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PRICING OF TIMER DIGITAL POWER OPTIONS BASED ON STOCHSTIC VOLATILITY

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Abstract. Timer options are financial instruments proposed by Société Générale Corporate and Investment Banking in 2007. Unlike vanilla options, where the expiry date is fixed, the expiry date of timer options is determined by the investor's choice, which is in linked to a variance budget. In this study, we derive a pricing formula for hybrid options that combine timer options, digital options, and power options, considering an environment where volatility of an underlying asset follows a fast-mean-reverting process. Additionally, we aim to validate the pricing accuracy of these analytical formulas by comparing them with the results obtained from Monte Carlo simulations. Finally, we conduct numerical studies on these options to analyze the impact of stochastic volatility on option's price with respect to various model parameters.

1. Introduction

Since financial markets have been growing and improving, not only a lot of investors but also market participants have become more interested in generating higher returns, but there have been growing concern over possible risk of the unpredictable market, which springs from the global financial crisis, the COVID–19 pandemic or the Russia–Ukraine conflict. It allows many researchers to give attention to the financial modelling to predict the prices of diverse derivatives in the real market. For example, geometric Brownian motion (GBM) governed by the model dynamics of the risky asset price has been introduced by Black-Scholes [4]. In addition, in order to capture and reflect the empirical results seen in the financial market, stochastic volatility(SV) models have been proposed by Hull and White [10], Heston [9], Fouque et al. [7]. SV model has been considered to be very helpful for derivative pricing for several years, because it has shown the existence of nonflat implied volatility, demonstrating that they make up for the disadvantage of Black-Scholes model and

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reflecting the empirical evidences that the implied volatility of equity options exhibits a smile or skew phenomenon. Nevertheless, the level of implied volatility can grow higher than the realized volatility due to the risk premium from price uncertainty of the risky asset. The high implied volatility implies that the option is overestimated.

Since then, there have been lots of researches that dealt with the volatility risk for the derivative pricing. In particular, *timer option*, one of innovative financial securities has been first launched by Société Générale Corporate and Investment Banking (SG CIB) in April 2007, which have a random expiration depending on the realized accumulated variance under the variance budget. A *timer option* enables investors to take into the volatility level consideration to exercise their options with a random maturity contrary to a standard European vanilla option that can exercise only in the fixed maturity. According to Sawyer [19], timer option is helpful for investors to seek more underestimated derivatives compared to vanilla options. If the volatility is high, the timer options are exercised at an early stage. On the other hand, it takes much time for the timer options to arrive at its maturity in case that the volatility is low. Under these circumstances, if the price of derivatives changes drastically in the uncertain market, described by the global financial crisis in 2007-2008, then it can directly lead to the dramatic changes of the volatility, which ultimately makes the option get exercised fast. As shown in Bernard and Cui [3], timer options have a significant role in hedging or implementing replication techniques for the variance swap or the volatility swaps. There have been many studies about the valuation for timer options. For instance, Bernard and Cui [3] first dealt with the timer options based on the stochastic volatility, verifying the analytic solutions for the options can be obtained by an efficient almost exact Monte Carlo method. Zheng and Zeng [26] examined the pricing formula for the timer options based on the 3/2 model by making use of a closed-form partial transform. Furthermore, Li [15] used the joint probability density function for the first-passage time that the realized variance hit the variance budget at the first time to derive the analytic pricing formula of the timer options with the Heston model.

Digital opiton is a type of trading option in which the investors consider a fixed strike price for a security. The option gains a profit if the asset's market price goes beyond the strike price prior to the maturity time. This trading option allows traders to give investors a fixed payment by accurately forecasting the prices of derivatives. Whereas binary options may be used in theoretical asset pricing, they are vulnerable to fraud in their applications and therefore, prohibited by regulators in lots of jurisdictions as a form of gambling [20]. The outlets of a lot of binary options have been revealed as fraudulent [21]. The U.S. FBI has been scrutinizing closely binary option scams all over the world, and the Israeli police have tied the industry to criminal syndicates [22], [23], [2]. The European Securities and Markets Authority (ESMA) have proscribed retail binary options trading [18]. Australian Securities & Investments Commission (ASIC) takes account of binary options as a "high-risk" and "unpredictable" investment option [11], and finally also prohibited binary options sale to retail investors in 2021 (cf. [1]).

Power options are one of the options that the payoff function of the underlying asset relies on an index of a positive integer with respect to the asset price at the maturity. Zhang et al. [25] found that the type of option can offer the flexibility and a greater amount of leverage to investors unlike the standard European vanilla options. As shown in Macovschi [16], the author dealt with the power options based on the Heston stochastic volatility model and a pure jump Levy model. Kim et al. [13] investigated the semi-analytic pricing formula of the power options with the Heston model, implementing numerical methods for the value of power put options and capped power call options. Zhang et al. [25] derived the price of the power options, assuming that, for Liu's uncertain stock model, the option value for the underlying stock price solves an uncertain differential equation compared with to the Black-Scholes framwork, and derive the analytic formulas for the power options through the approach of uncertain calculus under uncertainty theory.

In this article, integrating the index in terms of the underlying asset of the standard digital power options into the timer options, **we investigate timer-digital power options (TDPOs) under the SV**.

SV models have widely been used in the pricing problems of the diverse options, supplementing somes weaknesses of the existing Black-Scholes model that the SV models capture and reflect the empirical evidence in the real market verifying demonstrating that the implied volatility of equity options exhibits smile and skew phenmenon at the same time. In fact, as seen in Choi et al. [6], the assumption of constant volatility fails to capture an extraordinary stochastic phenomenon since the global crisis in 2007-2008. Therefore, the empirical studies that the volatility of the underlying asset price is a stochastic process have become enable for us to show the market dynamics more effectively and accurately. Fouque et al. [7] has taken into the several types of options consideration based upon the SV model incorporated by a fast mean-reverting process. From then on, many researches for the financial derivatives with the SV model have been conducted by Wong and Chan [24], Chiarella et al. [5], Kim et al. [14]. Especially, Kim et al. [14] utilized the SV model to external barrier options and then derive the analytic pricing formula by making use of the technique of the asymptotic analysis.

In this paper, we have the main contributions of this paper as follows: First of all, we set up the model dynamics for the timer-digital power options (TDPOs) and obtain the partial differential equations (PDEs) for the price of TDPOs. Second, we derive the approximated formulas for the PDEs by taking advantage of the method of asymptotic analysis, which is very significant one for us to handle the TDPO prices. Third, we demonstrate the accuracy of the corrected option price by using the Monte-Carlo method. Finally, we carry out the numerical experiments of the value of TDPOs and observe some economical meanings with respect to model parameters.

The rest of this paper is organized as follows. In Section 2, we construct the dynamics for the market model of the TDPOs and obtain the PDEs for the price of TDPOs. Section 3 deals with the first-order approximation for the value of the TDPOs by utilizing the asymptotic analysis. In section 4, we demonstrate the pricing accuracy of the our solutions for the TDPOs from Monte-Carlo simulation and also provide the numerical implications in terms of model parameters. Section 5 presents the concluding remarks.

2. Model formulation

In this section, we construct the stochastic dynamics for the underlying asset price and we deduce the PDEs by using the well-known Feynman-Kac formula, described in Øksendal [17]. First of all, let us consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a nonempty set, $\mathcal{F}$ is a σ-algebra over Ω, and $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$. In this probability space, the underlying asset price, denoted by X_t, follows

$$
\begin{aligned}
\mathrm{d}X_t &= \mu X_t \mathrm{d}t + f(Y_t) X_t \mathrm{d}W_t^1, \\
\mathrm{d}Y_t &= \alpha(m - Y_t)\mathrm{d}t + \beta \mathrm{d}W_t^2,
\end{aligned}
\tag{1}
$$

where μ is constant mean return rate of X, f is any bounded smooth function, α, β are positive constants, m is the long-run mean level of Y, and $\{W_t^1, t \geq 0\}$ and $\{W_t^2, t \geq 0\}$ are standard Brownian motions satisfying $\mathrm{d}\left\langle W^1, W^2 \right\rangle_t = \rho \mathrm{d}t$ and $|\rho| \leq 1$.

The fast mean-reverting Ornstein-Uhlenbeck process, denoted as Y_t, can be explicitly described through the equation $Y_t = m + (Y_0 - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} \mathrm{d}W_t^2$. This leads to the distribution of Y_t being $\mathcal{N}(m + (Y_0 - m)e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t}))$. In addition, as t goes to ∞, Y_t converges to its invariant distribution, which is $\mathcal{N}(m, u^2)$, where $u^2 = \frac{\beta^2}{2\alpha}$. The term "mean-reverting" refers to the typical time it takes for a process to revert back to its mean level, characterized by the invariant distribution of Y_t. In (1), α is referred to as the mean-reverting rate. When α is sufficiently large, the procee Y_t in (1) tends to revert to its long-run mean m regardless of the time. Therefore, we introduce a small parameter ϵ, defined as the inverse of the mean-reversion rate.

Utilizing the Girsanov theorem presented in Øksendal [17], under the risk-neutral measure $\widehat{\mathbb{P}}$, the model dynamics (1) is transformed into

$$
\begin{aligned}
\mathrm{d}X_t &= r X_t \mathrm{d}t + f(Y_t) X_t \mathrm{d}\widehat{W}_t^1, \\
\mathrm{d}Y_t &= \left(\frac{1}{\epsilon}(m - Y_t) - \frac{u\sqrt{2}}{\sqrt{\epsilon}} \Lambda(Y_t) \right) \mathrm{d}t + \frac{u\sqrt{2}}{\sqrt{\epsilon}} \mathrm{d}\widehat{W}_t^2,
\end{aligned}
\tag{2}
$$

where r is a risk free interest rate and the transformed Brownian motions $\widehat{W}_t^1$ and $\widehat{W}_t^2$ are correlated with ρ.

The maturity of a timer option τ is determined based on the variance budget $\mathbb{V}$ determined by the investor. To be more precise, the random maturity τ is identified as the initial moment when the cumulative realized variance aligns with the predetermined variance budget $\mathbb{V}$ for the first time. In other words,

$$\tau := \inf\left\{t(>0) : V_t = \mathbb{V}\right\}, \tag{3}$$

where V_t is the accumulated process defined by $V_t := \int_0^t f^2(Y_s)\mathrm{d}s$.

In this paper, to deal with the TDPOs, we propose a payoff function for the TDPO, denoted as $h(X_\tau)$, is described by

$$h(X_\tau) = \mathcal{H}(X_\tau^c - K), \tag{4}$$

where c is a nonnegative integer, K is a stirke price and $\mathcal{H}$ is a Heviside function. Then, at time $t \wedge \tau$, this option price is expressed by

$$P(t\wedge\tau, x, y, v) = \widehat{\mathbb{E}}\left[e^{-r(\tau - t\wedge\tau)}h(X_\tau) \mid X_{t\wedge\tau} = x, Y_{t\wedge\tau} = y, V_{t\wedge\tau} = v\right], \tag{5}$$

where $\widehat{\mathbb{E}}[\cdot]$ denotes expectation under the risk-neutral measure $\widehat{\mathbb{P}}$. According to Li [15], the timer option's price at any time $t \wedge \tau$ is the same as the option's price at the initial time. In other words, $P(t \wedge \tau, x, y, v)$ can be expressed by

$$P(t\wedge\tau, x, y, v) = \widehat{\mathbb{E}}\left[e^{-r\tau_{\mathbb{V}-v}}h(X_\tau) \mid X_0 = x, Y_0 = y\right]. \tag{6}$$

Now, by applying the Feynman-Kac formula and the fact that timer option does not depend on time t referring to Ha *et al.* [8], we obtain the partial differential equation (PDE) as follows:

$$\begin{aligned} &\mathcal{L}^\epsilon P(x,y,v) = 0, && v < \mathbb{V}, \\ &P(x,y,\mathbb{V}) = \mathcal{H}(X_\tau^c - K), \end{aligned} \tag{7}$$

where the differential operator $\mathcal{L}^\epsilon$ is given by

$$\begin{aligned} \mathcal{L}^\epsilon &:= \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2, \\ \mathcal{L}_0 &:= (m-y)\frac{\partial}{\partial y} + u^2\frac{\partial^2}{\partial y^2}, \\ \mathcal{L}_1 &:= -u\sqrt{2}\Lambda(y)\frac{\partial}{\partial y} + u\sqrt{2}\rho f(y)x\frac{\partial^2}{\partial x\partial y}, \\ \mathcal{L}_2 &:= r\left(x\frac{\partial}{\partial x} - \cdot\right) + f^2(y)\frac{\partial}{\partial v} + \frac{1}{2}f^2(y)x^2\frac{\partial^2}{\partial x^2}. \end{aligned} \tag{8}$$

Referring to Fouque *et al.* [7], if we expand P in powers of $\sqrt{\epsilon}$, then we have

$$P(x,y,v) = P_0(x,y,v) + \sqrt{\epsilon}P_1(x,y,v) + \epsilon P_2(x,y,v) + \epsilon\sqrt{\epsilon}P_3(x,y,v) + \cdots, \tag{9}$$

where $P_0(x,y,\mathbb{V}) = \mathcal{H}(X_\tau^c - K)$ and $P_i(x,y,\mathbb{V}) = 0$ if $i \geq 1$. In (9), we focus on the first two terms: P_0, leading-order price, and P_1, correction term. Next, the following theorem provides that the first two terms P_0 and P_1 do not depend on the volatility y, and it also presents a homogeneous PDE for P_0 and a non-homogeneous PDE for P_1.

Theorem 2.1. *If P_0 and P_1 do not grow as much as $\frac{\partial P_0}{\partial y} \sim e^{\frac{y^2}{2}}$ and $\frac{\partial P_1}{\partial y} \sim e^{\frac{y^2}{2}}$ as $y \to \infty$, then $P_0(x,y,v)$ and $P_1(x,y,v)$ are independent of y. Then, the leading-order price P_0 and correction term P_1 satisfy the following PDE problems*

$$\begin{aligned} &\langle \mathcal{L}_2 \rangle(\sigma) P_0(x,v) = 0, \quad v < \mathbb{V}, \\ &P_0(x,\mathbb{V}) = \mathcal{H}(X_\tau^c - K), \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\langle \mathcal{L}_2 \rangle(\sigma) P_1(x,v) = \mathcal{A} P_0, \qquad v < \mathbb{V}, \\ &P_1(x,\mathbb{V}) = 0, \end{aligned} \tag{11}$$

respectively, where

$$\langle \mathcal{L}_2 \rangle(\sigma) := r\left(x\frac{\partial}{\partial x} - \cdot\right) + \sigma^2 \frac{\partial}{\partial v} + \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2}, \tag{12}$$

$$\sigma := \sqrt{\langle f^2(y) \rangle}, \tag{13}$$

$$\begin{aligned} \mathcal{A} :=\ & u\sqrt{2}\,\langle \Lambda(y)\phi'(y) \rangle \left(\frac{\partial}{\partial v} + \frac{x^2}{2}\frac{\partial^2}{\partial x^2}\right) \\ & - u\sqrt{2}\rho\,\langle f(y)\phi'(y) \rangle \left(x\frac{\partial^2}{\partial x \partial v} + x^2\frac{\partial^2}{\partial x^2} + \frac{x^3}{2}\frac{\partial^3}{\partial x^3}\right), \end{aligned} \tag{14}$$

$\phi(y)$ is a solution to the Poisson equation $\mathcal{L}_0 \phi(y) = f^2(y) - \langle f^2(y) \rangle$, and σ in (13) *is defined as the effective volatility.*

Proof. Substituting (9) into (7), we get the following equation for $\sqrt{\epsilon}$.

$$\begin{aligned} \frac{1}{\epsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) \\ + \sqrt{\epsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) \cdots = 0. \end{aligned}$$

Since $\epsilon(>0)$ is arbitrary, the coefficients of each term should all be zero. In other words, the following equations is satisfied:

$$\begin{aligned} &\mathcal{L}_0 P_0 = 0,\ \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0,\ \mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0, \\ &\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0,\ \cdots. \end{aligned}$$

From these equations, we obtain that P_0 and P_1 are independent of y based on the growth condition and hold the two PDEs (10) and (11) for P_0 and P_1. □

Lemma 2.2. *In* (10)*, the leading order term $P_0(x,v)$ is given by*

$$P_0(x,v) = e^{-r\left(\frac{\mathbb{V}-v}{\sigma^2}\right)} \mathcal{N}(d), \tag{15}$$

where $\mathcal{N}(\cdot) = \frac{1}{2\pi}\int_{-\infty}^{\cdot} e^{-z^2/2} \mathrm{d}z$ and $d = \frac{\log\left(x/K^{\frac{1}{c}}\right) + \left(r - \frac{\sigma^2}{2}\right)\left(\frac{\mathbb{V}-v}{\sigma^2}\right)}{\sqrt{\mathbb{V}-v}}$.

Proof. To obtain the solution to the PDE (10), let us consider the following change of variables:

$$\xi = \frac{\mathbb{V}-v}{\sigma^2},\ z = \log\left(\frac{x}{K}\right),\ \nu = \frac{P_0}{K},\ \omega = e^{-(\alpha z+\beta\xi)}\nu,$$
$$\alpha = -\frac{k-1}{2},\ \beta = -\frac{\sigma^2(k+1)^2}{8},\ \text{and } k = \frac{2r}{\sigma^2}. \tag{16}$$

Then, the PDE (10) is transformed into the following heat equation

$$\frac{\partial\omega}{\partial\xi} = \frac{\sigma^2}{2}\frac{\partial^2\omega}{\partial z^2}, \quad 0<\xi<\frac{\mathbb{V}}{\sigma^2},$$
$$\omega(0,z) = e^{-\alpha z}K^{-1}, \quad z > \frac{1-c}{c}\log(K).$$

Referring to Kevorkian [12], ω is given by

$$\omega(z,\xi) = \int_{\frac{1-c}{c}\log K}^{\infty} \frac{1}{\sqrt{2\pi\xi\langle f^2\rangle}} e^{-\frac{(s-z)^2}{2\xi\langle f^2\rangle}}\omega(s,0)\mathrm{d}s.$$

Now, utilizing the change of variables in (16), ω can be derived as follows:

(17)

$$\omega(z,\xi) = K^{-1}\exp\left(\frac{\xi\sigma^2(k-1)^2}{8} + \frac{k-1}{2}z\right)\mathcal{N}\left(\frac{-\frac{1-c}{c}\log(K) + z + \left(r-\frac{\sigma^2}{2}\right)\xi}{\sigma\sqrt{\xi}}\right).$$

Finally, subtituting (17) into (16), we have

$$P_0 = e^{-r\left(\frac{\mathbb{V}-v}{\sigma^2}\right)}\mathcal{N}\left(\frac{\log\left(x/K^{\frac{1}{c}}\right) + \left(r-\frac{\sigma^2}{2}\right)\left(\frac{\mathbb{V}-v}{\sigma^2}\right)}{\sqrt{\mathbb{V}-v}}\right).$$

□

Lemma 2.3. *In* (11)*, the correction term* $P_1^\epsilon(x,v) = \sqrt{\epsilon}P_1(x,v)$ *is given by*

$$P_1^\epsilon(x,v) = -\sqrt{\epsilon}\left(\frac{\mathbb{V}-v}{\sigma^2}\right)\mathcal{A}P_0. \tag{18}$$

Proof. By the commuting property as in Fouque et al. [7], we have the following relation:

$$\langle\mathcal{L}_2\rangle\left(x^n\frac{\partial^n P_0}{\partial x^n}\right) = x^n\frac{\partial^n}{\partial x^n}\langle\mathcal{L}_2\rangle P_0.$$

Therefore, we can obtain the following result:

$$\langle\mathcal{L}_2\rangle\left(-\frac{\mathbb{V}-v}{\langle f^2\rangle}\mathcal{A}P_0\right) = \mathcal{A}P_0 - \frac{\mathbb{V}-v}{\langle f^2\rangle}\langle\mathcal{L}_2\rangle\left(\mathcal{A}P_0\right) = \mathcal{A}P_0,$$

where $\mathcal{A}$ is presented in (14). □

Finally, by incorporating the results of the leading-order price, P_0 in (15) and the correction term, P_1^ϵ in (18), we obtain the following approximated pricing formula of TDPO, denoted as $\widetilde{P}^\epsilon$.

$$P(x,y,v) \approx \widetilde{P}^\epsilon(x,v) \equiv P_0(x,v) + P_1^\epsilon(x,v). \tag{19}$$

Then, referring to Fouque *et al.* [7], if the payoff function, h, is continuously differentiable, then the error between P solution to PDE (7) and $\widetilde{P}^\epsilon$ in (19) can be described by

$$\left| P(x,y,v) - \widetilde{P}^\epsilon(x,v) \right| \leq \mathcal{O}(\epsilon) \tag{20}$$

holds for $0 < \epsilon \ll 1$. In this research, we numerically provide the accuracy of the approxiated solution in (19) utilizing the Monte Carlo simulations instead of deriving the mathematical proof (see Table 1 in Section 3).

3. Implications

In this section, we provide the accuracy of pricing formula of the TDPO described in (17) by comparing it with the generated results from the Monte Carlo simulations, instead of the driving the mathematical proof for the error estimate of the formula for TDPO. For these numerical experiments, we selected the model parameters as follows: $v = 0.01, r = 0.01, \mathbb{V} = 0.0265, \rho = -0.1, \langle \Lambda\phi' \rangle = 0.2, \langle f\phi' \rangle = 0.01, u = 0.1, K = 0.7$, and $\sigma = 0.1$, referring to Ha *et al.* [8]. In addition, all these computations are implemented using an Apple M2 Pro and 16 GB memory.

ϵ	$\widetilde{P}^\epsilon(x,v)$	P_{MC}	$\lvert\widetilde{P}^\epsilon(x,v) - P_{\mathrm{MC}}\rvert$	RE [%]
0.100	0.927268	0.927268	0.016289	1.756646
0.050	0.911179	0.926778	0.015598	1.683087
0.010	0.911447	0.923837	0.012391	1.341216
0.005	0.911510	0.921877	0.010367	1.124547
0.001	**0.911594**	**0.917466**	**0.005872**	**0.639973**

TABLE 1. Error comparison between price of timer-digital power option $P(x,y,v)$ and Monte–Carlo price P_{MC} with respect to ϵ. Referring to Ha *et al.* [8], we selected baseline parameters as follows: $x = 1, v = 0.01, r = 0.01, \mathbb{V} = 0.0265, \rho = -0.1, \langle \Lambda\phi' \rangle = 0.2, \langle f\phi' \rangle = 0.01, u = 0.1, K = 0.7$, and $\sigma = 0.1$. All the computations for Table 1 are implemented using a system with Apple M2 Pro and 16 GB memory.

We conduct a validation of our approximated solution presented in (19) via Monte Carlo simulations, which are considered as true or benchmark solution of TDPO given in (5). As observed In Table 1, it can be seen that the price discrepancy between the approximated price $\widetilde{P}^{\epsilon}(x, v)$ and the Monte Carlo price P_{MC} approaches zero sufficiently, which is evidenced by the data presented in the forth column of Table 1). In addition, the relative error, defined by RE $= \frac{|\widetilde{P}^{\epsilon}(x,v)-P_{\mathrm{MC}}|}{P_{\mathrm{MC}}} \times 100$, tends to zero as the parameter ϵ decreases, a trend that is clearly illustrated in the fifth column of Table 1. Therefore, our approximated pricing formula presented in (19) is derived accuratly and efficiently.

Figure 1 represents the behavior of the correction term, P_1, in terms of the underlying asset price or realized variance, with respect to model parameters: $x = 1, v = 0.01, r = 0.01, \mathbb{V} = 0.0265, \rho = -0.1, \langle \Lambda\phi' \rangle = 0.2, \langle f\phi' \rangle = 0.01, u = 0.1, K = 0.7, \sigma = 0.1$ and $\epsilon = 0.001$. First, in Figure 1(a), we examine the impact of the power of the stock price on the correction term with respect to the underlying asset price, x. In this figure, one can see that the correction term, P_1^{ϵ}, have a hump phenomenon as x approaches $K^{1/c}$, and if x greater than $K^{1/c}$, then the graph tends to decreases rapidly. It is worth noting that when $c = 1$, TDPO reduces the standard timer-digital options. Second, in Figure 1(b), we analyze the influence of the power of the stock price on correction term with respect to the realized variance level, v. In the case where $c = 1$, the graph exhibits a monotonically increasing shape as v is closed to $\mathbb{V}$, while in the cases of $c = 2$ and $c = 3$, the graph displays convexity. Interestingly, in this figure, it can be seen that the stochastic volatility is more significant as the parameter c increases, especially when the realized variance v deviates from the given variance budget $\mathbb{V}$ compared to when $v \approx \mathbb{V}$. Therefore, from both figures, we can find out that the parameter c, the power of the stock price, and the stochastic volatility play a key role in the price of TDPO.

4. Conclusion

In this study, we propose the timer-digital power options under a generalized stochastic volatility model. Unlike conventional options with fixed expiration dates, timer options offer investors the flexibility to choose their expiration date based on a predefined variance budget. This research focuses on deriving a pricing formula for timer-digital power options. In addition, we provide the pricing accuracy of obtained analytical formulas by utilizing Monte Carlo simulations. Furthermore, numerical investigations are conducted to investigate the impact of the stochastic volatility on the option's price with respect to the various model parameters.

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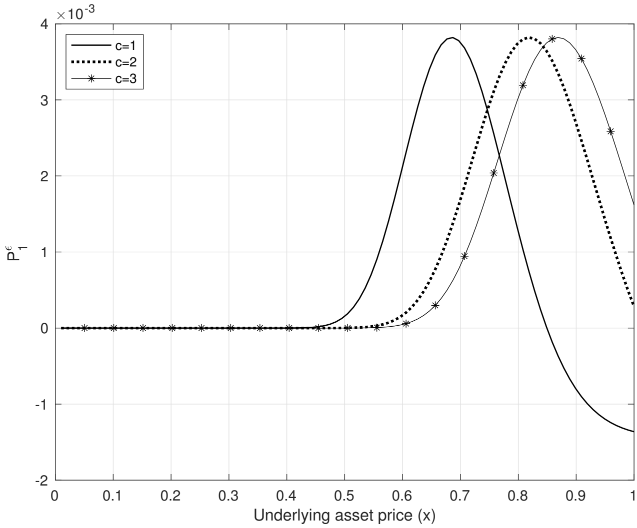
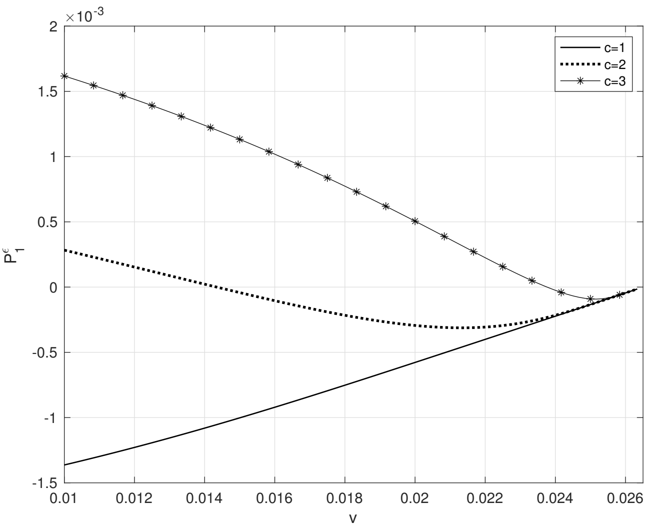


(a)



(b)

FIGURE 1. Behavior of correction term P_1^ϵ with respect to model parameters: $x = 1, v = 0.01, r = 0.01, \mathbb{V} = 0.0265, \rho = -0.1, \langle \Lambda\phi' \rangle = 0.2, \langle f\phi' \rangle = 0.01, u = 0.1, K = 0.7, \sigma = 0.1$, and $\epsilon = 0.001$.

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