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SOME RATIONAL CURVES OF MAXIMAL GENUS IN $\mathbb{P}^3$

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Abstract. For a reduced, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ of degree d, it was shown that the arithmetic genus g of C has an upper bound $\pi_0(d, r)$ by G. Castelnuovo. And he also classified the curves that attain the extremal value. These curves are arithmetically Cohen-Macaulay and contained in a surface of minimal degree. In this paper, we investigate the arithmetic genus of curves lie on a surface of minimal degree - the Veronese surface, smooth rational normal surface scrolls and singular rational normal surface scrolls. We also provide a construction of curves on singular rational normal surface scroll $S(0, 2) \subset \mathbb{P}^3$ which attain the maximal arithmetic genus.

1. Introduction

Let $C \subset \mathbb{P}^r$ be a reduced, irreducible and nondegenerate curve of arithmetic genus g and degree d. In the classical paper [2], G. Castelnuovo proved that there is an upper bound $\pi_0(d, r)$ of g which is explicitly determined in terms of d and r. Moreover, if $d \geq 2r+1$ then the curves that attain the possibly maximal value $g = \pi_0(d, r)$, so called *Castelnuovo curve*, are very well understood (see Theorem 1.2). Castelnuovo's ingenious idea has been further extended by G. Fano([4]), Eisenbud-Harris([3]) for $g = \pi_1(d, r)$, and I. Petrakiev([11]) for $g = \pi_2(d, r)$. For d, r and $0 \leq m \leq r-2$, let

$$\lambda_m = \left[\frac{d-1}{r-1+m}\right] \quad \text{and} \quad \epsilon_m = d - 1 - \lambda_m(r - 1 + m) \tag{1}$$

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where $[n]$ is the largest integer not exceeding n. Also set $\mu_m = \max\{0, [\frac{m-r+2+\epsilon_m}{2}]\}$ and the function

$$h_m(\ell) = \begin{cases} \ell(r+m-1) - m + 1, & 1 \le \ell \le \lambda_m \\ d - \mu_m, & \ell = \lambda_m + 1 \\ d, & \ell \ge \lambda_m + 2 \end{cases}$$

Then we define

$$\pi_m(d,r) = \sum_{\ell=1}^{\infty} (d - h_m(\ell)).$$

Remark 1.1. It is easy to see that

$$\pi_0(d,r) = \binom{\lambda_0}{2}(r-1) + \lambda_0\epsilon_0.$$

Theorem 1.2 (G. Castelnuovo, [2]). *Let $C \subset \mathbb{P}^r$ be a reduced, irreducible and nondegenerate curve of arithmetic genus g and degree d. Then the inequality $g \le \pi_0(d,r)$ holds always. Moreover, if $d \ge 2r+1$ then the equality $g = \pi_0(d,r)$ holds if and only if C is arithmetically Cohen-Macaulay and contained in a surface of minimal degree.*

The aim of this short paper is to reconsider the arithmetic genus of curves in surfaces of minimal degree as in the cases where the Veronese surface, smooth rational normal surface scrolls and singular rational normal surface scrolls. In each case, we calculate explicitly the arithmetic genus of curves on surfaces of minimal degree as divisors (see Section 2). We also provide a construction of rational curves on a rational normal surface scroll $S(0,2) \subset \mathbb{P}^3$ in which those curves attain the possibly maximal arithmetic genus in Castelnuovo's bound $\pi(d,3)$ (see Corollary 2.6 and Theorem 3.3).

2. Genus of curves in surfaces of minimal degree

In this section, we calculate explicitly the arithmetic genus of curves lie on surfaces of minimal degree as divisors and provide genus formulas in some cases. For the experts, there won't be much new results.

2.1. The Veronese surface

Let $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$ be the Veronese embedding of degree 2. Then we call $S := \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ the Veronese surface of degree 4 . Let C be a curve defined by a homogeneous polynomial F_l of degree $l \ge 3$ in $\mathbb{P}^2$. Then it is well known that the arithmetic genus $g(C)$ of C is

$$g(C) = \frac{(l-1)(l-2)}{2} = \begin{cases} 2k^2 - 5k + 3, & l = 2k-1 \\ 2k^2 - 3k + 1, & l = 2k \end{cases}.$$

Since the map ν_2 is an isomorphism, the image $\widetilde{C} := \nu_2(C)$ is also a reduced, irreducible and nondegenerate curve of degree $2l$ and arithmetic genus $g(C)$. Indeed, if $\widetilde{C}$ is contained in a hyperplane $\widetilde{H}$ in $\mathbb{P}^5$ then C should be contained a conic $Q_{\widetilde{H}}$ in $\mathbb{P}^2$ corresponds to $\widetilde{H}$ via the map ν_2. But it is impossible. Similarly the degree of $\widetilde{C}$ is given by the length of finite scheme $|\,\widetilde{C}\cap\widetilde{G}| = |\,C\cap Q_{\widetilde{G}}| = 2l$ by Bézout's theorem for a general hyperplane $\widetilde{G}$ in $\mathbb{P}^5$ and a conic $Q_{\widetilde{G}}$ corresponds to $\widetilde{G}$ via the map ν_2. On the other hand, (1) and Remark 1.1 enable us to calculate that if $l = 2k-1$ then $\lambda_0 = k-1$ and $\epsilon_0 = 1$, and if $l = 2k$ then $\lambda_0 = k-1$ and $\epsilon_0 = 3$. Thus the maximal genus $\pi_0(2l,5)$ of curves of degree $2l$ in $\mathbb{P}^5$ is

$$\pi_0(2l,5) = \begin{cases} 2k^2 - 5k + 3, & l = 2k-1 \\ 2k^2 - 3k + 1, & l = 2k. \end{cases}$$

Therefore we conclude that every curve in the Veronese surface has the maximal arithmetic genus. Note that every curve contained in the Veronese surface is always arithmetically Cohen-Macaulay (cf. [10, Proposition 2.9]).

2.2. Rational normal surface scroll: smooth case

For the vector bundle

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$$

on $\mathbb{P}^1$ where $1 \le \alpha \le \beta$, the rational normal surface scroll $S := S(\alpha,\beta) \subset \mathbb{P}^r$ of degree $\alpha+\beta = r-1$ is defined by the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$. Then the divisor class group of S is freely generated by the hyperplane divisor H and a ruling line F of S. For integers $a \ge 1$ and b, let $C \equiv aH + bF \subset S$ be a curve of degree $d = a(r-1) + b$.

Lemma 2.1. *Let C be a curve just stated as above. Then C is nondegenerate irreducible and arithmetically Cohen-Macaulay if and only if*

$$\begin{cases} a = 1 & \textit{and} \;\; b = 1 \\ a \ge 2 & \textit{and} \;\; \max\{-a\alpha, 2-r\} \le b \le 1. \end{cases}$$

Proof. Since C is nondegenerate, we have

$$H^0(S, \mathcal{O}_S(H - C)) = H^0(S, \mathcal{O}_S((1-a)H - bF)) = 0,$$

and hence $a = 1$ and $b \ge 1$, or $a \ge 2$. And the irreducibility of C implies that $b \ge -a\alpha$ (see [8, §V. Corollary 2.18]). Therefore an irreducible curve $C = aH + bF$ satisfies the following condition

$$\begin{cases} a = 1 & \text{and} & b \ge 1 \\ a \ge 2 & \text{and} & b \ge -a\alpha. \end{cases} \tag{2}$$

Now suppose that C is arithmetically Cohen-Macaulay and consider the exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_C \to \mathcal{O}_S(-C) \to 0.$$

Since S is also arithmetically Cohen-Macaulay, $H^1(\mathbb{P}^r, \mathcal{I}_S(j)) = H^2(\mathbb{P}^r, \mathcal{I}_S(j)) = 0$. Thus we have

$$H^1(\mathbb{P}^r, \mathcal{O}_S((j-a)H - bF)) \cong H^1(\mathbb{P}^r, \mathcal{I}_C(j)) = 0.$$

On the other hand, $H^1(\mathbb{P}^r, \mathcal{O}_S((j-a)H - bF))$ is obtained as

$$\begin{cases} 0, & 3 \le j-a \\ H^1\big(\mathbb{P}^1, Sym^{j-a}(\mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)) \otimes \mathcal{O}_{\mathbb{P}^1}(-b)\big), & 0 \le j-a \le 2 \\ 0, & j-a = -1 \\ H^1\big(\mathbb{P}^1, Sym^{a-j-2}(\mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)) \otimes \mathcal{O}_{\mathbb{P}^1}(b+r-3)\big), & j-a \le -2 \end{cases}$$

by the natural projection map $\pi : S \to \mathbb{P}^1$ and the adjunction formula. Then $H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((j-a)\alpha - b)\big) \subset H^1\big(\mathbb{P}^1, Sym^{j-a}(\mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)) \otimes \mathcal{O}_{\mathbb{P}^1}(-b)\big)$ and $H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((a-j-2)\alpha + b + r - 3)\big) \subset H^1\big(\mathbb{P}^1, Sym^{a-j-2}(\mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)) \otimes \mathcal{O}_{\mathbb{P}^1}(b+r-3)\big)$ enable us to obtain $2 - r \le b \le 1$ (cf. see Section 2 in [9]). Combining (2), we have the desired assertion. The converse statement comes directly from the calculation above. □

Lemma 2.2. *Let C be a curve in Lemma 2.1. Assume that C is arithmetically Cohen-Macaulay. Then the arithmetic genus g of C attains the possibly maximal value in Castelnuovo's genus bound as following:*

$$g = \begin{cases} 0, & \text{if } a = 1 \text{ and } b = 1 \\ \binom{a-1}{2}(r-1) + (a-1)(b+r-2), & \text{if } a \ge 2 \text{ and } \max\{-a\alpha, 2-r\} \le b \le 1 \end{cases}$$

Proof. First by adjunction formula, we have

$$2g - 2 = (K_s + C) \cdot C = ((a-2)H + (b+r-3)F)(aH + bF).$$

Thus it holds that

$$\begin{aligned} g &= \frac{a(a-2)(r-1) + a(b+r-3) + b(a-2) + 2}{2} \\ &= \begin{cases} \binom{a}{2}(r-1), & \text{if } b = 1 \\ \binom{a-1}{2}(r-1) + (a-1)(b+r-2), & \text{if } \max\{-a\alpha, 2-r\} \le b \le 1 \end{cases} \end{aligned}$$

To finish the proof, we recall the genus formula in Remark 1.1. Since $d = a(r-1) + b$

$$\lambda_0 = \left[\frac{d-1}{r-1}\right] = a + \left[\frac{b-1}{r-1}\right] = \begin{cases} a, & \text{if } b = 1 \\ a-1, & \text{if } \max\{-a\alpha, 2-r\} \le b < 1 \end{cases}$$

and

$$\epsilon_0 = \begin{cases} 0, & \text{if } b = 1 \\ b + r - 2, & \text{if } \max\{-a\alpha, 2-r\} \le b < 1 \end{cases}$$

Thus one can verify that $\pi_0(d, r) - g = 0$ for each case. □

2.3. Rational normal surface scroll: singular case

Notation and Remark 2.3. For the vector bundle

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha), \quad \text{for} \quad \alpha \geq 2$$

on $\mathbb{P}^1$, as in the smooth case, the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ defines the birational morphism $\phi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^r$, $r = \alpha + 1$ and its image is the rational normal surface scroll $S := S(0, \alpha) \subset \mathbb{P}^r$ of degree α. Then it is well known that $\operatorname{Pic}\big(\mathbb{P}(\mathcal{E})\big)$ is freely generated by the hyperplane class $[\widetilde{H}] := [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)]$ and the class of fibre $[\widetilde{F}] := [\pi^*\mathcal{O}_{\mathbb{P}^1}(1)]$ of the projection $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. Note that the divisor class group of S is freely generated by the strict image F of $\widetilde{F}$ via ϕ. Let $E = \phi^{-1}(V)$ for the vertex V of S be the exceptional divisor of $\mathbb{P}(\mathcal{E})$. Then it is easy to see that $E \equiv \widetilde{H} - \alpha\widetilde{F}$. Now for an effective Weil divisor C on S and its proper transform $\widetilde{C} = a\widetilde{H} + b\widetilde{F}$, we define
(A) the total transform ϕ^*C of C on $\mathbb{P}(\mathcal{E})$ is

$$\phi^*C = \widetilde{C} + \frac{b}{\alpha}E.$$

Since the total transform ϕ^*C is in general a $\mathbb{Q}$-divisor, we define more conveniently
(B) the integral total transform C^* of C is

$$C^* = [\phi^*C] = \widetilde{C} + \left[\frac{b}{\alpha}\right]E$$

where $[n]$ is the largest integer not exceeding n. For details, we refer the readers to [5].

Lemma 2.4. *Let C and C^* be curves stated as in Notation and Remark 2.3. Suppose that $C \equiv dF$ is an effective divisor on S. Then*

(i) *$C^* \equiv (t+1)\widetilde{H} - (\alpha - p - 1)\widetilde{F}$ where $d = t\alpha + 1 + p$ for $0 \leq p < \alpha$ and*

(ii) *C is arithmetically Cohen-Macaulay and the arithmetic genus of C coincides with that of C^*.*

Proof. See [5, Lemma 4.1 and Example 5.2] and [10, Proposition 2.9]. □

Proposition 2.5. *Let C be a nondegenerate effective divisor on $S(0, \alpha) \subset \mathbb{P}^{\alpha+1}$. Then,*

(1) *C is an arithmetically Cohen-Macaulay divisor linearly equivalent to bF on S for $b \geq \alpha + 1$.*

(2) *The arithmetic genus of C is*

$$\alpha\binom{t}{2} + tp \quad \text{where} \quad b = \alpha t + 1 + p, \quad 0 \leq p < \alpha. \tag{3}$$

Proof. Since C is nondegenerate and all ruling lines F intersect at the vetex V on $S(0,\alpha)$, the condition $b \geq \alpha+1$ is necessary. Then by Notation and Remark 2.3 and Lemma 2.4, it suffices to show that the arithmetic genus $g(C^*)$ of the integral total transform $C^* \equiv (t+1)\widetilde{H} - (\alpha-p-1)\widetilde{F}$ where $b = \alpha t + 1 + p$, $0 \leq p < \alpha$ of C is the desired formula. Note that $K_{\mathbb{P}(\mathcal{E})} = -2\widetilde{H} + (\alpha-2)\widetilde{F}$. Then by adjunction formula, we have

$$\begin{aligned} 2g(C^*) - 2 &= \left(K_{\mathbb{P}(\mathcal{E})} + C^*\right) \cdot C^* \\ &= ((t-1)\widetilde{H} + (p-1)\widetilde{F})((t+1)\widetilde{H} - (\alpha-p-1)\widetilde{F}) \\ &= \alpha(t^2-1) + (t+1)(p-1) - (t-1)(\alpha-p-1) \\ &= \alpha t(t-1) + 2tp - 2 \end{aligned}$$

Thus we have $g(C^*) = \alpha\binom{t}{2} + tp$. □

Corollary 2.6. *Let C be an effective divisor bF for $b \geq 1$ on the rational normal surface scroll $S(0,\alpha) \subset \mathbb{P}^r$. Then C attains always the possibly maximal arithmetic genus.*

Proof. Since $d = \alpha t + 1 + p$, $0 \leq p < \alpha$ and $r = \alpha + 1$, we have

$$\lambda_0 = \left[\frac{d-1}{r-1}\right] = t \quad \text{and} \quad \epsilon_0 = d - 1 - \lambda_0(r-1) = p.$$

This shows that the arithmetic genus $g = \alpha\binom{t}{2} + tp$ of C coincides with the maximal genus bound $\pi_0(d,r) = \binom{\lambda_0}{2}(r-1) + \lambda_0\epsilon_0$. □

3. Construction of curves of maximal genus

In this section, we provide a construction of some rational curves of degree $d \geq 7$ and of maximal genus in $\mathbb{P}^3$ lie on the rational normal surface scroll $S(0,2)$. We begin with the standard definition of a rational normal curve.

Notation and Remark 3.1. (A) Let $T := \mathbb{K}[s,t]$ be the homogeneous coordinate ring of $\mathbb{P}^1$. For each $k \geq 1$, we denote by T_k the k-th graded component of T. For the standard basis $\{s^d, s^{d-1}t, \cdots, st^{d-1}, t^d\}$ of the $\mathbb{K}$-vector space T_d, the rational normal curve $\widetilde{C} \subset \mathbb{P}^d$ is defined to be the image of the map $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ parameterized by

$$\widetilde{C} = \{[s^d(P) : s^{d-1}t(P) : \cdots : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}.$$

(B) Let $C_d \subset \mathbb{P}^r$ be a rational curve of degree $d \geq r$. Then there exists a subset $\{f_0, f_1, \ldots, f_r\} \subset T_d$ of $\mathbb{K}$-linearly independent forms of degree d such that the curve C can be written as a parametrization

$$C_d = \{[f_0(P) : f_1(P) : \cdots : f_r(P)] \mid P \in \mathbb{P}^1\}.$$

(C) For a smooth curve $D \subset \mathbb{P}^r$ and an integer $k \geq 2$, we consider

$$D^k := \overline{\bigcup_{p_1,\ldots,p_k \in D} < p_1, p_2, \ldots, p_k >}$$

the k-th join of D with itself where $< p_1, p_2, \ldots, p_k >$ is the linear span of k-points lying on D. Then there is an ascending chain

$$D \subsetneqq D^2 \subsetneqq \cdots \subsetneqq D^{s-1} \subsetneqq D^s = \mathbb{P}^r$$

where the number s is the smallest integer such that $D^s = \mathbb{P}^r$. Note that the linear projection map $\pi_q : D \to \mathbb{P}^{r-1}$ of D from a point $q \in \mathbb{P}^r \setminus D^2$ is an isomorphism. For details, we refer the reader to [12].

Let $C \subset \mathbb{P}^3$ be a rational curve given by the parametrization

$$C = \{[s^d(P) : s^2t^{d-2}(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\} \quad d \geq 4. \tag{4}$$

Lemma 3.2. *Let C be a curve just stated as above. Then C is a singular rational curve of degree d.*

Proof. First consider the rational normal curve $\widetilde{C} \subset \mathbb{P}^d$ defined in Notation and Remark 3.1.(A). For a set

$$\{P_1 = [0,1,0,\ldots,0,0], P_2 = [0,0,1,0,\ldots,0,0], \ldots, P_{d-3} = [0,0,\cdots,0,1,0,0,0]\}$$

of $(d-3)$-standard coordinate points in $\mathbb{P}^d$, let Λ be its linear span. Then Λ is a $(d-4)$-dimensional linear subspace of $\mathbb{P}^d$ and C is obtained by the linear projection map $\pi_\Lambda : \widetilde{C} \to \mathbb{P}^3$ of $\widetilde{C}$ from Λ. Since $\Lambda \cap C^2$ is not an empty set, the map π_Λ is not an isomorphism (see Notation and Remark 3.1.(C)). Indeed, π_Λ is birational and hence C is a curve as desired. ☐

Theorem 3.3. *Let C be a rational curve as stated in Lemma 3.2. Then*

(1) *C is contained in the rational normal surface scroll $S(0,2)$ as a divisor linear equivalent to dF where F is a ruling line of $S(0,2)$.*

(2) *C has the arithmetic genus*

$$g = \begin{cases} (k-1)(k-2), & \text{if } d = 2k-1 \\ (k-1)^2, & \text{if } d = 2k \end{cases}$$

In particular, the genus g of C is possibly maximal.

Proof. (1) Let $\mathbb{K}[X_0, X_1, X_2, X_3]$ be the homogeneous coordinate ring of $\mathbb{P}^3$. Then one can see that the parametrization (4) of C satisfies the quadratic equation $X_2^2 - X_1X_3$ which defines the rational normal surface scroll $S(0,2)$. Thus C is contained in $S(0,2)$ as a divisor linearly equivalent to dF. Indeed, letting $C \equiv bF$ for some integer b (see Notation and Remark 2.3), it is obvious that $b = d$ by degree counting of C. Now recall the notions in Notation and Remark 2.3. Then the integral total transform C^* of C is

$$C^* = \begin{cases} k\widetilde{H} - \widetilde{F}, & \text{if } d = 2k-1 \\ k\widetilde{H}, & \text{if } d = 2k \end{cases}.$$

Since $K_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(2))} = -2\widetilde{H}$, the adjunction formula enables us to get

$$\begin{aligned} 2g(C^*) - 2 &= \big(K_{\mathbb{P}(\mathcal{E})} + C^*\big)\cdot C^* \\ &= \begin{cases} \big((k-2)\widetilde{H} - \widetilde{F}\big)\cdot(k\widetilde{H} - \widetilde{F}), & \text{if } d = 2k-1 \\ \big((k-2)\widetilde{H}\big)\cdot(k\widetilde{H}), & \text{if } d = 2k \end{cases} \\ &= \begin{cases} 2k(k-2) - 2(k-1), & \text{if } d = 2k-1 \\ 2k(k-2), & \text{if } d = 2k \end{cases} \end{aligned}$$

Then we have desired arithmetic genus of C as $g = g(C^*)$ (cf. [5, Lemma 4.1]). Finally, the maximality of arithmetic genus comes directly from Corollary 2.6. □

Example 3.4. Let $C \subset \mathbb{P}^3$ $(4 \le d \le 12)$ be a rational curve of degree d given by the parametrization

$$C = \{[s^d(P) : s^2t^{d-2} : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}.$$

Then by means of the Computer Algebra System Singular [7], we obtain the arithmetic genus of C as following:

d	4	5	6	7	8	9	10	11	12
g	1	2	4	6	9	12	16	20	25

Table 1: The arithmetic genus of C

Note that one can obtain the same value as in Table 1 by using the formula in Theorem 3.3.

Remark 3.5. In Theorem 1.2, the condition $d \ge 2r+1$ is necessary for the Castelnuovo curves. On the other hand, it is interesting that for the rational curves C of degree $d \ge r+1$ in our main Theorem 3.3, it holds that C attains the maximal arithmetic genus and lies on a surface of minimal degree. Note that if $d = r+1$ and the curve attains the maximal arithmetic genus $\pi(r+1, r) = 1$ then it is either a simple linear projection of a rational normal curve or the elliptic normal curve (cf. [1] and [6]). The former lies on a rational normal surface scroll but the latter is not. To the authors, it is a next challenging problem to describe precisely the curves of degree $r+2 \le d \le 2r$ which attain the maximal arithmetic genus and do not lie on a surface of minimal degree.

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