L \otimes L \otimes L \otimes L \otimes L,\ 1 \le i \le 3,$$

$$\begin{aligned} 1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) &= x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5, \\ \omega_1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) &= (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5, \\ \omega_2(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) &= (-1)^{(|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \\ & \quad x_4 \otimes x_5 \otimes x_1 \otimes x_2 \otimes x_3, \\ \omega_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) &= (-1)^{(|x_1|+|x_2|)(|x_3|+|x_5|)+|x_5|(|x_4|+|x_3|)} \\ & \quad x_5 \otimes x_3 \otimes x_1 \otimes x_2 \otimes x_4. \end{aligned}$$

We call Δ the cobracket. If $\Delta \circ \alpha = \alpha^{\otimes^3} \circ \Delta$, then we say L is comultiplicative.

Now we study 3-Hom-Lie supercoalgebras by means of structure constants. Let (L, Δ, α) be a 3-Hom-Lie supercoalgebra with a basis $e_1, \dots, e_m$. Assume

$$\Delta(e_l) = \sum_{1 \le r < s < t \le n} c^l_{rst} e_r \wedge e_s \wedge e_t,\ c^k_{rst} \in \mathbb{K},\ 1 \le l \le m.$$

Then we have

$$\begin{aligned} (\alpha \otimes \alpha \otimes \Delta) \circ \Delta(e_l) &= \sum_{r<s<t} c^l_{rst} \alpha(e_r) \wedge \alpha(e_s) \wedge \Delta(e_t) \\ &= \sum_{r<s<t} \sum_{i<j<k} c^t_{ijk} c^l_{rst} \alpha(e_r) \wedge \alpha(e_s) \wedge e_i \wedge e_j \wedge e_k, \end{aligned}$$

$$\begin{aligned} & (1-\omega_1-\omega_2-\omega_3) \circ (1 \otimes 1 \otimes \Delta) \circ \Delta(e_l) \\ = {} & (1-\omega_1-\omega_2-\omega_3)\Big(\sum_{r<s<t} \sum_{i<j<k} c^t_{ijk} c^l_{rst} \alpha(e_r) \wedge \alpha(e_s) \wedge e_i \wedge e_j \wedge e_k\Big) \end{aligned}$$

$$= \sum_{r<s<t}\sum_{i<j<k} c^t_{ijk}c^l_{rst}\alpha(e_r)\wedge\alpha(e_s)\wedge e_i\wedge e_j\wedge e_k$$
$$-\sum_{r<s<t}\sum_{i<j<k} c^t_{rsk}c^l_{ijt}(-1)^{(|e_r|+|e_s|)(|e_i|+|e_j|)}e_i\wedge e_j\wedge\alpha(e_r)\wedge\alpha(e_s)\wedge e_k$$
$$-\sum_{r<s<t}\sum_{i<j<k}(-1)^{(|e_r|+|e_s|+|e_i|)(|e_j|+|e_k|)}c^l_{jkt}c^t_{rsi}e_j\wedge e_k\wedge\alpha(e_r)\wedge\alpha(e_s)\wedge e_i$$
$$-\sum_{r<s<t}\sum_{i<j<k}(-1)^{(|e_s|+|e_r|)(|e_i|+|e_k|)+|e_k|(|e_j|+|e_i|)}$$
$$c^l_{kit}c^t_{rsj}e_k\wedge e_i\wedge\alpha(e_r)\wedge\alpha(e_s)\wedge e_j$$
$$= \sum_{r<s<t}\sum_{i<j<k}[c^t_{ijk}c^l_{rst}-c^t_{rsk}c^l_{ijt}-c^l_{jkt}c^t_{rsi}-c^l_{kit}c^t_{rsj}]\alpha(e_r)\wedge\alpha(e_s)\wedge e_i\wedge e_j\wedge e_k,$$

$$c^k_{i_1i_2i_3} = sgn(\sigma)c^k_{i_{\sigma(1)}i_{\sigma(2)}i_{\sigma(3)}},\ 1\le i_1,i_2,i_3\le m.$$

Therefore,

$$\sum_{k=1}^{n}(c^t_{ijk}c^l_{rst}-c^t_{rsk}c^l_{ijt}-c^l_{jkt}c^t_{rsi}-c^l_{kit}c^t_{rsj})=0,\ 1\le i,j,k,l\le m.$$

Following the above discussions, we obtain the structural description of 3-Hom-Lie supercoalgebras in terms of structure constants.

Definition 2.11. A 3-Hom-Lie superbialgebra with derivation compatibility condition is a quadruple $(L,[\cdot,\cdot,\cdot],\Delta,\alpha)$ such that:
(1) $(L,[\cdot,\cdot,\cdot],\alpha)$ is a 3-Hom-Lie superalgebra,
(2) (L,Δ,α) is a 3-Hom-Lie supercoalgebra,
(3) Δ and $[\cdot,\cdot,\cdot]$ satisfy the following derivation compatibility condition:

$$\begin{aligned}\Delta([x,y,z]) = {} & ad^{(3)}(\alpha(x),\alpha(y))\Delta(z)-(-1)^{|z||y|}ad^{(3)}(\alpha(x),\alpha(z))\Delta(y)\\ & +(-1)^{|x|(|y|+|z|)}ad^{(3)}(\alpha(y),\alpha(z))\Delta(x),\end{aligned}\tag{2.10}$$

where

$$\begin{aligned}& ad^{(3)}(\alpha(x),\alpha(y))(u\otimes v\otimes w)\\ = {} & (ad(\alpha(x),\alpha(y))\otimes\alpha\otimes\alpha)(u\otimes v\otimes w)\\ & +(-1)^{|u|(|x|+|y|)}(\alpha\otimes ad(\alpha(x),\alpha(y))\otimes\alpha)(u\otimes v\otimes w)\\ & +(-1)^{(|u|+|v|)(|x|+|y|)}(\alpha\otimes\alpha\otimes ad(\alpha(x),\alpha(y)))(u\otimes v\otimes w).\end{aligned}$$

The condition (1) in Definition 2.11 is equivalent to that $(L^*,[\cdot,\cdot]^*,\alpha^*)$ is a 3-Hom-Lie supercoalgebra with $[\cdot,\cdot]^*: L^*\to L^*\otimes L^*\otimes L^*$ defined by (2.6).

The condition (2) in Definition 2.11 is equivalent to that (L^*,Δ^*,α^*) is a 3-Hom-Lie superalgebra with the 3-Lie superbracket $\Delta^*: L^*\otimes L^*\otimes L^*\to L^*$ defined by (2.6).

An alternate way of writing the condition (3) is for every $x, y, z \in L$, $\xi, \eta, \zeta \in L^*$,

$$\begin{aligned}\langle \Delta^*(\xi,\eta,\zeta), [x,y,z]\rangle &= \langle \xi\otimes\eta\otimes\zeta, \Delta([x,y,z])\rangle \\ &= \langle \xi\otimes\eta\otimes\zeta, ad^{(3)}(\alpha(x),\alpha(y))\Delta(z)\rangle \\ &\quad - (-1)^{|z||y|}\langle \xi\otimes\eta\otimes\zeta, ad^{(3)}(\alpha(x),\alpha(z))\Delta(y)\rangle \\ &\quad + (-1)^{|x|(|y|+|z|)}\langle \xi\otimes\eta\otimes\zeta, ad^{(3)}(\alpha(y),\alpha(z))\Delta(x)\rangle.\end{aligned}$$

For any $1 \le p \ne q \le n$, define an inclusion $\cdot_{pq} : \otimes^2 L \longrightarrow \otimes^n L$ by sending $r = \sum_i x_i \otimes y_i \in L \otimes L$ to

$$r_{pq} := \sum_i z_{i1}\otimes\cdots\otimes z_{in}, \qquad \text{where } z_{ij} = \begin{cases} x_i, & j = p, \\ y_i, & j = q, \\ 1, & i \ne p, q. \end{cases}$$

For example, when $n = 4$, we have

$$r_{12} = \sum_i x_i \otimes y_i \otimes 1 \otimes 1 \in L^{\otimes 4}, \quad r_{21} = \sum_i y_i \otimes x_i \otimes 1 \otimes 1 \in L^{\otimes 4}.$$

3. Coboundary 3-Hom-Lie superbialgebras

Definition 3.1. Let L be a 3-Lie superalgebra and $r \in L \otimes L$. The equation

$$[[r, r, r]] = 0$$

is called the 3-*Lie classical Yang-Baxter equation.*

This can be regarded as a natural extension of the classical Yang-Baxter equation

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

to the context of 3-Lie superalgebras.

When L is a 3-Hom-Lie superalgebra with the 3-Hom-Lie superbracket $[\cdot,\cdot,\cdot]$, for any $r = \sum_i x_i \otimes y_i = \sum_i \alpha(x_i) \otimes \alpha(y_i) \in L \otimes L$, we define $[[r, r, r]] \in \otimes^4 L$ by:

$$\begin{aligned}[[r,r,r]] &:= [r_{12}, r_{13}, r_{14}] + [r_{12}, r_{23}, r_{24}] + [r_{13}, r_{23}, r_{34}] + [r_{14}, r_{24}, r_{34}] \\ &= \sum_{i,j,k}\Big((-1)^{|x_i||x_j|}[x_i, x_j, x_k]\otimes\alpha(y_i)\otimes\alpha(y_j)\otimes\alpha(y_k) \\ &\quad + (-1)^{|x_k||x_j|}\alpha(x_i)\otimes[y_i, x_j, x_k]\otimes\alpha(y_j)\otimes\alpha(y_k) \\ &\quad + (-1)^{|x_j||y_i|}\alpha(x_i)\otimes\alpha(x_j)\otimes[y_i, y_j, x_k]\otimes\alpha(y_k) \\ &\quad + (-1)^{|y_j||y_k|}\alpha(x_i)\otimes\alpha(x_j)\otimes\alpha(x_k)\otimes[y_i, y_j, y_k]\Big).\end{aligned}$$

For any $r = \sum_i x_i \otimes y_i \in L \otimes L$, set

$$\Delta_1(x) := \sum_{i,j}[x, x_i, x_j]\otimes\alpha(y_j)\otimes\alpha(y_i);$$

$$\Delta_2(x) := \sum_{i,j}(-1)^{(|x|+|x_i|)|y_i|}\alpha(y_i)\otimes[x,x_i,x_j]\otimes\alpha(y_j);\tag{3.1}$$
$$\Delta_3(x) := \sum_{i,j}(-1)^{(|x|+|x_i|)(|y_j|+|y_i|)}\alpha(y_j)\otimes\alpha(y_i)\otimes[x,x_i,x_j]$$

for $x\in L$.

Lemma 3.2. *With the above notations, we have*

(i) Δ_1 *is a* 1*-cocycle associated to the representation* $(A\otimes A\otimes A, \mathrm{ad}_1\otimes\alpha\otimes\alpha)$;
(ii) Δ_2 *is a* 1*-cocycle associated to the representation* $(A\otimes A\otimes A, \alpha\otimes\mathrm{ad}_1\otimes\alpha)$;
(iii) Δ_3 *is a* 1*-cocycle associated to the representation* $(A\otimes A\otimes A, \alpha\otimes\alpha\otimes\mathrm{ad}_1)$.

Where

$$ad_1(x_1,x_2): L\to L,\ \ ad_1(x_1,x_2)x = [\alpha(x_1),\alpha(x_2),x].$$

Proof. For all $x,y,z\in L$, we have

$$\begin{aligned}
&\Delta_1([x,y,z])\\
=&\sum_{i,j}[[x,y,z],x_i,x_j]\otimes\alpha(y_j)\otimes\alpha(y_i)\\
=&\sum_{i,j}[[x,y,z],\alpha(x_i),\alpha(x_j)]\otimes\alpha^2(y_j)\otimes\alpha^2(y_i)\\
=&\sum_{i,j}(-1)^{(|x|+|y|+|z|)(|x_i|+|x_j|)}[\alpha(x_i),\alpha(x_j),[x,y,z]]\otimes\alpha^2(y_j)\otimes\alpha^2(y_i)\\
=&\sum_{i,j}(-1)^{(|x|+|y|+|z|)(|x_i|+|x_j|)}\Big([[x_i,x_j,x],\alpha(y),\alpha(z)]\\
&+(-1)^{(|x_i|+|x_j|)|x|}[\alpha(x),[x_i,x_j,y],\alpha(z)]\\
&+(-1)^{(|x_i|+|x_j|)(|x|+|y|)}[\alpha(x),\alpha(y),[x_i,x_j,z]]\Big)\otimes\alpha^2(y_j)\otimes\alpha^2(y_i)\\
=&\sum_{i,j}(-1)^{(|x|+|y|+|z|)(|x_i|+|x_j|)}\Big((-1)^{|x|(|x_i|+|x_j|)}[[x,x_i,x_j],\alpha(y),\alpha(z)]\\
&+(-1)^{(|x_i|+|x_j|)|x|}(-1)^{|y|(|x_i|+|x_j|)}[\alpha(x),[y,x_i,x_j],\alpha(z)]\\
&+(-1)^{(|x_i|+|x_j|)(|x|+|y|)}(-1)^{|z|(|x_i|+|x_j|)}[\alpha(x),\alpha(y),[z,x_i,x_j]]\Big)\\
&\otimes\alpha^2(y_j)\otimes\alpha^2(y_i)\\
=&(-1)^{|x|(|y|+|z|)}(\mathrm{ad}_{(\alpha(y),\alpha(z))}\otimes\alpha\otimes\alpha)\Delta_1(x)\\
&-(-1)^{|z||y|}(\mathrm{ad}_{(\alpha(x),\alpha(z))}\otimes\alpha\otimes\alpha)\Delta_1(y)+(\mathrm{ad}_{(\alpha(x),\alpha(y))}\otimes\alpha\otimes\alpha)\Delta_1(z)\\
=&(-1)^{|x|(|y|+|z|)}(\mathrm{ad}_1(y,z)\otimes\alpha\otimes\alpha)\Delta_1(x)-(-1)^{|z||y|}(\mathrm{ad}_1(x,z)\otimes\alpha\otimes\alpha)\Delta_1(y)\\
&+(\mathrm{ad}_1(x,y)\otimes\alpha\otimes\alpha)\Delta_1(z).
\end{aligned}$$

Therefore, Δ_1 is 1-cocycle associated to the representation $(A\otimes A\otimes A, \mathrm{ad}_1\otimes\alpha\otimes\alpha)$. The other two statements can be proved similarly. □

Proposition 3.3. *Let L be a 3-Hom-Lie superalgebra and $r \in L \otimes L$. Let $\Delta = \Delta_1 + \Delta_2 + \Delta_3$, where $\Delta_1, \Delta_2, \Delta_3$ are induced by r as in Eq. (3.1). Then $\Delta^* : L^* \otimes L^* \otimes L^* \to L^*$ defines a skew-symmetric operation.*

Proof. We only need to prove that for all $x \in L$,

$$\Delta(x) + \sigma_{12}\Delta(x) = 0, \ \Delta(x) + \sigma_{23}\Delta(x) = 0.$$

In fact, we have

$$\begin{aligned}\sigma_{12}\Delta_1(x) &= \sum_{i,j}(-1)^{|y_j|(|x|+|x_i|+|x_j|)}\alpha(y_j) \otimes [x, x_i, x_j] \otimes \alpha(y_i)\\ &= -\sum_{i,j}(-1)^{|y_i|(|x|+|x_i|+|x_j|)}(-1)^{|x_i||x_j|}\alpha(y_i) \otimes [x, x_j, x_i] \otimes \alpha(y_j),\end{aligned}$$

$$\begin{aligned}\sigma_{12}\Delta_1(x) + \Delta_2(x) &= -\sum_{i,j}(-1)^{|y_i|(|x|+|x_j|)}\alpha(y_i) \otimes [x, x_j, x_i] \otimes \alpha(y_j)\\ &\quad + \sum_{i,j}(-1)^{|y_i|(|x|+|x_i|)}\alpha(y_i) \otimes [x, x_i, x_j] \otimes \alpha(y_j)\\ &= ad_x(r + r_{21})\\ &= 0.\end{aligned}$$

Since

$$r + r_{21} = \sum_i (x_i \otimes y_i + (-1)^{|x_i||y_i|} y_i \otimes x_i) = 0,$$

we have $\sigma_{12}\Delta_1(x) = -\Delta_2(x)$.

Similarly, we have $\sigma_{12}\Delta_2(x) = -\Delta_1(x)$, $\sigma_{12}\Delta_3(x) = -\Delta_3(x)$ and $\sigma_{23}\Delta(x) = -\Delta(x)$. □

Lemma 3.4. *Let V be a $\mathbb{Z}_2$-vector space and $\Delta : V \to V \otimes V \otimes V$ a linear map. Then $\Delta^* : V^* \otimes V^* \otimes V^* \to V^*$ defines a 3-Hom-Lie superalgebra structure on V^* if and only if Δ^* is a skew-symmetric operation and Δ satisfies*

$$\begin{aligned}&(\Delta \otimes 1 \otimes 1)\Delta(x) + \sigma_{23}\sigma_{12}((1 \otimes \Delta \otimes 1)\Delta(x))\\ &+ \sigma_{13}\sigma_{24}((1 \otimes 1 \otimes \Delta)\Delta(x)) - (1 \otimes 1 \otimes \Delta)\Delta(x) = 0\end{aligned} \tag{3.2}$$

for $x \in L$.

Proof. The verification is straightforward. □

Before beginning, the following notation will be useful on the super 3-Hom-classical Yang-Baxter equation, we introduce a notation before the next theorem. For $a \in L$ and $1 \leq i \leq 5$, define the linear map $\otimes_i a : \otimes^4 L \longrightarrow \otimes^5 L$ by inserting a at the i-th position. For example, for any

$$t = t_1 \otimes t_2 \otimes t_3 \otimes t_4,$$

$$t \otimes_2 a = t_1 \otimes a \otimes t_2 \otimes t_3 \otimes t_4.$$

Theorem 3.5. *Let* $(L, [\cdot,\cdot,\cdot], \alpha)$ *be a multiplicative* 3*-Hom-Lie superalgebra and* $r \in L^{\otimes^2}$ *satisfying* $\alpha^{\otimes^2}(r) = r$*. Then*

$$
\begin{aligned}
&\sum_i (-1)^{(|x|+|x_i|)|x_i|}(ad_1(x_i,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha)([[r,r,r]]_1^\alpha \otimes_2 \alpha(y_i))\\
&+\sum_i (-1)^{(|x|+|x_i|)|y_i|}(\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha\otimes\alpha)([[r,r,r]]_1^\alpha \otimes_1 \alpha(y_i))\\
&+\sum_i (\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)([[r,r,r]]_2^\alpha \otimes_5 \alpha(y_i))\\
&+\sum_i (-1)^{(|x|+|x_i|)|x_i|}(\alpha\otimes\alpha\otimes ad_1(x_i,x)\otimes\alpha\otimes\alpha)([[r,r,r]]_2^\alpha \otimes_4 \alpha(y_i))\\
&+\sum_i (-1)^{|y_i||x|}(\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha)([[r,r,r]]_2^\alpha \otimes_3 \alpha(y_i))\\
&+\sum_i (-1)^{|y_i||x_k|}(\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x_i,x)\otimes\alpha)([[r,r,r]]_3^\alpha \otimes_5 \alpha(y_i))\\
&+\sum_i (-1)^{|y_i||x_k|}(\alpha\otimes\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x,x_i))([[r,r,r]]_3^\alpha \otimes_4 \alpha(y_i))\\
&+\sum_i (-1)^{(|x|+|x_i|)|x_i|}(\alpha\otimes\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x_i,x))\\
&([[r,r,r]]_3^\alpha \otimes_3 \alpha(y_i)) = 0,
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
[[r,r,r]]_1^\alpha &:= [r_{12},r_{13},r_{14}] + [r_{12},r_{23},r_{24}] - (-1)^{|x_j||y_j|}[r_{13},r_{32},r_{34}]\\
&\quad + (-1)^{|x_j||y_j|+|x_k||y_k|}[r_{14},r_{42},r_{43}],\\
[[r,r,r]]_2^\alpha &:= (-1)^{|x_j||y_j|}[r_{12},r_{31},r_{14}] - (-1)^{|x_i||y_i|+|x_j||y_j|}[r_{21},r_{32},r_{24}]\\
&\quad - (-1)^{|x_i||y_i|+|x_j||y_j|}[r_{31},r_{32},r_{34}]\\
&\quad - (-1)^{|x_i||y_i|+|x_j||y_j|}[r_{41},r_{42},r_{34}],\\
[[r,r,r]]_3^\alpha &:= -(-1)^{|x_k||y_k|}[r_{12},r_{13},r_{41}] + (-1)^{|x_i||y_i|+|x_k||y_k|}[r_{21},r_{23},r_{42}]\\
&\quad - (-1)^{|x_i||y_i|+|x_j||y_j|+|x_k||y_k|}[r_{31},r_{32},r_{43}]\\
&\quad - (-1)^{|x_i||y_i|+|x_j||y_j|+|x_k||y_k|}[r_{41},r_{42},r_{43}].
\end{aligned}
$$

Let $\Delta : L \to L^{\otimes^3}$ *be defined as* $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ *as in* (3.1)*. Then* $(L, [\cdot,\cdot,\cdot], \Delta, \alpha, r)$ *is a multiplicative coboundary local cocycle* 3*-Hom-Lie superbialgebra.*

Proof. Let $r = \sum_i x_i \otimes y_i$. First we will prove that $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ commutes with α. For $x \in L$, using (3.1), $\alpha[\cdot,\cdot,\cdot] = [\cdot,\cdot,\cdot] \circ \alpha^{\otimes^3}$ and the assumption $\alpha^{\otimes^2}(r) = r$, we have

$$\Delta_1(\alpha(x)) = \sum_{i,j} [\alpha(x), x_i, x_j] \otimes \alpha(y_j) \otimes \alpha(y_i)$$

$$= \sum_{i,j} [\alpha(x), \alpha(x_i), \alpha(xj)] \otimes \alpha^2(y_j) \otimes \alpha^2(y_i)$$
$$= \alpha^{\otimes^3} \Delta_1(x).$$

We can check similarly $\Delta_2(\alpha(x)) = \alpha^{\otimes^3}\Delta_2(x)$ and $\Delta_3(\alpha(x)) = \alpha^{\otimes^3}\Delta_3(x)$. So we obtain:

$$\Delta(\alpha(x)) = \Delta_1(\alpha(x)) + \Delta_2(\alpha(x)) + \Delta_3(\alpha(x)) = \alpha^{\otimes^3}\Delta(x).$$

Second, we show that Δ is anti-symmetric (see Proposition 3.3).

Finally, we show that the 3-Hom-super-co-Jacobi identity of Δ in (3.2) is equivalent to (2.11).

We know Δ contains three terms, then the 3-Hom-super-co-Jacobi identity gives us 36 terms. We will organise these terms in the following order, $G_i, 1 \leq i \leq 5$, denote the sum of these terms where x is at the i-th position in the 5-tensors. Thus

$$G_1 + G_2 + G_3 + G_4 + G_5 = 0.$$

There are 6 terms in G_1:

$$G_1 = G_{11} + G_{12} + G_{13} + G_{14} + G_{15} + G_{16},$$

where

$$G_{11} = \sum_{i,j,k,l} [[x, x_i, x_j], x_k, x_l] \otimes \alpha(y_l) \otimes \alpha(y_k) \otimes \alpha^2(y_j) \otimes \alpha^2(y_i),$$
$$G_{12} = \sum_{i,j,k,l} (-1)^{|y_i|(|x_j|+|x_k|)} [[x, x_i, x_j], x_k, x_l] \otimes \alpha(y_l) \otimes \alpha^2(y_i) \otimes \alpha(y_k) \otimes \alpha^2(y_j),$$
$$G_{13} = \sum_{i,j,k,l} (-1)^{|x_k|(|y_j|+|y_i|)} [[x, x_i, x_j], x_k, x_l] \otimes \alpha(y_l) \otimes \alpha^2(y_j) \otimes \alpha^2(y_i) \otimes \alpha(y_k),$$
$$G_{14} = -\sum_{i,j,k,l} \alpha([x, x_i, x_j]) \otimes \alpha^2(y_j) \otimes [\alpha(y_i), x_k, x_l] \otimes \alpha(y_l) \otimes \alpha(y_k),$$
$$G_{15} = -\sum_{i,j,k,l} (-1)^{|y_k|(|y_i|+|x_k|)} \alpha([x, x_i, x_j]) \otimes \alpha^2(y_j) \otimes \alpha(y_k) \otimes [\alpha(y_i), x_k, x_l] \otimes \alpha(y_l),$$
$$G_{16} = -\sum_{i,j,k,l} (-1)^{(|y_k|+|y_l|)(|y_i|+|x_k|)} \alpha([x, x_i, x_j]) \otimes \alpha^2(y_j) \otimes \alpha(y_l) \otimes \alpha(y_k) \otimes [\alpha(y_i), x_k, x_l].$$

By the equation in Remark 2.2, we have

$$\begin{aligned}
& G_{11} + G_{12} + G_{13} \\
= & \sum_{i,j,k,l} (-1)^{|x|(|x_i|+|x_j|+|x_k|)} [[x_i, x_j, x_k], \alpha(x), \alpha(x_l)] \\
& \otimes \alpha^2(y_l) \otimes \alpha^2(y_k) \otimes \alpha^2(y_j) \otimes \alpha^2(y_i) \\
= & \sum_{i,j,k,l} (-1)^{|x_l|(|x_i|+|x_j|+|x_k|)} (-1)^{|x_i|(|x_j|+|x_k|)} (-1)^{|x||x_l|+|x_j||x_k|} \\
& (ad_1(x_l, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)[x_k, x_j, x_i] \otimes \alpha(y_l) \otimes \alpha(y_k) \otimes \alpha(y_j) \otimes \alpha(y_i)
\end{aligned}$$

$$= \sum_i (-1)^{(|x|+|x_k|)|x_l|}(ad_1(x_l,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha)[r_{12},r_{13},r_{14}]\otimes_2\alpha(y_l).$$

Furthermore, we have

$$\begin{aligned} G_{14} = & \sum_{i,j,k,l}(-1)^{(|x_i|+|x|)|x_j|}(-1)^{|x_k||x_l|}(ad_1(x_j,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha) \\ & \alpha(x_i)\otimes\alpha(y_j)\otimes[y_i,x_l,x_k]\otimes\alpha(y_l)\otimes\alpha(y_k) \\ = & \sum_j(-1)^{(|x_i|+|x|)|x_j|}(ad_1(x_j,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha)[r_{12},r_{23},r_{24}]\otimes_2\alpha(y_j). \end{aligned}$$

Similarly,

$$\begin{aligned} G_{15} = & -\sum_j(-1)^{|y_k|(|y_i|+|x_k|)}(-1)^{(|x_i|+|x|)|x_j|} \\ & (ad_1(x_j,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha)(\alpha(x_i)\otimes\alpha(y_j)\otimes\alpha(y_k)\otimes[y_i,x_k,x_l]\otimes\alpha(y_l)) \\ = & -\sum_j(-1)^{(|x_i|+|x|)|x_j|}(-1)^{|x_k||y_k|}(ad_1(x_j,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha) \\ & [r_{13},r_{32},r_{34}]\otimes_2\alpha(y_j), \\ G_{16} = & \sum_j(-1)^{(|y_k|+|y_l|)(|y_i|+|x_k|)}(-1)^{|x_j|(|x|+|x_i|)}(-1)^{|x_k||x_l|} \\ & (ad_1(x_j,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha)(\alpha(x_i)\otimes\alpha(y_j)\otimes\alpha(y_l)\otimes\alpha(y_k)\otimes[y_i,x_l,x_k]) \\ = & \sum_j(-1)^{(|x_i|+|x|)|x_j|}(-1)^{|x_l||y_l|+|x_k||y_k|}(ad_1(x_j,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha) \\ & [r_{14},r_{42},r_{43}]\otimes_2\alpha(y_j). \end{aligned}$$

In the above equalities we used $\alpha^{\otimes 2}(r)=r=\sum_i x_i\otimes y_i=\sum_i\alpha(x_i)\otimes\alpha(y_i)$, we used the skew-supersymmetry of $[\cdot,\cdot,\cdot]$ and $\alpha([x_i,x_j,x_k])=[\alpha(x_i),\alpha(x_j),\alpha(x_k)]$, we know $(-1)^{|x||x_i|}(-1)^{|x||x_j|}=1$, where $|r|=0$.

Therefore, we obtain

$$G_1=\sum_i(-1)^{(|x_i|+|x|)|x_i|}(ad_1(x_i,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha)[[r,r,r]]_1^\alpha\otimes_2\alpha(y_i).$$

In a similar manner, we have

$$G_2=\sum_i(-1)^{(|x|+|x_i|)|y_i|}(\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha\otimes\alpha)[[r,r,r]]_1^\alpha\otimes_1\alpha(y_i).$$

There are 8 terms in G_3:

$$G_3=G_{31}+G_{32}+G_{33}+G_{34}+G_{35}+G_{36}+G_{37}+G_{38},$$

where

$$\begin{aligned} G_{31} = & \sum_{i,j,k,l}(-1)^{(|x|+|x_i|+|x_j|+|x_k|)(|y_k|+|y_l|)}\alpha(y_l)\otimes\alpha(y_k) \\ & \otimes[[x,x_i,x_j],x_k,x_l]\otimes\alpha^2(y_j)\otimes\alpha^2(y_i), \end{aligned}$$

$$G_{32} = -\sum_{i,j,k,l}(-1)^{(|x|+|x_i|)(|y_i|+|y_j|)}\alpha^2(y_j)\otimes\alpha^2(y_i)$$
$$\otimes[[x,x_i,x_j],x_k,x_l]\otimes\alpha(y_l)\otimes\alpha(y_k),$$
$$G_{33} = \sum_{i,j,k,l}(-1)^{|y_j|+|x_k|+|x_l|+|y_l|)(|x|+|x_i|+|x_j|)}[\alpha(y_j),x_k,x_l]$$
$$\otimes\alpha(y_l)\otimes\alpha[x,x_i,x_j]\otimes\alpha(y_k)\otimes\alpha^2(y_i),$$
$$G_{34} = \sum_{i,j,k,l}(-1)^{(|y_k|+|y_j|+|x_k|+|x_l|)(|x|+|x_i|+|x_j|)}(1)^{|y_k|(|y_j|+|x_k|)}$$
$$\alpha(y_k)\otimes[\alpha(y_j),x_k,x_l]\otimes\alpha[x,x_i,x_j]\otimes\alpha(y_l)\otimes\alpha^2(y_i),$$
$$G_{35} = \sum_{i,j,k,l}(-1)^{(|y_l|+|y_k|)(|y_j|+|y_k|+|x|+|x_i|+|x_j|)}\alpha(y_l)\otimes\alpha(y_k)$$
$$\otimes\alpha[x,x_i,x_j]\otimes[\alpha(y_j),x_k,x_l]\otimes\alpha^2(y_i),$$
$$G_{36} = \sum_{i,j,k,l}(-1)^{(|y_j|+|x|+|x_i|+|x_j|)(|y_i|+|x_k|+|x_l|+|y_l|)}[\alpha(y_i),x_k,x_l]$$
$$\otimes\alpha(y_l)\otimes\alpha[x,x_i,x_j]\otimes\alpha^2(y_j)\otimes\alpha(y_k),$$
$$G_{37} = \sum_{i,j,k,l}(-1)^{(|y_i|+|x_k|)|y_k|}(-1)^{(|y_j|+|x|+|x_i|+|x_j|)(|y_i|+|x_k|+|x_l|+|y_k|)}$$
$$\alpha(y_k)\otimes[\alpha(y_i),x_k,x_l]\otimes\alpha[x,x_i,x_j]\otimes\alpha^2(y_j)\otimes\alpha(y_l),$$
$$G_{38} = \sum_{i,j,k,l}(-1)^{(|y_l|+|y_k|)(|y_i|+|x_k|+|y_j|+|x|+|x_i|+|x_j|)}\alpha(y_l)\otimes\alpha(y_k)$$
$$\otimes\alpha[x,x_i,x_j]\otimes\alpha^2(y_j)\otimes[\alpha(y_i),x_k,x_l].$$

We have

$$G_{31}+G_{32}$$
$$= \sum_{i,j,k,l}\alpha^2(y_l)\otimes\alpha^2(y_k)\otimes((-1)^{|x|(|x_k|+|x_l|)}[\alpha(x),[x_k,x_l,x_i],\alpha(x_j)]$$
$$+(-1)^{(|x|+|x_i|)(|x_k|+|x_l|)}[\alpha(x),\alpha(x_i),[x_k,x_l,x_j]])\otimes\alpha^2(y_j)\otimes\alpha^2(y_i)$$
$$= -\sum_{i,j,k,l}(-1)^{(|x_j|+|x|)(|x_k|+|x_l|+|x_i|)}(-1)^{|x|(|x_i|+|x_j|)}(-1)^{|x_k||x_l|}$$
$$(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)\alpha(y_l)\otimes\alpha(y_k)\otimes[x_l,x_k,x_i]\otimes\alpha(y_j)\otimes\alpha(y_i)$$
$$-\sum_{i,j,k,l}(-1)^{(|x|+|x_i|)(|x_k|+|x_l|)}(-1)^{|x_k||x_l|}$$
$$(\alpha\otimes\otimes\alpha\otimes ad_1(x,x_i)\alpha\otimes\alpha)\alpha(y_l)\otimes\alpha(y_k)\otimes[x_l,x_k,x_j]\otimes\alpha(y_j)\otimes\alpha(y_i)$$
$$= -\sum_{j}(-1)^{(|x|+|x_i|)|x_j|}(-1)^{|x_l||y_l|+|x_k||y_k|}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)$$
$$[r_{31},r_{32},r_{34}]\otimes_4\alpha(y_j)$$

$$-\sum_i(-1)^{|x_l||y_l|+|x_k||y_k|}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[r_{31},r_{32},r_{34}]\otimes_5\alpha(y_i).$$

Furthermore, we have

$$\begin{aligned}
&G_{33}+G_{36}\\
=&\sum_{i,j,k,l}(-1)^{(|y_j|+|x_k|+|x_l|+|y_l|)(|x|+|x_i|+|x_j|)}(-1)^{|x_l|(|y_j|+|x_k|)}\\
&(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[x_l,y_j,x_k]\otimes\alpha(y_l)\otimes\alpha(y_j)\otimes\alpha(y_k)\otimes\alpha(y_i)\\
&+(-1)^{(|y_j|+|x|+|x_i|+|x_j|)(|y_i|+|x_k|+|x_l|+|y_l|)}(-1)^{|x_l|(|y_i|+|x_k|)}(-1)^{|x_j|(|x|+|x_i|)}\\
&\sum_{i,j,k,l}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)\\
&([x_l,y_i,x_k]\otimes\alpha((y_l))\otimes\alpha((x_i))\alpha(y_j)\otimes\alpha(y_k))\\
=&\sum_i(-1)^{|x_j||y_j|}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[r_{12},r_{31},r_{14}]\otimes_5\alpha(y_i)\\
&+\sum_j(-1)^{(|x|+|x_i|)|x_j|}(-1)^{|x_i||y_i|}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)\\
&[r_{12},r_{31},r_{14}]\otimes_4\alpha(y_j),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&G_{34}+G_{37}\\
=&\sum_{i,j,k,l}(-1)^{(|y_i|+|x_k|+|x_l|+|y_k|)(|y_j|+|x|+|x_i|+|x_j|)}(-1)^{|y_k|(|y_i|+|x_k|)}(-1)^{|x_j|(|x|+|x_i|)}\\
&(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)(\alpha(y_k)\otimes[y_i,x_k,x_l]\otimes\alpha(x_i)\otimes\alpha(y_j)\otimes\alpha(y_l))\\
&+\sum_{i,j,k,l}(-1)^{(|y_k|+|y_j|+|x_k|+|x_l|)(|x|+|x_i|+|x_j|)}(-1)^{|y_k|(|y_j|+|x_k|)}(-1)^{|x_j|(|x|+|x_i|)}\\
&(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)(\alpha(y_k)\otimes[y_j,x_k,x_l]\otimes\alpha((y_j))\otimes\alpha(y_l)\otimes\alpha(y_i)\\
=&-\sum_j(-1)^{|x_j|(|x|+|x_i|)}(-1)^{|x_i||y_i|+|x_k||y_k|}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)\\
&[r_{21},r_{32},r_{24}]\otimes_4\alpha(y_j)\\
&-\sum_i(-1)^{|x_j||y_j|+|x_k||y_k|}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[r_{21},r_{32},r_{24}]\otimes_5\alpha(y_i),
\end{aligned}$$

$$\begin{aligned}
&G_{35}+G_{38}\\
=&-\sum_{i,j,k,l}(-1)^{(|y_i|+|x_k|)(|y_j|+|y_k|+|x|+|x_i|+|x_j|)}(-1)^{|x_l|(|y_j|+|x_k|)}(-1)^{|y_j||x_k|}\\
&(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)(\alpha(y_l)\otimes\alpha(y_k)\otimes\ (x_j)\otimes[x_l,x_k,y_j]\otimes\alpha(y_i)\\
&-\sum_{i,j,k,l}(-1)^{(|y_k|+|y_l|)(|y_i|+|x_k|+|y_j|+|x|+|x_i|+||x_j|)}
\end{aligned}$$

$$(-1)^{|x_j|(|x|+|x_i|)}(-1)^{|x_l|(|y_i|+|x_k|)}(-1)^{|y_i||x_k|}$$
$$(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)(\alpha(y_l)\otimes\alpha(y_k)\otimes\alpha(x_i)\otimes\alpha(y_j)\otimes[x_l,x_k,y_i])$$
$$= -\sum_i(-1)^{|x_l||y_l|+|x_k||y_k|}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[r_{41},r_{42},r_{34}]\otimes_5\alpha(y_i)$$
$$-\sum_j(-1)^{|x_j|(|x|+|x_i|)}(-1)^{|x_l||y_l|+|x_k||y_k|}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)$$
$$[r_{41},r_{42},r_{34}]\otimes_4\alpha(y_j).$$

Therefore, we obtain

$$G_3 = \sum_i(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[[r,r,r]]_2^\alpha\otimes_5\alpha(y_i)$$
$$+\sum_i(-1)^{(|x|+|x_i|)|x_i|}(\alpha\otimes\alpha\otimes ad_1(x_i,x)\otimes\alpha\otimes\alpha)[[r,r,r]]_2^\alpha\otimes_4\alpha(y_i).$$

We similarly obtain

$$G_4 = \sum_i(-1)^{|y_i||x|}(\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha)[[r,r,r]]_2^\alpha\otimes_3\alpha(y_i)$$
$$+\sum_i(-1)^{|y_i||x_k|}(\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x_i,x)\otimes\alpha)[[r,r,r]]_3^\alpha\otimes_5\alpha(y_i),$$
$$G_5 = \sum_i(-1)^{|y_i||x_k|}(\alpha\otimes\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x,x_i))[[r,r,r]]_3^\alpha\otimes_4\alpha(y_i)$$
$$+\sum_i(-1)^{(|x|+|x_i|)|x_i|}(-1)^{|y_i||x_k|}(\alpha\otimes\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x_i,x))$$
$$[[r,r,r]]_3^\alpha\otimes_3\alpha(y_i).$$

This completes the proof. □

With the notations above, if r is skew-symmetric, then it can be checked that:

$$\begin{aligned}
[[r,r,r]]_1^\alpha := {}& [r_{12},r_{13},r_{14}]+[r_{12},r_{23},r_{24}]-(-1)^{|x_j||y_j|}[r_{13},r_{32},r_{34}]\\
&+(-1)^{|x_j||y_j|+|x_k||y_k|}[r_{14},r_{42},r_{43}]\\
={}& [r_{12},r_{13},r_{14}]+[r_{12},r_{23},r_{24}]+[r_{13},r_{23},r_{34}]+[r_{14},r_{24},r_{34}]\\
={}& [[r,r,r]]^\alpha,\\
-[[r,r,r]]_2^\alpha := {}& -(-1)^{|x_j||y_j|}[r_{12},r_{31},r_{14}]+(-1)^{|x_i||y_i|+|x_j||y_j|}[r_{21},r_{32},r_{24}]\\
&+(-1)^{|x_i||y_i|+|x_j||y_j|}[r_{31},r_{32},r_{34}]\\
&+(-1)^{|x_i||y_i|+|x_j||y_j|}[r_{41},r_{42},r_{34}]\\
={}& [r_{12},r_{13},r_{14}]+[r_{12},r_{23},r_{24}]+[r_{13},r_{23},r_{34}]+[r_{14},r_{24},r_{34}]\\
={}& [[r,r,r]]^\alpha,\\
[[r,r,r]]_3^\alpha := {}& -(-1)^{|x_k||y_k|}[r_{12},r_{13},r_{41}]+(-1)^{|x_i||y_i|+|x_k||y_k|}[r_{21},r_{23},r_{42}]
\end{aligned}$$

$$
\begin{aligned}
&- (-1)^{|x_i||y_i|+|x_j||y_j|+|x_k||y_k|}[r_{31}, r_{32}, r_{43}] \\
&- (-1)^{|x_i||y_i|+|x_j||y_j|+|x_k||y_k|}[r_{41}, r_{42}, r_{43}] \\
= & [r_{12}, r_{13}, r_{14}] + [r_{12}, r_{23}, r_{24}] + [r_{13}, r_{23}, r_{34}] + [r_{14}, r_{24}, r_{34}] \\
= & [[r, r, r]]^{\alpha},
\end{aligned}
$$

$$[[r, r, r]]^{\alpha}_1 = [[r, r, r]]^{\alpha}, \; [[r, r, r]]^{\alpha}_2 = -[[r, r, r]]^{\alpha}, \; [[r, r, r]]^{\alpha}_3 = [[r, r, r]]^{\alpha}.$$

We can obtain an equivalence description of (3.3):

$$
\begin{aligned}
&\sum_i (-1)^{(|x|+|x_i|)|x_i|}(ad_1(x_i, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)([[r, r, r]]^{\alpha} \otimes_2 \alpha(y_i)) \\
&+ \sum_i (-1)^{(|x|+|x_i|)|y_i|}(\alpha \otimes ad_1(x, x_i) \otimes \alpha \otimes \alpha \otimes \alpha)([[r, r, r]]^{\alpha} \otimes_1 \alpha(y_i)) \\
&- \sum_i (\alpha \otimes \alpha \otimes ad_1(x, x_i) \otimes \alpha \otimes \alpha)([[r, r, r]]^{\alpha} \otimes_5 \alpha(y_i)) \\
&- \sum_i (-1)^{(|x|+|x_i|)|x_i|}(\alpha \otimes \alpha \otimes ad_1(x_i, x) \otimes \alpha \otimes \alpha)([[r, r, r]]^{\alpha} \otimes_4 \alpha(y_i)) \\
&- \sum_i (-1)^{|y_i||x|}(\alpha \otimes \alpha \otimes \alpha \otimes ad_1(x, x_i) \otimes \alpha)([[r, r, r]]^{\alpha} \otimes_3 \alpha(y_i)) \\
&+ \sum_i (-1)^{|y_i||x_k|}(\alpha \otimes \alpha \otimes \alpha \otimes ad_1(x_i, x) \otimes \alpha)([[r, r, r]]^{\alpha} \otimes_5 \alpha(y_i)) \\
&+ \sum_i (-1)^{|y_i||x_k|}(\alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes ad_1(x, x_i))([[r, r, r]]^{\alpha} \otimes_4 \alpha(y_i)) \\
&+ \sum_i (-1)^{(|x|+|x_i|)|x_i|}(\alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes ad_1(x_i, x))([[r, r, r]]^{\alpha} \otimes_3 \alpha(y_i)) = 0.
\end{aligned}
$$

The above discussion can be summarized in the next Corollary 3.6 wich can be regarded as a 3-Hom-Lie superalgebra analogue of the face that $\alpha^{\otimes^2}(r) = r$ and a skew-symmetric solution of the classical Yang-Baxter equation gives a local cocycle 3-Hom-Lie superbialgebra.

Corollary 3.6. *Let L be a 3-Hom-Lie superalgebra, $\alpha^{\otimes^2}(r) = r$ and $r \in L \otimes L$ skew-symmetric. If*

$$[[r, r, r]]^{\alpha} = 0,$$

and $\Delta = \Delta_1 + \Delta_2 + \Delta_3 : L \to L \otimes L \otimes L$, in which $\Delta_1, \Delta_2, \Delta_3$ are included by r as in Eq. (3.1)*. Then $(L, [\cdot, \cdot, \cdot], \Delta, \alpha, r)$ is a local cocycle 3-Hom-Lie superbialgebra.*

Theorem 3.7. *Let L be a 3-Lie superalgebra, $\alpha : L \to L$ be a 3-Lie superalgebra endomorphism and $r \in L \otimes L$ be a solution of 3-Lie CYBE. Then for each integer $n \geq 0$, $(\alpha^{\otimes^2})^n(r)$ is a solution of 3-Lie CHYBE in the 3-Hom-Lie superalgebra $L_\alpha = (L, [\cdot, \cdot, \cdot]_\alpha = \alpha \circ [\cdot, \cdot, \cdot], \alpha)$.*

Proof. We can prove that L_α is a 3-Hom-Lie superalgebra (in fact, the Hom-super-Jacobi identity for $[\cdot, \cdot, \cdot]_\alpha$ is α^3 applied to the Jacobi identity of $[\cdot, \cdot, \cdot]$).

It remains to show that $(\alpha^{\otimes^2})^n(r)$ satisfies the 3-Lie CHYBE in the 3-Hom-Lie superalgebra L_α.

For $r = \sum_i x_i \otimes y_i$, using $\alpha([\cdot,\cdot,\cdot]) = [\cdot,\cdot,\cdot] \circ \alpha^{\otimes^2}$ and the definition $[\cdot,\cdot,\cdot]_\alpha = \alpha([\cdot,\cdot,\cdot])$, we have to prove that

$$[[(\alpha^{\otimes^2})^n(r), (\alpha^{\otimes^2})^n(r), (\alpha^{\otimes^2})^n(r)]]^\alpha = 0,$$

$$\begin{aligned}
&[[(\alpha^{\otimes^2})^n(r), (\alpha^{\otimes^2})^n(r), (\alpha^{\otimes^2})^n(r)]]^\alpha \\
= &\sum_{i,j,k} \Big((-1)^{|x_i||x_j|}[\alpha^n(x_i), \alpha^n(x_j), \alpha^n(x_k)]_\alpha \otimes \alpha(\alpha^n(y_i)) \otimes \alpha(\alpha^n(y_j)) \otimes \alpha(\alpha^n(y_k)) \\
&+(-1)^{|x_k||x_j|}\alpha(\alpha^n(x_i)) \otimes [\alpha^n(y_i), \alpha^n(x_j), \alpha^n(x_k)]_\alpha \otimes \alpha(\alpha^n(y_j)) \otimes \alpha(\alpha^n(y_k)) \\
&+(-1)^{|x_j||y_i|}\alpha(\alpha^n(x_i)) \otimes \alpha(\alpha^n(x_j)) \otimes [\alpha^n(y_i), \alpha^n(y_j), \alpha^n(x_k)]_\alpha \otimes \alpha(\alpha^n(y_k)) \\
&+(-1)^{|y_j||y_k|}\alpha(\alpha^n(x_i)) \otimes \alpha(\alpha^n(x_j)) \otimes \alpha(\alpha^n(x_k)) \otimes [\alpha^n(y_i), \alpha^n(y_j), \alpha^n(y_k)]_\alpha \Big) \\
= &\alpha^{n+1}([[r, r, r]]) \\
= &0.
\end{aligned}$$

This result shows that given a 3-Hom-Lie superalgebra endomorphism, each classical r-matrix induces an infinite family of solutions of the 3-Lie super CHYBE. □

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