East Asian Math. J.
Vol. 40 (2024), No. 1, pp. 085–094
http://dx.doi.org/10.7858/eamj.2024.007

EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR p-LAPLACIAN PROBLEMS WITH A SINGULAR WEIGHT

Chan-Gyun Kim and Yong-Hoon Lee*

Abstract. In this paper we study the existence and multiplicity of positive solutions for p-Laplacian problems with a singular weight. Proofs mainly make use of Global Continuation Theorem and Fixed Point Index argument.

1. Introduction

Consider

$$\begin{cases} (\varphi_p(u'(t)))' + \lambda h(t) f(u(t)) = 0, \ t \in (0,1), \\ u(0) = 0, \ u(1) = b > 0, \end{cases} \tag{E_λ}$$

where $\varphi_p(s) = |s|^{p-2}s, \ p > 1, \ \lambda \in [0, \infty) =: \mathbb{R}_+$ is a parameter, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $f(z) > 0$ for all $z > 0$, and $h \in C((0,1),(0,\infty))$ may be singular at $t = 0$ and/or 1.

Throughout this paper, the following hypotheses are assumed, unless otherwise stated.

(F_1) for all $R > 0$, there exists $A_R > 0$ such that

$$f(z) \le A_R z^{p-1} \text{ for } z \in [0, R],$$

(F_2) $f_\infty := \lim_{u\to\infty} \dfrac{f(u)}{\varphi_p(u)} = \infty$,

(F_3) there exists $r_f > 0$ such that f is nondecreasing on $(0, r_f)$.

Let us denote

$$\mathcal{A} = \{h \in C((0,1),(0,\infty)) : \ \textstyle\int_0^1 s^{p-1} h(s)ds < \infty\}$$

and

$$\mathcal{B} = \{h \in C((0,1),(0,\infty)) : \ \textstyle\int_0^1 \varphi_p^{-1}\left(\int_s^1 h(\tau)d\tau\right) ds < \infty\}.$$

Received December 19, 2023; Accepted December 27, 2023.
2010 *Mathematics Subject Classification.* 34B15, 34B18, 34B27.
Key words and phrases. p-Laplacian, Existence, Multiplicity, Singular weight.
* This work was financially supported by a 2-Year Research Grant of Pusan National University.

* Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

In [5], Kim and Lee studied the following problem

$$\begin{cases} (\varphi_p(u'(t)))' + \lambda h(t) f(u(t)) = 0, \ t \in (0,1), \\ u(0) = a > 0, \ u(1) = 0. \end{cases} \tag{1}$$

Here $h \in L^1(0,1)$. Under the assumptions that f is nondecreasing and satisfies (F_2), they showed that there exists $\lambda^* > 0$ such that (1) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, at least one positive solution $\lambda = \lambda^*$ and no positive solutions for $\lambda \in (\lambda^*, \infty)$. Later on, in [4], the authors obtained the same result with more general assumptions that $h \in \mathcal{B}$ and (F_2) and (F_3). Motivated by these papers, we study the existence, multiplicity and nonexistence of positive solutions of (E_λ).

The usual norm in a Banach space $C^1[0,1]$ is denoted by

$$\|u\|_1 = \|u\|_\infty + \|u'\|_\infty \text{ for } u \in C^1[0,1].$$

Here $\|v\|_\infty = \max_{t\in[0,1]} |v(t)|$ for $v \in C[0,1]$. We will call u a positive C^1-solution if u is a positive solution and $u \in C^1[0,1]$.

We don't know if all solutions of (E_λ) are $C^1[0,1]$. If we confine our attention to $C^1[0,1]$ as the solution space, we get the following result.

Theorem 1.1. *Assume $h \in \mathcal{A}$, $(F_1), (F_2)$ and (F_3). Then there exists $\lambda^* > 0$ such that (E_λ) has at least two positive C^1-solutions for $\lambda \in (0, \lambda^*)$, at least one positive C^1-solution for $\lambda = \lambda^*$ and no positive C^1-solutions for $\lambda \in (\lambda^*, \infty)$.*

2. Preliminaries

Theorem 2.1. ([7], Global Continuation Theorem) *Let X be a Banach space and $\mathcal{K}$ an order cone in X. Consider*

$$x = H(\mu, x), \tag{2}$$

where $\mu \in \mathbb{R}_+$ and $x \in \mathcal{K}$. If $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ is completely continuous and $H(0,x) = 0$ for all $x \in \mathcal{K}$. Then $\mathcal{C}_+(\mathcal{K})$, the component of the solution set of (2) containing (0,0) is unbounded.

Theorem 2.2. ([6], Generalized Picone Identity) *Let us define*

$$l_p[y] = (\varphi_p(y'))' + b_1(t)\varphi_p(y),$$
$$L_p[z] = (\varphi_p(z'))' + b_2(t)\varphi_p(z).$$

If y and z are any functions such that $y,\ z, \varphi_p(y'),\ \varphi_p(z')$ are differentiable on I and $z(t) \neq 0$ for $t \in I$, the generalized Picone identity can be written as

$$\begin{aligned} \frac{d}{dt}\left\{ \frac{|y|^p \varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y') \right\} &= (b_1 - b_2)|y|^p \\ &\quad - \left[|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y) y' \varphi_p\left(\frac{z'}{z}\right) \right] \\ &\quad - y l_p(y) + \frac{|y|^p}{\varphi_p(z)} L_p(z). \end{aligned}$$

Remark 1. By Young's inequality, we get

$$|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)\varphi_p\left(\frac{z'}{z}\right) \geq 0,$$

and the equality holds if and only if $sgn\ y' = sgn\ z'$ and $|\frac{y'}{y}|^p = |\frac{z'}{z}|^p$.

Theorem 2.3. ([3]) *Let X be a Banach space, $\mathcal{K}$ a cone in X and $\mathcal{O}$ bounded open in X. Let $0 \in \mathcal{O}$ and $A : \mathcal{K} \cap \bar{\mathcal{O}} \to \mathcal{K}$ be completely continuous. Suppose that $Ax \neq \nu x$ for all $x \in \mathcal{K} \cap \partial\mathcal{O}$ and all $\nu \geq 1$. Then $i(A, \mathcal{K} \cap \mathcal{O}, \mathcal{K}) = 1$.*

3. Main result

For the sake of convenience, we transform problem (E_λ) into a zero Dirichlet boundary problem. More precisely, introducing $v(t) = u(t) - bt$, we may rewrite (E_λ) to the following problem

$$\begin{cases} (\varphi_p(v'(t)+b))' + \lambda h(t) f(v(t)+bt) = 0, \ t \in (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{$\hat{E}_\lambda$}$$

Now, we define an operator corresponding to problem $(\hat{E}_\lambda)$. First, we define an operator corresponding to the case $L^1(0,1)$. For $g \in L^1(0,1)$, define

$$\zeta_g(x) = \int_0^1 \left[\varphi_p^{-1}\left(x + \int_s^1 g(\tau)d\tau\right) - b\right] ds.$$

Then, we can easily check the following facts that ζ_g is well-defined, strictly increasing, continuous in $(-\infty, \infty)$, $\zeta_g(-\infty) = -\infty$ and $\zeta_g(\infty) = \infty$. Thus ζ_g has the unique zero and denote it by $\xi(g)$. Note that $\xi(0) = b^{p-1}$, and $\xi : L^1(0,1) \to \mathbb{R}$ is a bounded function, i.e., for all $M > 0$, there exists $C_M > 0$ such that $|\xi(g)| \leq C_M$ for all g with $\|g\|_{L^1(0,1)} \leq M$. Let

$$\mathcal{K} = \{u \in C^1[0,1] : u \text{ is concave on } (0,1)\}.$$

Then $\mathcal{K}$ is an order cone. Let us define $F : L^1((0,1),(0,\infty)) \to \mathcal{K}$ by

$$F(g)(t) = \int_0^t \left(\varphi_p^{-1}\left[\xi(g) + \int_s^1 g(\tau)d\tau\right] - b\right) ds.$$

Then F is well defined and $F(g)(0) = F(g)(1) = 0$. Define $G : \mathbb{R}_+ \times \mathcal{K} \to L^1(0,1)$ by

$$G(\lambda, u)(t) = \lambda h(t) f(u(t) + bt), \ t \in (0,1).$$

Lemma 3.1. *Assume $h \in \mathcal{A}$ and (F_1). Then G is well defined, sends bounded sets into bounded sets and continuous.*

Proof. For $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$, by (F_1), there exists $A_u > 0$ such that

$$f(z) \leq A_u z^{p-1} \text{ for } 0 \leq z \leq \|u\|_\infty + b \text{ and } 0 \leq u(\tau) = \int_0^\tau u'(s)ds \leq \tau\|u'\|_\infty.$$

By this facts,

$$\begin{aligned}\int_0^1 G(\lambda,u)(\tau)d\tau &= \int_0^1 \lambda h(\tau)f(u(\tau)+b\tau)d\tau \\ &\le \lambda A_u \int_0^1 h(\tau)(u(\tau)+b\tau)^{p-1}d\tau \\ &\le \lambda A_u(\|u'\|_\infty+b)^{p-1}\int_0^1 \tau^{p-1}h(\tau)d\tau<\infty.\end{aligned}$$

Thus, G is well-defined and send bounded sets into bounded sets. Moreover, by Lebesgue dominated convergence Theorem, G is continuous. □

Define $H:\mathbb{R}_+\times\mathcal{K}\to\mathcal{K}$ by $H(\lambda,u)=F(G(\lambda,u))$, i.e.,

$$H(\lambda,u)(t)=\int_0^t\left(\varphi_p^{-1}\left[\xi(G(\lambda,u))+\int_s^1 G(\lambda,u)(\tau)d\tau\right]-b\right)ds.$$

Then we can easily see that H is well-defined and $H(\mathbb{R}_+\times\mathcal{K})\subset\mathcal{K}$. Furthermore, u is a positive solution of $(\hat{E}_\lambda)$ if and only if $u=H(\lambda,u)$ on $\mathcal{K}$.

By the similar arguments in the proof of Lemma 3 in [1], we can prove the complete continuity of H on $\mathbb{R}_+\times\mathcal{K}$. We only state the result as follows.

Lemma 3.2. *Assume $h\in\mathcal{A}$ and (F_1). Then $H:\mathbb{R}_+\times\mathcal{K}\to\mathcal{K}$ is completely continuous.*

Since $H(0,u)=0$ for all $u\in\mathcal{K}$, by Lemma 3.2 and Global Continuation Theorem (Theorem 2.1), we know that there exists an unbounded continuum $\mathcal{C}$ of positive solutions of $(\hat{E}_\lambda)$ emanating from $(0,0)$. Equivalently, there exists an unbounded continuum $\mathcal{C}'$ of positive solutions of (E_λ) emanating from $(0,bt)$.

Using the generalized Picone identity and the properties of the p-sine function ([2], [8]), we obtain the following lemmas which determine the shape of the unbounded continuum $\mathcal{C}'$ (equivalently $\mathcal{C}$).

Lemma 3.3. *Assume (F_2). Then there exists $\bar{\lambda}>0$ such that if u is a positive solution of (E_λ), then $\lambda\le\bar{\lambda}$.*

Proof. Let u be a positive solution of (E_λ). Since u is concave and $u(1)=b$, we have $u(t)\ge\frac{1}{4}b$ for all $t\in(\frac{1}{4},\frac{3}{4})$. It follows from (F_2) that there exists $L>0$ such that $f(z)>Lz^{p-1}$ for $z\ge\frac{1}{4}b$. Then, we have

$$(\varphi_p(u'(t)))'+\lambda Lh(t)\varphi_p(u(t))<0,\ t\in\left(\frac{1}{4},\frac{3}{4}\right).$$

Putting $m:=\min_{t\in[\frac{1}{4},\frac{3}{4}]}h(t)>0$, it follows that

$$(\varphi_p(u'(t)))'+\lambda Lm\varphi_p(u(t))<0,\ t\in\left(\frac{1}{4},\frac{3}{4}\right).$$

It is easy to check that $w(t) = S_q\left(2\pi_p(t - \frac{1}{4})\right)$ is a solution of

$$\begin{cases} (\varphi_p(w'(t)))' + (2\pi_p)^p \varphi_p(w(t)) = 0, \ t \in (\frac{1}{4}, \frac{3}{4}) \\ w(\frac{1}{4}) = w(\frac{3}{4}) = 0, \end{cases}$$

where S_q is the q-sine function with $\frac{1}{p} + \frac{1}{q} = 1$ and $\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$. Taking $y = w$ and $z = u$ in Theorem 2.2 and integrating from 1/4 to 3/4, by Remark 1,

$$\int_{1/4}^{3/4} ((2\pi_p)^p - \lambda Lm)\,|w|^p dt \geq 0,$$

which implies

$$\lambda \leq \frac{(2\pi_p)^p}{Lm} =: \bar{\lambda}.$$

□

Lemma 3.4. *Assume $h \in \mathcal{A}$, (F_1) and (F_2). Let $I = [\alpha, \beta] \subseteq (0, \infty)$. Then there exists $b_I > 0$ such that for all positive C^1-solutions u of (E_λ) with $\lambda \in I$, we have*

$$\|u\|_1 \leq b_I.$$

Proof. First we will show that there exists $M_I > 0$ such that $\|u\|_\infty < M_I$ for all possible solutions of (E_λ) with $\lambda \in I$. Assume on the contrary that there exists a sequence (u_n) of positive solutions of (E_{λ_n}) with $\lambda_n \in I$ and $||u_n||_\infty \to \infty$ as $n \to \infty$. It follows from the concavity of u_n that

$$u_n(t) \geq \tfrac{1}{4}\|u_n\|_\infty \text{ for all } t \in \left(\tfrac{1}{4}, \tfrac{3}{4}\right) \text{ and all } n.$$

Take $C_K = \frac{(2\pi_p)^p}{\alpha m} + 1$, where $m := \min_{t\in[\frac{1}{4},\frac{3}{4}]} h(t) > 0$. By (F_2), there exists $K > 0$ such that $f(z) > C_K \varphi_p(z)$ for all $z > K$. From the assumption, we get $||u_N||_\infty > 4K$ for sufficiently large N. Therefore, we have

$$f(u_N(t)) > C_K \varphi_p(u_N(t)), \ t \in \left(\frac{1}{4}, \frac{3}{4}\right).$$

This implies

$$(\varphi_p(u_N'(t)))' + \alpha C_K m \varphi_p(u_N(t)) < 0, \ t \in \left(\frac{1}{4}, \frac{3}{4}\right).$$

As in the proof of Lemma 3.3, if we take $w(t) = S_q\left(2\pi_p(t - \frac{1}{4})\right)$, we obtain

$$C_K \leq \frac{(2\pi_p)^p}{\alpha m}.$$

This contradicts the choice of C_K. Thus there exists $M_I > 0$ such that $\|u\|_\infty < M_I$ for all possible solutions of (E_λ) with $\lambda \in I$. We will show that there exists $L_I > 0$ such that $\|u'\|_\infty < L_I$ for all possible C^1-solutions u of (E_λ) with $\lambda \in I$.

By (F_1), there exists $A_I > 0$ such that $f(z) \le A_I z^{p-1}$ for $0 \le z \le M_I$. And there exists $\delta \in (0, \frac{1}{2})$ such that

$$\beta A_I \int_0^\delta s^{p-1} h(s) ds < \frac{1}{2}. \tag{3}$$

Let u be a C^1-solution of (E_λ) with $\lambda \in I$. Since u is concave,

$$u(t) \le u'(\delta)(t-\delta) + u(\delta). \tag{4}$$

In (4), if $t = 0$, then $0 = u(0) \le u'(\delta)(-\delta) + u(\delta)$. This implies

$$u'(\delta) \le \frac{u(\delta)}{\delta} \le \frac{M_I}{\delta}. \tag{5}$$

Similarly if $t = 1$, then $0 < b = u(1) \le u'(\delta)(1-\delta) + u(\delta)$ and we obtain

$$-u'(\delta) \le \frac{u(\delta)}{1-\delta} \le \frac{M_I}{\delta}. \tag{6}$$

By (5) and (6), $|u'(\delta)| \le \frac{M_I}{\delta}$. For $t \in (0, \delta)$, integrating (E_λ) from t to δ, by (3) and using the fact $0 \le u(t) = \int_0^t u'(s) ds \le t\|u'\|_\infty$, we have

$$\begin{aligned}
|\varphi_p(u'(t))| &\le |\varphi_p(u'(\delta))| + \lambda \int_t^\delta h(s) f(u(s)) ds \\
&\le |\varphi_p(u'(\delta))| + \lambda A_I \int_t^\delta h(s) u(s)^{p-1} ds \\
&\le |\varphi_p(u'(\delta))| + \beta A_I \int_0^\delta s^{p-1} h(s) ds \|u'\|^{p-1}_{L^\infty(0,\delta)} \\
&\le \left(\frac{M_I}{\delta}\right)^{p-1} + \frac{1}{2}\|u'\|^{p-1}_{L^\infty(0,\delta)} \text{ for } t \in (0,\delta).
\end{aligned}$$

Thus, we have

$$\|u'\|^{p-1}_{L^\infty(0,\delta)} \le 2\left(\frac{M_I}{\delta}\right)^{p-1}.$$

On the other hand, again integrating (E_λ) from δ to 1, we have

$$|\varphi_p(u'(1))| \le \left(\frac{M_I}{\delta}\right)^{p-1} + \beta M_1 \int_\delta^1 h(s) ds,$$

where $M_1 = \sup_{0 \le z \le M_I} f(z) > 0$. Thus, the proof is complete. □

By Lemma 3.3 and Lemma 3.4, we get the following proposition.

Proposition 3.5. *Assume (F_1), (F_2), and let $h \in \mathcal{A}$. Then there exists an unbounded continuum $\mathcal{C}'$ emanating from $(0, bt)$ in the closure of the set of positive solutions of (E_λ) in $\mathbb{R}_+ \times \mathcal{K}$ such that for all $R(\ge b)$, there exists $(\lambda_R, u_R) \in \mathcal{C}'$ with $\|u_R\|_1 = R$. Furthermore, $\lambda_R \to 0$ as $R \to \infty$.*

Assume that problem $(\hat{E}_\lambda)$ has a positive solution say, v_* at $\lambda_* > 0$, i.e., v_* satisfies

$$(\varphi_p(v_*'(t) + b))' + \lambda_* h(t) f(v_*(t) + bt) = 0, \quad t \in (0,1). \tag{7}$$

Consider fixed $\lambda \in (0, \lambda_*)$. For $N > 0$, put

$$\Omega_N = \{u \in C_0^1[0,1] \quad : \quad 0 < u(t) < u_*(t),\ t \in (0,1), 0 < u'(0) < u_*'(0), \\ u_*'(1) < u'(1) < 0 \text{ and } \|u'\|_\infty < N\}.$$

Then, Ω_N is bounded and open in $C_0^1[0,1]$. Consider the following modified problem

$$\begin{cases} (\varphi_p(v'(t) + b))' + \lambda h(t) f(\gamma(t, v(t)) + bt) = 0,\ t \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{M_λ}$$

where $\gamma : (0,1) \times \mathbb{R} \to \mathbb{R}_+$ by $\gamma(t,z) = \begin{cases} v_*(t) & \text{if } z > v_*(t), \\ z & \text{if } 0 \le z \le v_*(t), \\ 0 & \text{if } z < 0. \end{cases}$

Lemma 3.6. *Assume $h \in \mathcal{A}, (F_2), (F_3)$ and let $\lambda \in (0, \lambda_*)$. Then, there exists $N_0 > 0$ such that $v \in \Omega_{N_0} \cap \mathcal{K}$ for all positive C^1-solutions v of (M_λ).*

Proof. Let v be a positive C^1-solution of (M_λ). Clearly, $v > 0$ in $(0,1)$, since v is concave on $(0,1)$ and $v(0) = v(1) = 0$. We first show $v(t) \le v_*(t)$ for $t \in (0,1)$. If it is not true, there exists an interval $[t_1, t_2] \subset [0,1]$ such that $v(t) > v_*(t)$ for $t \in (t_1, t_2)$, $v(t_1) = v_*(t_1)$ and $v(t_2) = v_*(t_2)$. Since $v - v_* \in C_0[t_1, t_2]$, there exists $A \in (t_1, t_2)$ such that

$$v'(A) = v_*'(A) \ \text{ and } \ v(A) > v_*(A). \tag{8}$$

On $[t_1, t_2]$, we have

$$\lambda_* f(v_*(t) + bt) > \lambda f(v_*(t) + bt) = \lambda f(\gamma(t, v(t)) + bt).$$

This implies

$$(\varphi_p(v'(t) + b))' + \lambda_* h(t) f(v_*(t) + bt) > 0 \text{ for } t \in (t_1, t_2). \tag{9}$$

From (7) and (9), we have

$$(\varphi_p(v_*'(t) + b))' - \varphi_p(v'(t) + b)' < 0 \text{ for } t \in (t_1, t_2). \tag{10}$$

For $t \in (A, t_2)$, integrating (10) from A to t, we have $v_*'(t) \le v'(t)$ by (8). Again integrating this inequality from A to t_2, we get $v_*(A) \ge v(A)$ and this contradicts (8).

Second, we show $v(t) < v_*(t)$ for all $t \in (0,1)$. If it is not true, then by the first argument, we have only two cases; either there exists $[t_3, t_4] \subseteq [0,1]$ such that $v(t) = v_*(t)$, $t \in [t_3, t_4]$ or there exist $t_5 \in (0,1)$ and $\delta_1 > 0$ such that $v(t_5) = v_*(t_5)$, $v(t) < v_*(t)$, $t \in (t_5 - \delta_1, t_5 + \delta_1) \backslash \{t_5\}$ and $v'(t_5) = v_*'(t_5)$. In the first case, we can easily know that (10) is satisfied in $t \in (t_3, t_4)$. This is a contradiction. In the second case, by (7), we have

$$(\varphi_p(v_*'(t) + b))' + \lambda h(t) f(v_*(t) + bt) < 0,\ t \in (0,1).$$

This implies that there exists $\epsilon_1 > 0$ such that

$$\max_{t\in[t_5-\delta_1,t_5+\delta_1]}\{(\varphi_p(v_*'(t)+b))'+\lambda h(t)f(v_*(t)+bt)\}=-\epsilon_1<0. \quad (11)$$

Since f is uniformly continuous on $[0,\|v_*\|_\infty+b]$, there exists $\delta_2>0$ such that if $|x-y|<\delta_2$ and $x,y\in[0,\|v_*\|_\infty+b]$, then

$$|f(x)-f(y)|<\epsilon_1 C^{-1},$$

where $C=\lambda\max_{t\in[t_5-\delta_1,t_5+\delta_1]}h(t)>0$. And there exists a subinterval $[c,d]$ containing t_5 of $(t_5-\delta_1,t_5+\delta_1)$ such that

$$-\delta_2<v(t)-v_*(t)\le 0,\ t\in[c,d] \quad (12)$$

and

$$(v-v_*)'(c)>0 \text{ and } (v-v_*)'(d)<0. \quad (13)$$

By (12), $f(v(t)+bt)<f(v_*(t)+bt)+\epsilon_1C^{-1},\ t\in[c,d]$. This implies

$$(\varphi_p(v_*'(t)+b))'+\lambda h(t)f(v(t)+bt)\le-\epsilon_1+\epsilon_1=0,\ t\in[c,d]. \quad (14)$$

By (13) and (14), we have

$$\begin{aligned}0 &> [\varphi_p(v'(d)+b)-\varphi_p(v_*'(d)+b)]-[\varphi_p(v'(c)+b)-\varphi_p(v_*'(c)+b)]\\ &= \int_c^d[(\varphi_p(v'(t)+b))'-(\varphi_p(v_*'(t)+b))']dt\\ &= -\int_c^d[\lambda h(t)f(v(t)+bt)+\varphi_p(v_*'(t)+b)']\ge 0.\end{aligned}$$

This is a contradiction. Thus,

$$v(t)<v_*(t),\ t\in(0,1). \quad (15)$$

Third, we show

$$0>v'(1)>v_*'(1). \quad (16)$$

From $v_*(1)=v(1)=0$ and the continuity of f at b, it follows that, for c sufficiently close to 1,

$$\begin{aligned}&\lambda_*h(t)f(v_*(t)+bt)-\lambda h(t)f(v(t)+bt)\\ =\ & h(t)[\lambda(f(v_*(t)+bt)-f(v(t)+bt))+(\lambda_*-\lambda)f(v_*(t)+bt)]>0,\end{aligned}$$

for $t\in(c,1)$. This implies

$$(\varphi_p(v_*'(t)+b))'-(\varphi_p(v'(t)+b))'<0,\ t\in(c,1). \quad (17)$$

We claim that there exists $d\in(c,1)$ such that $v'(d)>v_*'(d)$. Indeed, otherwise, $v'(t)\le v_*'(t)$ for all $t\in(c,1)$. Integrating this from t to 1, we have $v(t)\ge v_*(t)$, for $t\in(c,1)$. This is a contradiction by (15). Thus, the claim is done. Integrating (17) from d to 1, we obtain

$$\varphi_p(v_*'(1)+b)-\varphi_p(v'(1)+b)\le\varphi_p(v_*'(d)+b)-\varphi_p(v'(d)+b)<0$$

and thus $v_*'(1)<v'(1)$. Since v is a positive C^1-solution of $(\hat{M}_\lambda)$, $v'(1)<0$.

Fourth, we show $0 < v'(0) < v_*'(0)$. Since $v_*(0) = 0$ and $v(t) < v_*(t)$, $t \in (0,1)$, there exists $c_1 \in (0,1)$ such that $v(t) + bt \le v_*(t) + bt \le r_f$ for $t \in (0, c_1)$. By (F_3), we have

$$\lambda_* h(t) f(v_*(t) + bt) > \lambda h(t) f(v(t) + bt) \text{ for } t \in (0, c_1),$$

which implies

$$(\varphi_p(v_*'(t) + b))' - (\varphi_p(v'(t) + b))' < 0,\ t \in (0, c_1).$$

This is the inequality corresponding to (17). By the similar manner to prove (16), we can show $0 < v'(0) < v_*'(0)$.

Finally, by the facts $0 < v(t) < v_*(t)$ for all $t \in (0,1)$, $v_*(0) = v_*(1) = 0$ and the concavity of v, we have $\|v'\|_\infty < N_0$ for some $N_0 > 0$. Consequently, $v \in \Omega_{N_0} \cap \mathcal{K}$ for all positive C^1-solutions v of (M_λ). □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\lambda^* = \sup\{\mu : (\hat{E}_\lambda)$ has at least two positive solutions for all $\lambda \in (0, \mu)\}$. Then, by Lemma 3.3 and Proposition 3.5, we have $0 < \lambda^* \le \bar{\lambda}$. By the choice of λ^* and the complete continuity of H, $(\hat{E}_\lambda)$ has at least two positive solutions for $\lambda \in (0, \lambda^*)$ and at least one positive solution at $\lambda = \lambda^*$. We shall show that $(\hat{E}_\lambda)$ has no positive solution for all $\lambda > \lambda^*$. On the contrary, assume that there exists $\lambda_* > \lambda^*$ such that $(\hat{E}_{\lambda_*})$ has a positive solution. We claim that $(\hat{E}_\lambda)$ has at least two positive solutions for $\lambda \in [\lambda^*, \lambda_*)$. Then this contradicts the definition of λ^* and the proof is done. Define an operator M by taking

$$Mv(t) = \int_t^1 \left(\varphi_p^{-1} \left[\xi(\hat{G}(\lambda, v)) + \lambda \int_s^1 h(\tau) f(\gamma(\tau, v) + b\tau) d\tau \right] - b \right) ds,$$

where $\hat{G}(\lambda, v)(t) = \lambda h(t) f(\gamma(\tau, v) + b\tau)$. Then $M : \mathcal{K} \to \mathcal{K}$ is completely continuous and u is a solution of (M_λ) if and only if $u = Mu$ on $\mathcal{K}$. By simple calculation, we can show that there exists $R_1 > 0$ such that $||Mv||_1 < R_1$ for all $v \in \mathcal{K}$. Taking R_1 big enough satisfying $B_{R_1} \supset \Omega_{N_0}$ and applying Theorem 2.3, we get

$$i(M, B_{R_1} \cap \mathcal{K}, \mathcal{K}) = 1.$$

By Lemma 3.6 and the excision property, we get

$$i(M, \Omega_{N_0} \cap \mathcal{K}, \mathcal{K}) = i(M, B_{R_1} \cap \mathcal{K}, \mathcal{K}) = 1. \tag{18}$$

Since problem $(\hat{E}_\lambda)$ is equivalent to problem (M_λ) on $\Omega_{N_0} \cap \mathcal{K}$, we conclude $(\hat{E}_\lambda)$ has a positive solution in $\Omega_{N_0} \cap \mathcal{K}$. Assume $H(\lambda, \cdot)$ has no fixed point in $\partial\Omega_{N_0} \cap \mathcal{K}$ (otherwise, the proof of our claim is done). Then $i(H(\lambda, \cdot), \Omega_{N_0} \cap \mathcal{K}, \mathcal{K})$ is well defined and by (18), we have

$$i(H(\lambda, \cdot), \Omega_{N_0} \cap \mathcal{K}, \mathcal{K}) = 1. \tag{19}$$

By Lemma 3.3, we may choose $\lambda_1 > \bar{\lambda}$ such that $(\hat{E}_{\lambda_1})$ has no solution in $\mathcal{K}$. By *a priori* estimate (Lemma 3.4) with $I = [\lambda, \lambda_1]$, there exists $R_2 > R_1$ such that for all possible positive C^1-solutions u of $(\hat{E}_\mu)$ with $\mu \in [\lambda, \lambda_1]$, we have

$$\|u\|_1 < R_2. \tag{20}$$

Define $T : [0,1] \times (\bar{B}_{R_2} \cap \mathcal{K}) \to \mathcal{K}$ by $T(\tau, v) = H(\tau\lambda_1 + (1-\tau)\lambda, v)$. Then T is completely continuous on $[0,1] \times \mathcal{K}$, and by (20), $T(\tau, v) \neq v$ for all $(\tau, v) \in [0,1] \times (\partial B_{R_2} \cap \mathcal{K})$. By the property of homotopy invariance, we have

$$i(H(\lambda, \cdot), B_{R_2} \cap \mathcal{K}, \mathcal{K}) = i(H(\lambda_1, \cdot), B_{R_2} \cap \mathcal{K}, \mathcal{K}) = 0.$$

By the additive property and (19), we have

$$i(H(\lambda, \cdot), (B_{R_2} \setminus \overline{\Omega}_{N_0}) \cap \mathcal{K}, \mathcal{K}) = -1.$$

Therefore $(\hat{E}_\lambda)$ has another positive solution in $(B_{R_2} \setminus \overline{\Omega}_{N_0}) \cap \mathcal{K}$ and this completes the claim. Thus, the proof is complete by the equivalence of problems $(\hat{E}_\lambda)$ and (E_λ). □

References

[1] R. P. Agarwal and H. Lü and D. O'Regan, Eigenvalues and the one-dimensional p-Laplacian, *J. Math. Anal. Appl.*, **266** (2002) 383-400.

[2] M. del Pino and M. Elgueta and R. Manásevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0, u(0) = u(T) = 0,\ p > 1$, *J. of Differential Equations*, **80** (1989), 1-13.

[3] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones. Academic Press, 1998, New York.

[4] C. G. Kim and Y. H. Lee, Existence of multiple positive solutions for p-Laplacian problems with a general indefinite weight, *Comm. Contemporary Math.*, **10** (2008), 337-362

[5] C. G. Kim and Y. H. Lee, Existence of multiple positive solutions for a p-Laplacian problem with L^1-indefinite weight, *Taiwanese J. Math.*, **15** (2011) 723-736.

[6] T. Kusano and T. Jaros and N. Yoshida, A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order, *Nonlinear Anal.*, **40** (2000), 381-395.

[7] E. Zeidler, Nonlinear Functional Analysis and its Applications I. Springer-Verlag, 1985, New York.

[8] M. Zhang, Nonuniform nonresonance of semilinear differential equations, *J. of Differential Equations*, **166** (2000), 33-50.

Chan-Gyun Kim

Department of Mathematics, Chinju National University of Education, Chinju 52673, Republic of Korea

Email address: cgkim75@cue.ac.kr

Yong-Hoon Lee

Department of Mathematics, Pusan National University, Busan 46241, Republic of Korea

Email address: yhlee@pusan.ac.kr