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COMBINATORIAL SUPERSYMMETRY: SUPERGROUPS, SUPERQUASIGROUPS, AND THEIR MULTIPLICATION GROUPS

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Abstract. The Clifford algebra of a direct sum of real quadratic spaces appears as the superalgebra tensor product of the Clifford algebras of the summands. The purpose of the current paper is to present a purely set-theoretical version of the superalgebra tensor product which will be applicable equally to groups or to their non-associative analogues — quasigroups and loops. Our work is part of a project to make supersymmetry an effective tool for the study of combinatorial structures. Starting from group and quasigroup structures on four-element supersets, our superproduct unifies the construction of the eight-element quaternion and dihedral groups, further leading to a loop structure which hybridizes the two groups. All three of these loops share the same character table.

1. Introduction

Supersymmetry is an important concept in mathematical physics and the treatment of structures such as Clifford algebras [14]. When it is applied to linear spaces, it entails a direct sum decomposition with two *homogeneous* summands, respectively, described as *even* and *odd*. In a combinatorial or set-theoretical context, it becomes simpler: just a disjoint union decomposition (3.1) into two uniands,[1] again described as *even* and *odd*. A set decomposed in this way is called a *superset*.[2]

The even and odd uniands of a superset may be regarded as the pre-images of 0 and 1, respectively, in the domain of a *parity function* to $\mathbb{Z}/_2$. Whenever

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[1]Recall that *uniands* are the sets that are joined in a union of sets.

[2]This usage of the term "superset", which fits into the standard terminological framework of "supermathematics," is not to be confused with its use in the context of a comparison $X \subseteq Y$ between two sets, where X is the subset, and Y is sometimes described as the superset.

a general algebra structure is carried by the group $\mathbb{Z}/_2$, then a *superalgebra* (thus, for example, a *supergroup*) is defined as the domain of a parity function to $\mathbb{Z}/_2$ which is a homomorphism of the algebra structure. For example, the symmetric group may be taken as a supergroup whose even subgroup is the alternating group (Example 3.5). Supergroup structure offers new insights into the close relationship between the quaternion group Q_8 and the dihedral group D_4 (which share the same character table). For example, both may appear as the even subgroup in supergroup structures on the (complex) Pauli group of quantum information theory, the group that is generated by the Pauli matrices (Example 3.8).

The current paper investigates the nonassociative versions of supergroups, *superquasigroups* (Definition 3.9), along with their *multiplication supergroups* (Theorem 3.18). The superquasigroup definition is based on the interpretation of the cyclic group $\mathbb{Z}/_2$ as a(n associative and totally symmetric) quasigroup, where addition plays the roles of quasigroup multiplication, right division, and left division.

The usual tensor product of superalgebras involves positive and negative signs [7, III.3.9]. Thus, it cannot be applied in our combinatorial context. To overcome this problem, we introduce the new concept of a *signed superquasigroup*, along with the somewhat simpler notion of a *signed super*(*semi*)*group* (§4). The smallest signed supergroup is the cyclic group of order 2, both of whose elements are even (Lemma 4.4). Our analogue of the tensor product of superalgebras is the *signed* (*super*)*product*. It comes in two versions. A simpler version, for supersemigroups (Definition 4.10), is in formal agreement with the definition of the tensor product of superalgebras on homogeneous elements. On the other hand, because of the right and left divisions in quasigroups, the signed product of signed superquasigroups (Definition 4.13) is more refined. The signed cyclic group of order 2 from Lemma 4.4 acts as a unit for our signed superproducts. Theorem 4.15 and its corollary then show that signed superproducts of signed superquasigroups are signed superquasigroups, and that signed superproducts of signed supergroups are signed supergroups.

When the usual linear supersymmetry is applied to real Clifford algebras, the Clifford algebra of the direct sum of two real quadratic spaces emerges as the superalgebra tensor product $\widehat{\otimes}$ of the respective Clifford algebras of the summands [7, Th. III.3.10]. By Sylvester's Law of Inertia, a finite-dimensional real vector space with a non-degenerate quadratic form is a direct sum of two kinds of one-dimensional space, namely with x^2 or $-x^2$ as the quadratic form. The unit groups (in the sense, say, of the "groups associated with Clifford algebras" of [10]) of the respective Clifford algebras of these 1-dimensional spaces are the 4-element cyclic group $C_4 = \langle i \rangle$ and the 4-element Boolean group $B_4 = \langle -1 \rangle \times \langle -1 \rangle$ (a.k.a. the Klein Vierergruppe). It follows that the unit group of any finite-dimensional real Clifford algebra is obtained indirectly from these two groups by an iterated superalgebra tensor product. This process is now rendered completely combinatorial, and extended to quasigroups, by taking the signed superproduct $\widehat{\times}$ of signed superquasigroups.

In Section 4.5, certain building blocks for the combinatorial process are listed. The following *basic* signed superquasigroup structures on the superset $\{\pm 1\} \uplus \{\pm i\}$ come into consideration:

- The Boolean group B_4, with $i^2 = 1$;
- The cyclic group C_4, with $i^2 = -1$, as a subgroup of the circle group S^1 of complex numbers of unit modulus;
- The *right* superquasigroup $C_4^{/}$ whose quasigroup product is given by the complex scalar product $\langle x|y\rangle = x\,\overline{y}$;
- The *left* superquasigroup $C_4^{\backslash}$ whose quasigroup product is given by the complex scalar product $\langle x|y\rangle = \overline{x}\,y$.

By Galois theory, the latter two *chiral* superquasigroups are isomorphic under conjugation between the two square roots of -1. According to Theorem 3.18, their multiplication supergroups implement the dihedral group D_4 as a signed supergroup (Example 3.19).

Signed superproducts of these basic signed superquasigroups are worked out in Sections 4.6 and 4.7. They are all related in various ways to the quaternion group Q_8 and dihedral group D_4. In §4.6.1, the quaternion group emerges directly as the supersquare $C_4 \widehat{\times} C_4$ of the cyclic group. In §4.6.2, the dihedral group D_4 emerges directly as the supersquare $B_4 \widehat{\times} B_4$ of the Boolean group. This realization of D_4 as a supergroup differs from the multiplication supergroup $\mathrm{MLT}\, C_4^{/}$ that was seen in Example 3.19, which actually appears as the signed superproduct $B_4 \widehat{\times} C_4$ of the Boolean group with the cyclic group in §4.7.1. A signed superproduct is also involved in a signed superquasigroup which is exhibited in §4.7.2. The quasigroup is principally isotopic to a loop of order 8, the so-called *quatedral* loop. As our proposed name for the loop is meant to suggest, it is a hybridization of Q_8 with D_4. (In `Gap`, it appears innocuously as `NilpotentLoop(8,116)` — Remark 4.20.)

1.1. Plan of the paper

Section 2 provides needed background on magmas, quasigroups, and loops, including the multiplication groups of quasigroups. (For other topics not treated in Section 2, readers may consult [1, 11, 13].) Section 3 is devoted to general supersets, supergroups and superquasigroups, in conjunction with a treatment of the multiplication supergroups of superquasigroups. Section 4 specializes to signed supergroups and superquasigroups. The key tool of the paper, the signed superproduct, is presented as the combinatorial incarnation of the tensor product of linear superalgebras.

1.1.1. *Category theory formulation.* Most of the definitions and results of this paper may be couched naturally in categorical language — see Remark 3.2(b) — generally becoming more transparent in the process. Nevertheless, we will maintain a combinatorial approach for the benefit of readers who may not be comfortable with categorical thinking.

1.1.2. *Notational remark.* Algebraic or diagrammatic notation, in which the functions follow their arguments (either on the line or as a superfix), is taken as the default option throughout the paper. This convention, followed by classics of non-associative algebra such as [1], and also employed in the `Gap` package, mitigates the inevitable proliferation of brackets, and enables formulas to be read in natural order from left to right without backtracking or threading.

2. Quasigroups

2.1. Magmas and combinatorial quasigroups

2.1.1. *Magmas.*

Definition 2.1. A *magma* M or $(M, \cdot)$ is a (possibly empty) set M that is equipped with a binary operation

$$M \times M \to M; (m, m') \mapsto m \cdot m' \tag{2.1}$$

which by default may be described as *multiplication.*

Remark 2.2. (a) It is often convenient to denote the product $m \cdot m'$ in (2.1) by juxtaposition, as mm'.

(b) Magmas are sometimes described as *binars*, or in Ore's terminology as *groupoids* (now abrogated by its alternative use in category theory).

2.1.2. *Right and left multiplications.* Currying of the binary operation in a magma yields families of unary operations, parametrized by the elements of the (underlying set of the) magma.

Definition 2.3. Consider an element q of a magma $(M, *)$.

(a) The function

$$R_*(q) \colon M \to M; x \mapsto xq \tag{2.2}$$

is known as *right multiplication* by q. Thus $xR_*(q) = x * q$.

(b) The function

$$L_*(q) \colon M \to M; x \mapsto qx \tag{2.3}$$

is known as *left multiplication* by q. Thus $xL_*(q) = q * x$.

The right and left multiplications of a magma $(M, \cdot)$, with multiplication denoted simply by juxtaposition, may be written simply as $R(q)$ and $L(q)$.

Definition 2.4. An element e of a magma $(M, *)$ is an *identity element* if $R_*(e) \colon M \to M$ and $L_*(e) \colon M \to M$ fix each element of M.

Lemma 2.5. *A magma has at most one identity element.*

Proof. Note that $e = eR_*(f) = e * f = fL_*(e) = f$ if e and f are identity elements. □

2.1.3. *Combinatorial quasigroups.*

Definition 2.6. A (*combinatorial*) *quasigroup* Q or $(Q,\cdot)$ is a magma where the equation $x_1x_2 = x_3$ has a unique solution x_k in Q, with $k \in \{1,2,3\} \smallsetminus \{i,j\}$, for each two-element subset $\{i,j\}$ of $\{1,2,3\}$ and for each choice x_i, x_j of elements of Q.

Lemma 2.7. *Let* $(Q,*)$ *be a combinatorial quasigroup.*

(a) *For each* $q \in Q$*, the right multiplication* (2.2) *is bijective.*
(b) *For each* $q \in Q$*, the left multiplication* (2.3) *is bijective.*

Proof. It suffices to prove (a), as the proof of (b) is dual (left/right switch). If Q is empty, the result is vacuously true. So, consider the case where Q is nonempty.

For the injectivity, suppose $x'R_*(q) = x''R_*(q)$ for $x', x'' \in Q$. Then both x' and x'' are solutions x to the equation $x * q = x' * q$. The uniqueness condition from Definition 2.6 then implies that $x' = x''$.

For the surjectivity, consider an element y of Q. The existence condition from Definition 2.6 implies that there is a solution x in Q to the equation $x * q = y$. Then $y = xR_*(q)$ lies in the image of $R_*(q)\colon Q \to Q$. □

Lemma 2.7 has a converse.

Proposition 2.8. *A magma is a* (*combinatorial*) *quasigroup if and only if all its left and right multiplications are bijective.*

Proof. The forward (only if) direction is just Lemma 2.7. Conversely, suppose that the multiplications in a magma $(M,\cdot)$ are all bijective. We then have the following scheme of solutions to the various equations of Definition 2.6:

$$x_1 = x_3R(x_2)^{-1}, \quad x_2 = x_3L(x_1)^{-1},$$
$$x_3 = x_1R(x_2) = x_2L(x_1),$$

all of which are unique. □

2.2. Equational quasigroups and loops

Definition 2.9. A(n *equational*) *quasigroup* Q or $(Q,*,/,\backslash)$ is a set Q equipped with three magma structures:

- *multiplication* $(Q,*)$;
- *right division* $(Q,/)$;
- *left division* $(Q,\backslash)$,

such that the labelled identities

$$\mathsf{w}\backslash(\mathsf{w}*\mathsf{v}) \overset{\text{(IL)}}{=} \mathsf{v} \overset{\text{(IR)}}{=} (\mathsf{v}*\mathsf{w})/\mathsf{w}, \tag{2.4}$$

$$\mathsf{w}*(\mathsf{w}\backslash\mathsf{v}) \overset{\text{(SL)}}{=} \mathsf{v} \overset{\text{(SR)}}{=} (\mathsf{v}/\mathsf{w})*\mathsf{w} \tag{2.5}$$

hold.

Remark 2.10. Note the symmetry of (2.4) and (2.5) about the axis through the central elements v.

Proposition 2.11. *Consider a* (*possibly empty*) *set* Q*.*

(a) *Suppose that* $(Q, *)$ *is a combinatorial quasigroup. Then its magma structure augments to the structure* $(Q, *, /, \backslash)$ *of an equational quasigroup.*

(b) *If* $(Q, *, /, \backslash)$ *is an equational quasigroup, then its magma reduct* $(Q, *)$ *is a combinatorial quasigroup.*

Proof. (a) By Lemma 2.7, the multiplications of $(Q, *)$ are bijective. Now, for elements x, y of Q, define

$$x/y = xR(y)^{-1} \quad \text{and} \quad x\backslash y = yL(x)^{-1}.$$

The identities of (2.4) and (2.5) are then immediate.

(b) The identity (IR) expresses the injectivity of the right multiplication $R_*(\mathsf{w})$ in $(Q, *)$. The identity (SR) expresses the surjectivity of the right multiplication $R_*(\mathsf{w})$. Thus the right multiplications of $(Q, *)$ are bijective. Symmetrically, the left multiplications of $(Q, *)$ are bijective (compare Remark 2.10). Then, Proposition 2.8 shows that the magma $(Q, *)$ is a combinatorial quasigroup. □

By virtue of Proposition 2.11, one may normally omit the qualifications "combinatorial" and "equational" when referring to quasigroups, and also when referring to the objects of this final definition:

Definition 2.12. A quasigroup is a *loop* if it has an identity element.

2.3. Multiplication groups

According to Proposition 2.8, the right and left multiplications of a quasigroup $(Q, \cdot)$ lie inside the group $Q!$ of permutations of the underlying set Q, i.e., the group of bijective mappings from Q to Q.

Definition 2.13. Let $(Q, \cdot)$ be a quasigroup. Then its *multiplication group* $\operatorname{Mlt} Q$ or $\operatorname{Mlt}(Q, \cdot)$ is the subgroup

$$\langle R(q), L(q) \mid q \in Q \rangle_{Q!} \tag{2.6}$$

of $Q!$ generated by all the right and left multiplications of $(Q, \cdot)$.

In certain cases, the generation process implicit in (2.6) produces no new elements.

Proposition 2.14. *For* $2 < n$*, consider the quasigroup* $(\mathbb{Z}/_n, -)$ *of residues modulo* n *under subtraction. Then its multiplication group is the dihedral group*

$$D_n = \{\, R(n), L(n) \mid n \in \mathbb{Z}/_n \,\} \tag{2.7}$$

of order $2n$*.*

Proof. The action diagram

(2.8)
$$\begin{tikzcd}
(q_1 - x) - q_2 \arrow[r, equal] & (q_1 - q_2) - x & & x - (q_1 + q_2) \arrow[r, equal] & (x - q_1) - q_2 \\
q_1 - x \arrow[u, mapsto, "R(q_2)"'] \arrow[d, mapsto, "L(q_2)"] & & x \arrow[ll, mapsto, "L(q_1)"'] \arrow[ul, mapsto, "L(q_1 - q_2)"'] \arrow[dl, mapsto, "R(q_1 - q_2)"] \arrow[ur, mapsto, "R(q_1 + q_2)"] \arrow[dr, mapsto, "L(q_1 + q_2)"'] \arrow[rr, mapsto, "R(q_1)"] & & x - q_1 \arrow[u, mapsto, "R(q_2)"] \arrow[d, mapsto, "L(q_2)"'] \\
q_2 - (q_1 - x) \arrow[r, equal] & x - (q_1 - q_2) & & (q_1 + q_2) - x \arrow[r, equal] & q_2 - (x - q_1)
\end{tikzcd}$$

for the right and left multiplications in $(\mathbb{Z}/_n, -)$ exhibits the closure of the set on the right hand side of (2.7) under composition. While the equation (2.7) is actually best seen as a direct, purely algebraic *definition* of D_n, one may note the relations

$$\begin{aligned}
R(1)^n &\overset{(2.8)}{=} R\big(\overbrace{1 + 1 + \cdots + 1}^{n}\big) = R(n) = R(0) = 1, \\
L(1)^2 &\overset{(2.8)}{=} R(1 - 1) = R(0) = 1, \\
L(1)R(1)L(1) &\overset{(2.8)}{=} L(1 - 1)L(1) = L(0)L(1) \overset{(2.8)}{=} R(0 - 1) = R(1)^{-1}
\end{aligned}$$

which allow the correlation $R(1) \mapsto r, L(1) \mapsto s$ with the familiar presentation $\langle r, s \,|\, r^n = s^2 = 1,\ srs = r^{-1} \rangle$ of D_n. $\square$

3. Superquasigroups

3.1. Supersets and supermagmas

3.1.1. *Supersets and parity.*

Definition 3.1. Consider a set S.

(a) The set S becomes a *superset* when equipped with a specified disjoint union decomposition

$$S = S_0 \uplus S_1 \tag{3.1}$$

in which the respective uniands are identified as the *even part* S_0 and *odd part* S_1.

(b) As a superset according to (3.1), S has a *supercardinality* or *superorder* that is defined as $\operatorname{scrd} V = \#S_0|\#S_1$. Here, $\#X$ is used for the usual cardinality $|X|$ of a set X, to avoid an excess of vertical strokes.

(c) Elements x of S_0 are described as having *even parity*: $|x| = 0$.

(d) Elements x of S_1 are described as having *odd parity*: $|x| = 1$.

(e) If T_0, T_1 are respective subsets of the even and odd uniands S_0, S_1 from (3.1), then $T = T_0 \uplus T_1$ is said to be a *supersubset* or *subsuperset* of S.

Remark 3.2. (a) Equivalently, one may define a superset S to be the domain of a *parity function*

$$p_S \colon S \to \mathbb{Z}/_2; x \mapsto |x|. \tag{3.2}$$

Then, we have S_r as the inverse image $p_S^{-1}\{ r \}$ for $r = 0, 1$. Note that this is consistent with Definition 3.1(c), (d).

(b) A categorical approach to supersets (from which we refrain in this paper) would start from (3.2), regarding it as an object of the slice category of sets over $\mathbb{Z}/_2$.

3.1.2. *Supermagmas and supergroups.*

Definition 3.3. Suppose that (the underlying set of) a magma S is a superset $S = S_0 \uplus S_1$. Suppose that whenever x and y are elements of S, then their product $x \cdot y$ has

$$|x \cdot y| = |x| + |y| \tag{3.3}$$

with addition modulo 2. Then S is said to be a *supermagma*.

Remark 3.4. (a) The condition (3.3) of Definition 3.3 says that the parity function (3.2) is a magma homomorphism to the additive group of residues modulo 2. Thus, within a supermagma, the even part forms a submagma.

(b) As applied in Definition 3.3 to general magmas, the "super-" prefix will equally be applied to some special classes of magmas, such as *supersemigroups*, *supermonoids*, and *supergroups*. Note, however, that the term "supergroup" here should not be confused with its usage in the algebraic group context of [16], nor with the notion of a "finite super-group" as in [2, §2.2]. Nevertheless, in connection with the latter, compare Remark 4.6.

3.1.3. *Examples of supergroups.*

Example 3.5. Consider the symmetric group $S_n = X!$ on the set $X = \{ 0, \dots, n-1 \}$ of finite cardinality n. It becomes a supergroup $S_n = A_n \uplus A_n(0\ 1)$. Its supercardinality is

$$r|s = \begin{cases} 1|0 & \text{if } n \leq 1; \\ (n!/2)|(n!/2) & \text{if } n > 1. \end{cases}$$

The even subgroup is the alternating group A_n.

Example 3.6. Inside the algebra $\mathbb{C}$ of complex numbers, the cyclic group C_4 generated by i becomes a supergroup $C_4 = \{ \pm 1 \} \uplus \{ \pm i \}$ of supercardinality $2|2$.

Example 3.7. Under each of the decompositions

$$\begin{aligned} &\text{(i)} && Q_8 = \{ \pm 1, \pm i \} \uplus \{ \pm j, \pm k \}, \\ &\text{(j)} && Q_8 = \{ \pm 1, \pm j \} \uplus \{ \pm k, \pm i \}, \\ &\text{(k)} && Q_8 = \{ \pm 1, \pm k \} \uplus \{ \pm i, \pm j \}, \end{aligned}$$

the quaternion group Q_8 becomes a supergroup of supercardinality $4|4$.

Example 3.8. In the algebra $\mathbb{C}_2^2$ of 2×2 complex matrices, consider the *Pauli matrices*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.4}$$

along with the 2×2 identity matrix I [9, §2.1.3]. The (*complex*) *Pauli group* consists of the set

$$G_1 = \{ \pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ \} \tag{3.5}$$

of complex 2×2 matrices [9, (10.81)]. (The suffix 1 in G_1 refers to the fact that (3.5) is the Pauli group for a single qubit in quantum information theory.) The matrix equations

$$\begin{cases} -YX = XY = iZ, \\ -ZY = YZ = iX, \\ -XZ = ZX = iY, \\ X^2 = Y^2 = Z^2 = 1 \end{cases} \tag{3.6}$$

summarize the multiplication in G_1. Note that the elements of the Pauli group form a real spanning set for the 8-dimensional real algebra $\mathbb{C}_2^2$ of complex 2×2 matrices.

The Pauli group supports supergroup structures

$$\{ \pm I, \pm iX, \pm iY, \pm iZ \} \uplus \{ \pm X, \pm Y, \pm Z, \pm iI \} \text{ and} \tag{3.7}$$

$$\{ \pm I, \pm iX, \pm Y, \pm Z \} \uplus \{ \pm X, \pm iY, \pm iZ, \pm iI \} . \tag{3.8}$$

The matrix equations (3.6) show that the even subgroup of (3.7) is Q_8. On the other hand, the even subgroup of (3.8) is D_4. (The latter statement is verified by noting that iX, of order 4, is inverted when conjugated by the involution Y.)

3.2. Superquasigroups

In §2.2, it was noted that a quasigroup Q may be defined as a set which carries three compatible magma structures: multiplication $(Q, \cdot)$, right division $(Q, /)$, and left division $(Q, \backslash)$. In particular, a group is Boolean (or in other words, elementary abelian of exponent 2) if and only if all three of these magma structures coincide.

Definition 3.9. A *superquasigroup* Q is a set Q carrying a superset structure whose parity function

$$p_Q \colon Q \to \mathbb{Z}/_2; x \mapsto |x|$$

is a quasigroup homomorphism from Q to the Boolean group $(\mathbb{Z}/_2, +)$. In particular, the domain Q of the quasigroup homomorphism p_Q is a quasigroup.

Proposition 3.10. *Let Q be a superquasigroup of supercardinality $m|n$. Then $n = 0$ or $n = m$.*

Proof. Suppose that $n > 0$, so the odd part Q_1 is nonempty. Let q be an odd element. The right multiplication $R(q)$ is injective, with $Q_0R(q) \subseteq Q_1$ and $Q_1R(q) \subseteq Q_0$. Then $|Q_0| = |Q_0R(q)| \leq |Q_1|$ and $|Q_1| = |Q_1R(q)| \leq |Q_0|$, whence $n = |Q_1| = |Q_0| = m$. □

Corollary 3.11. *If the order of a finite superquasigroup is odd, then there are no odd elements.*

Remark 3.12. Proposition 3.10 may be viewed as an easy special case of the general result that the classes of a quasigroup congruence are equipollent (compare [11, Exercise 3.10.7]).

3.3. Superproducts and superfunctions

3.3.1. *Superproducts.*

Definition 3.13. Suppose that $T = T_0 \uplus T_1$ and $T' = T'_0 \uplus T'_1$ are supersets, with respective supercardinalities $r|s$ and $r'|s'$. Then the superset

$$T \widehat{\times} T' = [(T_0 \times T'_0) \uplus (T_1 \times T'_1)] \uplus [(T_0 \times T'_1) \uplus (T_1 \times T'_0)] \tag{3.9}$$

of supercardinality $(rr' + ss')|(rs' + sr')$ is the *superproduct* of the supersets T and T'.

Lemma 3.14. *In the context of Definition 3.13, consider $t \in T$ and $t' \in T'$. Then the addition*

$$|(t, t')| = |t| + |t'|$$

in $\mathbb{Z}/_2$ recovers the partition (3.9)*. In the language of Remark 3.2*(a)*, the parity function of the superproduct takes the sum of the respective parity functions of the components.*

Remark 3.15. In connection with Remark 3.2(b), recall that products in slice categories are given by pullbacks. Thus, it may be noted that the even part of (3.9) is the product of the parity functions $T \to \mathbb{Z}/_2$ and $T' \to \mathbb{Z}/_2$ in the slice category of sets over $\mathbb{Z}/_2$.

3.3.2. *Superfunctions between supersets.*

Definition 3.16. Suppose that $f : T \to T'; t \mapsto t^f$ is a function from (the underlying set of) a superset T to (the underlying set of) a superset T'. Consider its graph $\mathrm{Gr}\, f = \{ (t, t^f) \mid t \in T \}$ as (the underlying set of) a supersubset of the superproduct $T \widehat{\times} T'$ of T with T'.

(a) The *even part* f_0 of f is the function $f_0 : T \to T'$ defined by its graph $\mathrm{Gr}\, f_0 = \big(\mathrm{Gr}\, f\big)_0$.
(b) The *odd part* f_1 of f is the function $f_1 : T \to T'$ defined by its graph $\mathrm{Gr}\, f_1 = \big(\mathrm{Gr}\, f\big)_1$.
(c) The decomposition of (the graph of) f into the disjoint parts

$$f_0|_{T_0} : T_0 \to T'_0; \quad f_0|_{T_1} : T_1 \to T'_1; \tag{3.10}$$

$$f_1|_{T_0} : T_0 \to T'_1; \quad f_1|_{T_1} : T_1 \to T'_0 \tag{3.11}$$

makes f a *superfunction.*

(d) The function f is *even* if $\operatorname{Gr} f_1$ is empty.

(e) The function f is *odd* if $\operatorname{Gr} f_0$ is empty.

3.4. Multiplication supergroups

The multiplication group of (the underlying quasigroup of) a superquasigroup becomes a supergroup.

Lemma 3.17. *Suppose that $Q = Q_0 \uplus Q_1$ is a superquasigroup.*

(a) *If q is an element of Q_0, then $R(q)\colon Q \to Q$ and $L(q)\colon Q \to Q$ are even functions.*

(b) *If q is an element of Q_1, then $R(q)\colon Q \to Q$ and $L(q)\colon Q \to Q$ are odd functions.*

(c) *The set $\{ R(q), L(q) \mid q \in Q \}$ has a decomposition*

$$\{ R(q), L(q) \mid q \in Q \} = \{ R(q), L(q) \mid q \in Q_0 \} \uplus \{ R(q), L(q) \mid q \in Q_1 \}$$

as a superset.

Proof. (a) Suppose that q is even. Consider an element x of Q.

(i) If x is even, then xq and qx are even. Thus

$$R(q)\colon Q_0 \to Q_0 \quad \text{and} \quad L(q)\colon Q_0 \to Q_0 .$$

(ii) If x is odd, then xq and qx are odd. Thus

$$R(q)\colon Q_1 \to Q_1 \quad \text{and} \quad L(q)\colon Q_1 \to Q_1 .$$

By comparison with (3.10) from Definition 3.16, it is then apparent that $R(q)$ and $L(q)$ are even functions.

(b) Suppose that q is odd. Consider an element x of Q.

(i) If x is even, then xq and qx are odd. Thus

$$R(q)\colon Q_0 \to Q_1 \quad \text{and} \quad L(q)\colon Q_0 \to Q_1 .$$

(ii) If x is odd, then xq and qx are even. Thus

$$R(q)\colon Q_1 \to Q_0 \quad \text{and} \quad L(q)\colon Q_1 \to Q_0 .$$

By comparison with (3.11) from Definition 3.16, it is then apparent that $R(q)$ and $L(q)$ are odd functions.

(c) now follows from (a) and (b). □

Theorem 3.18. *Suppose that $Q = Q_0 \uplus Q_1$ is a superquasigroup. Then the multiplication group $\operatorname{Mlt} Q$ of the quasigroup Q becomes a supergroup*

$$\operatorname{MLT} Q = (\operatorname{MLT} Q)_0 \uplus (\operatorname{MLT} Q)_1 , \tag{3.12}$$

where $(\operatorname{MLT} Q)_0$ is the set of even functions in $\operatorname{Mlt} Q$, while $(\operatorname{MLT} Q)_1$ is the set of odd functions in $\operatorname{Mlt} Q$.

Proof. We show that the sets occurring in the decomposition (3.12) account for all the elements of the multiplication group. In other words, for each function $f\colon (Q_0 \uplus Q_1) \to (Q_0 \uplus Q_1)$ in the multiplication group, either $\mathrm{Gr}\, f_0$ is empty or $\mathrm{Gr}\, f_1$ is empty. If Q_1 is empty, then the only option is $f|_{Q_0}\colon Q_0 \to Q_0$ from (3.10), so $\mathrm{Mlt}\, Q = (\mathrm{MLT}\, Q)_0$ and the case is closed.

Now suppose that the parity homomorphism $p\colon Q \to (\mathbb{Z}/_2, +)$ is surjective. Thus, there is a well-defined surjective homomorphism

$$\mathrm{Mlt}\, p\colon \mathrm{Mlt}\, Q \to \mathrm{Mlt}\,(\mathbb{Z}/_2, +) \cong (\mathbb{Z}/_2, +)$$

of groups that extends the surjective parity map

$$\{\, R(q), L(q) \mid q \in Q \,\} \to \mathbb{Z}/_2; R(q) \mapsto |q|, L(q) \mapsto |q|$$

of the superset

$$\{\, R(q), L(q) \mid q \in Q \,\} = \{\, R(q), L(q) \mid q \in Q_0 \,\} \uplus \{\, R(q), L(q) \mid q \in Q_1 \,\}$$

from Lemma 3.17(c) [11, (2.12)]. Here, $(\mathrm{MLT}\, Q)_r = (\mathrm{Mlt}\, p)^{-1} \{\, r \,\}$ for $r \in \mathbb{Z}/_2$, completing the verification of (3.12) for this case. □

Example 3.19. Consider the superquasigroup $C_4^{/} = \{\, \pm 1 \,\} \uplus \{\, \pm i \,\}$ equipped with multiplication $x * y = x\overline{y}$ — c.f. Example 3.6 above and Definition 4.17(d) below. Abstractly, its multiplication group is D_4: apply Proposition 2.14 and the quasigroup isomorphism

$$(\mathbb{Z}/_4, -) \to C_4^{/}; r \mapsto \exp(2\pi i r/4).$$

The multiplication supergroup of the superquasigroup $C_4^{/}$ is

$$\mathrm{MLT}(C_4^{/}, *) = \{\, R(\pm 1), L(\pm 1) \,\} \uplus \{\, R(\pm i), L(\pm i) \,\},$$

which is isomorphic to the supergroup $D_4 = \{\, \pm 1, \pm xy \,\} \uplus \{\, \pm x, \pm y \,\}$ discussed in §4.7.1 below.

4. Signed superquasigroups and superproducts

4.1. Signed supermagmas

The supermagmas that were presented in the Examples 3.6 and 3.7 have a special form.

Definition 4.1. Let $S = S_0 \uplus S_1$ be a supermagma of supercardinality $2k|2l$. Suppose that S has a subsuperset $T = T_0 \uplus T_1$ of supercardinality $k|l$ whose (even or odd) elements are written as $x = +x = (+1)x$, such that each element of S is of the form $+x$ or $-x = (-1)x$, with parities $|+x| = |-x|$. Further, suppose that

$$(\sigma x)(\sigma' x') = (\sigma\sigma')xx' \tag{4.1}$$

in S, for $\sigma, \sigma' \in \{\, \pm 1 \,\}$ and $x, x' \in T$. Then the supermagma

$$S = S_0 \uplus S_1 \text{ with } S_0 = \{\, \pm x \mid x \in T_0 \,\} \text{ and } S_1 = \{\, \pm x \mid x \in T_1 \,\} \tag{4.2}$$

is said to be *signed*, *with respect to* the subsuperset T as a *transversal*.

Lemma 4.2. *In Definition 4.1, the supermagma S is specified by its transversal superset T as the superset* (4.2) *that is equipped with the multiplication* (4.1).

Lemma 4.3. *The smallest signed supermagma is $\emptyset = \emptyset \uplus \emptyset$ — the empty superquasigroup — with supercardinality $0|0$ and transversal $\emptyset = \emptyset \uplus \emptyset$.*

Lemma 4.4. *The smallest signed supermonoid or supergroup is the cyclic group $C_2 = \{ \pm 1 \} \uplus \emptyset$, with supercardinality $2|0$ and transversals $\{ 1 \} \uplus \emptyset$ or $\{ -1 \} \uplus \emptyset$.*

Proposition 4.5. *Suppose that $G = G_0 \oplus G_1$ is a finite supergroup. Suppose that the even element z is a central involution in the underlying group G. Let $T = (T \cap G_0) \uplus (T \cap G_1)$ be a transversal to the subgroup $\{ 1, z \}$ of G. For each element t of T, set $t = +t$ and $zt = -t$. Then the supergroup G is a signed supergroup with respect to the transversal supersubset T.*

Remark 4.6. In the context of Proposition 4.5, the pair (G, z) is a "finite supergroup" in the sense of [2, §2.2].

The following definition provides a convenient way of obtaining new signed supermagmas (compare §4.7.2 below).

Definition 4.7. Suppose that $S = S_0 \uplus S_1$ is a signed supermagma with transversal $T = T_0 \uplus T_1$, as in Definition 4.1. Consider a pair (t_l, t_r) of elements of T. Then the (t_l, t_r)-*modification* $S_{(t_l,t_r)}$ of S is the supermagma on the superset $S = S_0 \uplus S_1$ with product

$$(\sigma x)(\sigma' x') = \begin{cases} (\sigma\sigma')xx' & \text{if } (x, x') \neq (t_l, t_r); \\ -(\sigma\sigma')t_l t_r & \text{if } (x, x') = (t_l, t_r) \end{cases} \tag{4.3}$$

for $\sigma, \sigma' \in \{ \pm 1 \}$ and $x, x' \in T$.

4.2. Supercommutativity and supercenters

Definition 4.8. Let S be a signed super(semi)group.

(a) Elements x, y of S *supercommute* if

$$x \cdot y = (-1)^{|x| \cdot |y|} y \cdot x. \tag{4.4}$$

(b) The *supercenter* of S is defined to be the set of elements of S which supercommute with every element of S.

Lemma 4.9. *Let S be a signed supergroup.*

(a) *Even elements of its supercenter are central.*
(b) *Even elements of its center are supercentral.*

Proof. (a) Suppose that z is an even element of the supercenter. Then by (4.4), we have

$$x \cdot z = (-1)^{|x| \cdot 0} z \cdot x = z \cdot x \tag{4.5}$$

for each element x of S.

(b) Suppose that z is an even element of the center. Then the equalities of (4.5) show that z is supercentral. □

4.3. Signed superproducts of supersemigroups

Definition 4.10. Suppose that $S = S_0 \uplus S_1$ and $S' = S'_0 \uplus S'_1$ are signed supersemigroups, with respective transversals $T = T_0 \uplus T_1$ and $T' = T'_0 \uplus T'_1$. Then their *signed* (*super*)*product* $S \widehat{\times} S'$ is specified according to Lemma 4.2 as the signed supermagma with transversal superset $T \widehat{\times} T'$ as a domain for the multiplication

$$(t \otimes t')(u \otimes u') = (-1)^{|t'|\cdot|u|}(tu \otimes t'u') \tag{4.6}$$

defined in terms of the componentwise products tu in S and $t'u'$ in S'. Here, ordered pairs $(t, t') \in T \times T'$ appearing in the uniands of the right hand side of (3.9) are written as tensors $t \otimes t'$, subject to the identifications

$$(-t) \otimes t' = t \otimes (-t') = -(t \otimes t') \tag{4.7}$$

in the signed supermagma $S \widehat{\times} S'$.

The examples of §4.6.1 and §4.6.2 below provide illustrations of how the signed product of signed (semi)groups actually works.

Proposition 4.11. *Consider the context of Definition 4.10. Then the signed product $S \widehat{\times} S'$ of two signed supersemigroups* (*supermonoids*) *S, S' is again a signed supersemigroup* (*supermonoid*).

Proof. Checking associativity within the signed product is a straightforward computation using (4.6). Indeed, for elements t, u, v of S and t', u', v' of S', both associations of $(t \otimes t')(u \otimes u')(v \otimes v')$ reduce to

$$(-1)^{|t'|\cdot|u|+|t'|\cdot|v|+|u'|\cdot|v|}(tuv \otimes t'u'v'). \tag{4.8}$$

In the superproduct of signed supermonoids, selection of the identity element is given componentwise as $1 \otimes 1$. □

Remark 4.12. The exponent

$$|t'| \cdot |u| + |t'| \cdot |v| + |u'| \cdot |v| \tag{4.9}$$

of (-1) in (4.8) may be "explained" as follows: To perform the reduction

$$(t \otimes t')(u \otimes u')(v \otimes v') \to (-1)^{|t'|\cdot|u|+|t'|\cdot|v|+|u'|\cdot|v|}(tuv \otimes t'u'v'),$$

the elements t' and u' have to be moved across to the right to join v' in the final product. During the course of the move, t' has to cross past both u and v, thereby generating the first two summands of (4.9). The third summand is generated as u' is moved past v.

4.4. Signed superproducts of superquasigroups

Signed superproducts of signed superquasigroups are more involved than signed superproducts of signed supersemigroups.

Definition 4.13. Suppose that $S = S_0 \uplus S_1$ and $S' = S_0' \uplus S_1'$ are signed superquasigroups, with respective transversals $T = T_0 \uplus T_1$ and $T' = T_0' \uplus T_1'$. Then their *signed (super)product* $S \widehat{\times} S'$ is specified according to Lemma 4.2 as the signed superquasigroup with transversal superset $T \widehat{\times} T'$ as a domain for the operations

$$(t \otimes t') \cdot (u \otimes u') = (-1)^{|t'|\cdot|u|}(tu \otimes t'u'), \tag{4.10}$$

$$(t \otimes t')/(u \otimes u') = (-1)^{(|t'|+|u'|)\cdot|u|}(t/u \otimes t'/u'), \tag{4.11}$$

$$(t \otimes t')\backslash(u \otimes u') = (-1)^{|t'|\cdot(|t|+|u|)}(t\backslash u \otimes t'\backslash u') \tag{4.12}$$

defined in terms of the componentwise operations in S and S'. Here, as before, the ordered pairs $(t, t') \in T \times T'$ that appear in the uniands of the right hand side of (3.9) are written as tensors $t \otimes t'$, subject to the identifications

$$(-t) \otimes t' = t \otimes (-t') = -(t \otimes t') \tag{4.13}$$

in the signed superquasigroup $S \widehat{\times} S'$ that are sometimes described as the *tensor relations.*

Remark 4.14. As in Remark 4.12, the left division formula (4.12) may be "explained" as follows, based on the case where the quasigroups are groups and $t\backslash u = t^{-1}u$. We have

$$\begin{aligned}(t \otimes t')\backslash(u \otimes u') &= (t \otimes t')^{-1} \cdot (u \otimes u') \overset{(1)}{=} \left(t'^{-1} \otimes t^{-1}\right) \cdot (u \otimes u') \\ &\overset{(2)}{=} (-1)^{|t'^{-1}|\cdot|t^{-1}|+|t'^{-1}|\cdot|u|} \left(t^{-1}u \otimes t'^{-1}u'\right) \\ &= (-1)^{|t'|\cdot(|t|+|u|)}(t\backslash u \otimes t'\backslash u'),\end{aligned}$$

the first summand in the exponent of -1 on the right hand side of (2) being acquired as t'^{-1} is moved past t^{-1} to join u', and the second as it is moved past u. The manipulation across (1) is motivated by the suggestion that the inverse of a (tensor) product is the (tensor) product of the inverses taken in the reverse order.

Theorem 4.15. *Consider the context of Definition 4.13. The signed product $S \widehat{\times} S'$ of two signed superquasigroups S, S' is again a signed superquasigroup.*

Proof. To show that $S \widehat{\times} S'$ is a quasigroup, the identities for an equational quasigroup will be verified. By quasigroup conjugacy — the symmetry of the theory of quasigroups [11, §1.3], it suffices to consider

$$\begin{aligned}&[(t \otimes t')/(u \otimes u')] \cdot (u \otimes u') \\ \overset{(4.11)}{=}\ &(-1)^{(|t'|+|u'|)\cdot|u|}(t/u \otimes t'/u') \cdot (u \otimes u')\end{aligned}$$

$$\overset{(4.10)}{=} (-1)^{(|t'|+|u'|)\cdot|u|}(-1)^{|t'/u'|\cdot|u|}\big[(t/u)\cdot u \otimes (t'/u')\cdot u'\big]$$
$$\overset{(3.3)}{=} (-1)^{(|t'|+|u'|)\cdot|u|}(-1)^{(|t'|+|u'|)\cdot|u|}(t\otimes t') = t\otimes t'$$

as a representative case. □

Corollary 4.16. *With inversion given by*

$$(t\otimes u)^{-1} = (-1)^{|t|\cdot|u|}(t^{-1}\otimes u^{-1}),$$

the signed product $S\widehat{\times}S'$ of two signed supergroups S, S' is a signed supergroup.

Proof. Combine Theorem 4.15 with Proposition 4.11. Note that

$$(t\otimes u)^{-1} = (1\otimes 1)/(t\otimes u) \overset{(4.11)}{=} (-1)^{(|1|+|u|)\cdot|t|}(1/t\otimes 1/u)$$
$$= (-1)^{|u|\cdot|t|}(t^{-1}\otimes u^{-1})$$

yields the inversion formula. □

4.5. Basic signed superquasigroups

Definition 4.17. (a) The *unit basic* super(quasi)group $C_2 = \{\pm 1\} \uplus \emptyset$ is the signed supergroup with multiplication table

	1
1	1

on its transversal $\{1\}\uplus\emptyset$. Compare Lemma 4.4.

(b) The *Boolean basic* super(quasi)group $B_4 = \{\pm 1\}\uplus\{\pm i\}$ is the signed supergroup with multiplication table

	1	i
1	1	i
i	i	1

on the transversal $\{1\}\uplus\{i\}$.

(c) The *cyclic basic* super(quasi)group $C_4 = \{\pm 1\}\uplus\{\pm i\}$ is the signed supergroup with multiplication table

	1	i
1	1	i
i	i	-1

on the transversal $\{1\}\uplus\{i\}$.

(d) The *right basic* superquasigroup $C_4^{/} = \{\pm 1\}\uplus\{\pm i\}$ is defined as the signed superquasigroup with multiplication table

	1	i
1	1	$-i$
i	i	1

on the transversal $\{1\}\uplus\{i\}$.

(e) The *left basic* superquasigroup $C_4^{\backslash} = \{\pm 1\} \uplus \{\pm i\}$ is defined as the signed superquasigroup with multiplication table

	1	i
1	1	i
i	$-i$	1

on the transversal $\{1\} \uplus \{i\}$.

(f) The *twisted Boolean basic* superquasigroup $\widetilde{B_4} = \{\pm 1\} \uplus \{\pm i\}$ is the signed superquasigroup with multiplication table

	1	i
1	1	$-i$
i	$-i$	1

on the transversal $\{1\} \uplus \{i\}$.

(g) The right and left basic superquasigroups are described as being *chiral*.

The name of the unit basic supergroup from Definition 4.17(a) comes from the following observation.

Proposition 4.18. *There are isomorphisms $C_2 \widehat{\times} Q \cong Q \cong Q \widehat{\times} C_2$ for any signed superquasigroup Q.*

Proof. The respective isomorphisms are given by $1 \otimes q \mapsto q \mapsto q \otimes 1$ for each element q of Q. □

4.6. Supersquares of basic superquasigroups

In this section, the functioning of the signed superproduct construction is demonstrated by taking the signed supersquares of three representative basic signed superquasigroups: cyclic, Boolean, and chiral.

4.6.1. *Cyclic times cyclic: the quaternion group.* The quaternion group Q_8 emerges from the supersquare $C_4 \widehat{\times} C_4$ of the cyclic basic supergroup C_4 and its transversal $T = \{1\} \uplus \{i\}$, using the following multiplication table on the supersquare $T \widehat{\times} T$ of the transversal T:

	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$1 \otimes 1$	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$i \otimes i$	$i \otimes i$	$-(-1 \otimes -1)$	$i \otimes -1$	$-(-1 \otimes i)$
$1 \otimes i$	$1 \otimes i$	$-(i \otimes -1)$	$1 \otimes -1$	$-(i \otimes i)$
$i \otimes 1$	$i \otimes 1$	$-1 \otimes i$	$i \otimes i$	$-1 \otimes 1$

in which the even element $i \otimes i$ represents $\mathbf{k}$, and the respective odd elements $i \otimes 1$ and $1 \otimes i$ represent $\mathbf{i}$ and $\mathbf{j}$ — compare Example 3.7(k). Proposition 4.11 ensures that the supersquare will be a supergroup. As a non-abelian abstract group, it has 8 elements, 6 of which have order 4, and is thus (isomorphic to) the quaternion group Q_8. As sample products in Q_8 obtained using the table, we have

$$\begin{aligned}\mathbf{j}\cdot\mathbf{i} &= (1\otimes i)(i\otimes 1) \overset{(4.6)}{=} (-1)^{|i|\cdot|i|}(i\otimes i)\\ &= -(i\otimes i) = -\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\mathbf{j}\cdot\mathbf{k} &= (1\otimes i)(i\otimes i) \overset{(4.6)}{=} (-1)^{|i|\cdot|i|}(i\otimes -1)\\ &= -(i\otimes -1) \overset{(4.13)}{=} -(-1)(i\otimes 1) = \mathbf{i}\end{aligned}$$

according to Definition 4.10. The signed product is $Q_8 = \langle\mathbf{k}\rangle \uplus \langle\mathbf{k}\rangle\,\mathbf{j}$ as a supergroup. Since the center of Q_8 is $\{\pm 1\}$, Lemma 4.9 shows that the supercenter of this signed supergroup is also $\{\pm 1\}$.

4.6.2. *Boolean times Boolean: the dihedral group.* The dihedral group D_4 (of order 8) is constructed as the supersquare $B_4 \widehat{\times} B_4$ of the Boolean basic supergroup B_4 and its transversal $T = \{1\} \uplus \{i\}$, based upon this multiplication table on the supersquare $T\widehat{\times}T$ of the transversal T:

	$1\otimes 1$	$i\otimes i$	$1\otimes i$	$i\otimes 1$
$1\otimes 1$	$1\otimes 1$	$i\otimes i$	$1\otimes i$	$i\otimes 1$
$i\otimes i$	$i\otimes i$	$-(1\otimes 1)$	$i\otimes -1$	$-(1\otimes i)$
$1\otimes i$	$1\otimes i$	$-(i\otimes 1)$	$1\otimes 1$	$-(i\otimes i)$
$i\otimes 1$	$i\otimes 1$	$1\otimes i$	$i\otimes i$	$1\otimes 1$

Proposition 4.11 ensures that the superproduct will be a supergroup. The group relations

$$\begin{gathered}(i\otimes i)^4 = -(1\otimes 1)^2 = 1\otimes 1,\\ (1\otimes i)^2 = 1\otimes 1, \text{ and}\\ (1\otimes i)(i\otimes i)(1\otimes i) = -(i\otimes 1)(1\otimes i) = -(i\otimes i) = (i\otimes i)^{-1}\end{gathered}$$

hold. Thus, setting $x = i\otimes i$ and $y = 1\otimes i$, it is seen that the presentation

$$\langle x, y \,|\, x^4 = y^2 = 1,\ yxy = x^{-1}\rangle$$

of D_4 is faithfully modeled by the 8-element group $B_4\widehat{\times}B_4$. Indeed, as a supergroup, the signed product is $D_4 = \langle x\rangle \uplus \langle x\rangle\, y$. Since the center of D_4 is $\{\pm 1\}$, Lemma 4.9 again shows that the supercenter of this signed supergroup is also $\{\pm 1\}$.

4.6.3. *Right times right.* The supersquare $C_4^{/}\widehat{\times}C_4^{/}$ of the right basic superquasigroup $C_4^{/}$, with its transversal $T = \{\,1\,\} \uplus \{\,i\,\}$, is based on the multiplication table

	$1\otimes 1$	$i\otimes i$	$1\otimes i$	$i\otimes 1$
$1\otimes 1$	$1\otimes 1$	$-i\otimes -i$	$1\otimes -i$	$-i\otimes 1$
$i\otimes i$	$i\otimes i$	$-(1\otimes 1)$	$i\otimes 1$	$-(1\otimes i)$
$1\otimes i$	$1\otimes i$	$-(-i\otimes 1)$	$1\otimes 1$	$-(-i\otimes i)$
$i\otimes 1$	$i\otimes 1$	$1\otimes -i$	$i\otimes -i$	$1\otimes 1$

on the supersquare $T\widehat{\times}T$ of the transversal T. The table reduces to

(4.14)

	$1\otimes 1$	$i\otimes i$	$1\otimes i$	$i\otimes 1$
$1\otimes 1$	$1\otimes 1$	$i\otimes i$	$-(1\otimes i)$	$-(i\otimes 1)$
$i\otimes i$	$i\otimes i$	$-(1\otimes 1)$	$i\otimes 1$	$-(1\otimes i)$
$1\otimes i$	$1\otimes i$	$i\otimes 1$	$1\otimes 1$	$i\otimes i$
$i\otimes 1$	$i\otimes 1$	$-(1\otimes i)$	$-(i\otimes i)$	$1\otimes 1$

by the tensor relations (4.13). Relabelling the supersquare transversal elements as

$$1\otimes 1 \mapsto 1,\ i\otimes i \mapsto i,\ 1\otimes i \mapsto e,\ i\otimes 1 \mapsto ei,$$

in particular noting the consistency of the last assignment with the second entry in the third row of the body of the reduced table (4.14), the full multiplication table on the 8-element supersquare appears as

(4.15)

$\cdot$	1	i	-1	$-i$	e	ei	$-e$	$-ei$
1	1	i	-1	$-i$	$-e$	$-ei$	e	ei
i	i	-1	$-i$	1	ei	$-e$	$-ei$	e
-1	-1	$-i$	1	i	e	ei	$-e$	$-ei$
$-i$	$-i$	1	i	-1	$-ei$	e	ei	e
e	e	ei	$-e$	$-ei$	1	i	-1	$-i$
ei	ei	$-e$	$-ei$	e	$-i$	1	i	-1
$-e$	$-e$	$-ei$	e	ei	-1	$-i$	1	i
$-ei$	$-ei$	e	ei	$-e$	i	-1	$-i$	1

.

In (4.15), the multiplications $(ei)e = -i \neq i = e(ei) = e(ie)$ witness the breakdown of commutativity and associativity. Also, it is apparent that the body of (4.15) is a Latin square, in accordance with the quasigroup property from Theorem 4.15.

The permutation $L(1)$ from (4.15) is the involution of negation on $\{\pm e, \pm ei\}$. Applying this permutation to the column headers in (4.15) yields the table

$*$	1	i	-1	$-i$	e	ei	$-e$	$-ei$
1	1	i	-1	$-i$	e	ei	$-e$	$-ei$
i	i	-1	$-i$	1	$-ei$	e	ei	$-e$
-1	-1	$-i$	1	i	$-e$	$-ei$	e	ei
$-i$	$-i$	1	i	-1	ei	$-e$	$-ei$	$-e$
e	e	ei	$-e$	$-ei$	-1	$-i$	1	i
ei	ei	$-e$	$-ei$	e	i	-1	$-i$	1
$-e$	$-e$	$-ei$	e	ei	1	i	-1	$-i$
$-ei$	$-ei$	e	ei	$-e$	$-i$	1	i	-1

which is recognizable as the multiplication table of the quaternion group Q_8, in which the relations $i * ei = e$, $ei * e = i$, $e * i = ei$ hold, and where the elements $\pm i, \pm e, \pm ei$ all have order 4. In summary, we have the following.

Theorem 4.19. *Consider the signed supersquare $S = C_4^/ \widehat{\times} C_4^/$ of the right basic superquasigroup $C_4^/$, in which the identifications*

$$1 \otimes 1 \mapsto 1,\ i \otimes i \mapsto i,\ 1 \otimes i \mapsto k,\ i \otimes 1 \mapsto j$$

have been made.

(a) *The superquasigroup S is the isotope of the quaternion group*

$$\{\pm 1, \pm i\} \uplus \{\pm j, \pm k\}$$

given by the multiplication xy^i.

(b) *The even part forms the cyclic subgroup $\langle i \rangle \cong C_4$.*

(c) *There is a subsuperquasigroup $\langle j \rangle = \{\pm 1\} \uplus \{\pm j\} \cong C_4^/$.*

(d) *There is a subsuperquasigroup $\langle k \rangle = \{\pm 1\} \uplus \{\pm k\} \cong C_4^/$.*

Proof. It suffices to note that within the quaternion supergroup from (a), the conjugation $y \mapsto y^i$ fixes the even part and negates the odd part. □

4.7. Superproducts of basic superquasigroups

4.7.1. *Boolean times cyclic: the dihedral group.* The signed product $B_4 \widehat{\times} C_4$ of the Boolean supergroup B_4 and the cyclic supergroup C_4, each with its transversal $T = \{1\} \uplus \{i\}$, is based on the multiplication table

	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$1 \otimes 1$	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$i \otimes i$	$i \otimes i$	$-(1 \otimes -1)$	$i \otimes -1$	$-(1 \otimes i)$
$1 \otimes i$	$1 \otimes i$	$-(i \otimes -1)$	$1 \otimes -1$	$-(i \otimes i)$
$i \otimes 1$	$i \otimes 1$	$1 \otimes i$	$i \otimes i$	$1 \otimes 1$

on the supersquare $T \widehat{\times} T$ of the transversal T. The table reduces to

	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$1 \otimes 1$	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$i \otimes i$	$i \otimes i$	$1 \otimes 1$	$-(i \otimes 1)$	$-(1 \otimes i)$
$1 \otimes i$	$1 \otimes i$	$i \otimes 1$	$-(1 \otimes 1)$	$-(i \otimes i)$
$i \otimes 1$	$i \otimes 1$	$1 \otimes i$	$i \otimes i$	$1 \otimes 1$

by the tensor relations (4.13). We proceed as in §4.6.2. Proposition 4.11 ensures that the product will be a supergroup. The group relations

$$
\begin{aligned}
&(1 \otimes i)^4 = -(1 \otimes 1)^2 = 1 \otimes 1, \\
&(i \otimes 1)^2 = 1 \otimes 1, \text{ and} \\
(i \otimes 1)(1 \otimes i)(i \otimes 1) = (i \otimes i)&(i \otimes 1) = -(1 \otimes i) = (1 \otimes i)^{-1}
\end{aligned}
$$

hold. Thus, setting $x = 1 \otimes i$ and $y = i \otimes 1$, it is seen that the presentation

$$\langle x, y \,|\, x^4 = y^2 = 1,\ yxy = x^{-1} \rangle$$

of D_4 is faithfully modeled by the 8-element group $B_4 \widehat{\times} C_4$. This time, as a supergroup, the signed product is $D_4 = \{ \pm 1, \pm xy \} \uplus \{ \pm x, \pm y \}$ (compare Example 3.19). Since the center of D_4 is $\{ \pm 1 \}$, Lemma 4.9 again shows that the supercenter of this signed supergroup is $\{ \pm 1 \}$.

4.7.2. *Right times cyclic.* Here, the signed superproduct $C_4^{/} \widehat{\times} C_4$ of the right superquasigroup $C_4^{/}$ and the cyclic supergroup C_4, taken with their transversals $T = \{ 1 \} \uplus \{ i \}$, is based on the multiplication table

	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$1 \otimes 1$	$1 \otimes 1$	$-i \otimes i$	$1 \otimes i$	$-i \otimes 1$
$i \otimes i$	$i \otimes i$	$-(1 \otimes -1)$	$i \otimes -1$	$-(1 \otimes i)$
$1 \otimes i$	$1 \otimes i$	$-(-i \otimes -1)$	$1 \otimes -1$	$-(-i \otimes i)$
$i \otimes 1$	$i \otimes 1$	$1 \otimes i$	$i \otimes i$	$1 \otimes 1$

on the supersquare $T \widehat{\times} T$ of the transversal T. The table reduces to

(4.16)

	$1 \otimes 1$	$i \otimes i$	$1 \otimes i$	$i \otimes 1$
$1 \otimes 1$	$1 \otimes 1$	$-(i \otimes i)$	$1 \otimes i$	$-(i \otimes 1)$
$i \otimes i$	$i \otimes i$	$1 \otimes 1$	$-(i \otimes 1)$	$-(1 \otimes i)$
$1 \otimes i$	$1 \otimes i$	$-(i \otimes 1)$	$-(1 \otimes 1)$	$i \otimes i$
$i \otimes 1$	$i \otimes 1$	$1 \otimes i$	$i \otimes i$	$1 \otimes 1$

by the tensor relations (4.13). Upon relabelling the superproduct transversal elements as

$$1 \otimes 1 \mapsto 1,\ i \otimes i \mapsto ie,\ 1 \otimes i \mapsto i,\ i \otimes 1 \mapsto e,$$

and in particular noting the consistency of the second assignment with the last entry in the third row of the body of the reduced table (4.16), we will consider the $(1,e)$-modification S of the 8-element superproduct $C_4^{/}\widehat{\times}C_4$. Its full multiplication table appears as

(4.17)

$\cdot$	1	i	-1	$-i$	e	ie	$-e$	$-ie$
1	1	i	-1	$-i$	e	$-ie$	$-e$	ie
i	i	-1	$-i$	1	ie	$-e$	$-ie$	e
-1	-1	$-i$	1	i	$-e$	ie	e	$-ie$
$-i$	$-i$	1	i	-1	$-ie$	e	ie	$-e$
e	e	ie	$-e$	$-ie$	1	i	-1	$-i$
ie	ie	$-e$	$-ie$	e	$-i$	1	i	-1
$-e$	$-e$	$-ie$	e	ie	-1	$-i$	1	i
$-ie$	$-ie$	e	ie	$-e$	i	-1	$-i$	1

Here $L(1) = (ie\ \ -ie)$. The table for $x * y = x \cdot y^{L(1)}$ is

(4.18)

$*$	1	i	-1	$-i$	e	ie	$-e$	$-ie$
1	1	i	-1	$-i$	e	ie	$-e$	$-ie$
i	i	-1	$-i$	1	ie	e	$-ie$	$-e$
-1	-1	$-i$	1	i	$-e$	$-ie$	e	ie
$-i$	$-i$	1	i	-1	$-ie$	$-e$	ie	e
e	e	ie	$-e$	$-ie$	1	$-i$	-1	i
ie	ie	$-e$	$-ie$	e	$-i$	-1	i	1
$-e$	$-e$	$-ie$	e	ie	-1	i	1	$-i$
$-ie$	$-ie$	e	ie	$-e$	i	1	$-i$	-1

By direct observation that the body of the table (4.17) is a Latin square, the magma $(S,\cdot)$ is a quasigroup. Its isotope $(S,*)$ is a loop. The loop is not commutative, since in the table (4.18), we have

$$(ie) * i = -e \neq e = i * (ie). \tag{4.19}$$

The loop is not associative, since no nonabelian group of order 8 has 4 elements of order 4. More directly, we have $[e * (ie)] * i = [-i] * i = 1 \neq -1 = e * [-e] = e * [(ie) * i]$. Further, note that $e\backslash 1 = e$, since $e^2 = 1$. But $(ie)e = -i \neq i$, so the loop does not have the right Bol property (compare [13, §I.4.2]). Similarly, it does not have the left Bol property, since $1/e = e$, while $e(ei) = e(ie) = -i \neq i$.

As a signed superloop, $(S,*)$ has the transversal $\{1, i\} \uplus \{e, ie\}$. The transversal multiplication table

(4.20)

	1	i	e	ie
1	1	i	e	ie
i	i	-1	ie	e
e	e	ie	1	$-i$
ie	ie	$-e$	$-i$	-1

may be extracted from (4.18).

Among the loops of order 8, the quaternion group Q_8 has 6 elements of order 4, while the dihedral group D_4 has 2 elements of order 4. Since the loop $(S, *)$ has 4 elements of order 4, and has emerged from a context where Q_8 and D_4 would normally be expected, we regard $(S, *)$ as a hybridization of the quaternion group Q_8 and dihedral group D_4. Thus, we refer to it as the *quatedral loop*. Its properties are examined in more detail in [12]. In particular, within the combinatorial character theory of finite loops and quasigroups [5], [6], [11, Ch. 6], it shares the common character table of Q_8 and D_4, with a unique non-linear irreducible character χ of dimension 2.

Recall that the *Frobenius-Schur indicator* of a character of a finite group or quasigroup is defined to be its average value on the squares of elements [4, §23]. The indicator values of $1, 0$, and -1, for the character of an ordinary finite-dimensional representation, correspond respectively to its realizability over the reals, complex numbers, and quaternions. On D_4 the character χ has Frobenius-Schur indicator 1 (corresponding to the natural representation by symmetries of a square), while for Q_8, the indicator is -1 (here corresponding to the embedding of Q_8 into the skewfield of quaternions). On the quatedral loop, the non-linear character χ has indicator 0, once again yielding an interpolation between Q_8 and D_4. For general quasigroups, the Frobenius-Schur indicator is not guaranteed to be integral, let alone to lie in the set $\{1, 0, -1\}$.

Remark 4.20. In the `LOOPS` package [8] of `GAP` [3], the quatedral loop is listed as `NilpotentLoop(8,116)` with

$$1 \mapsto 1,\ 2 \mapsto e,\ 3 \mapsto i,\ 4 \mapsto ie,\ 5 \mapsto -1,\ 6 \mapsto -e,\ 7 \mapsto -i,\ 8 \mapsto -ie$$

as the assignment of elements [15].

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