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GORENSTEIN FP_n-INJECTIVE MODULES WITH RESPECT TO A SEMIDUALIZING BIMODULE

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Abstract. Let S and R be rings and ${}_SC_R$ a semidualizing bimodule. We introduce the notion of G_C-FP_n-injective modules, which generalizes G_C-FP-injective modules and G_C-weak injective modules. The homological properties and the stability of G_C-FP_n-injective modules are investigated. When S is a left n-coherent ring, several nice properties and new Foxby equivalences relative to G_C-FP_n-injective modules are given.

1. Introduction

In 1970, Stenström [14] introduced the notion of FP-injective modules. A left R-module M is said to be FP-injective (or absolutely pure) if $\mathrm{Ext}^1_R(F, M) = 0$ for any finitely presented left R-module F. It plays an important role in characterizing coherent rings. Let $n \geq 0$ be an integer, in a recent article [3], Bravo and Pérez studied the relative homological algebra associated to the notions of n-finitely presented modules and n-coherent rings. In particular, they introduced the notion of FP_n-injective and FP_n-flat modules in terms of n-finitely presented modules and considered cotorsion pairs associated to these two classes. When $n = 1$ and ∞, FP_n-injective modules are exactly FP-injective and FP_∞-injective (or weak injective) modules (cf. [2, 6, 8]), respectively. Relative to a semidualizing bimodule C, the notions mentioned above were extended to C-FP_n-injective and C-FP_n-flat modules in [16], and it was shown that they possess many nice properties analogous to that of FP_n-injective and FP_n-flat modules.

On the other hand, as a nice generalization of injective, projective and flat modules, Gorenstein injective, Gorenstein projective and Gorenstein flat modules were introduced over any ring by Enochs and Jenda in [4, 5]. The generalized versions of these modules with respect to a semidualizing bimodule ${}_SC_R$ were also developed (see [10, 15]), which were named as G_C-injective,

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G_C-projective and G_C-flat modules. Inspired by the ideas of Sather-Wagstaff, Sharif and White using C-flat C-cotorsion modules to study G_C-flat modules [13], Hu and Zhang introduced in [12] C-FP-injective C-FP-projective modules and G_C-FP-injective modules. When ${}_SC_R$ is a faithfully semidualizing bimodule, it is proven that the G_C-FP-injective modules have nice properties analogous to that of G_C-injective modules under the condition that S is a left coherent ring. Moreover, the category of G_C-FP-injective modules was part of a weak AB-context, in the terminology of Hashimoto. To make the results of G_C-FP-injective modules work for general rings, Gao, Ma and Zhao [7] introduced the concept of G_C-weak injective modules. It is proven that many parts of the homological theory on G_C-FP-injective modules can be generalized directly to the similar theory on G_C-weak injective modules over general rings. The Foxby equivalences relative to G_C-weak injective modules were also given.

Motivated by the references mentioned above, we introduce G_C-FP_n-injective modules with respect to a semidualizing bimodule ${}_SC_R$ by choosing an appropriate class of modules, which is a bit different from the definition of G_C-FP-injective modules and G_C-weak injective modules. Many results of G_C-FP-injective and G_C-weak injective modules can be obtained as corollaries from the results of G_C-FP_n-injective modules when $n = 1$ and ∞. When S is a left n-coherent ring, several nice properties and new Foxby equivalence relative to G_C-FP_n-injective modules are given.

The paper is organized as follows. In Section 2, we give some terminologies and some preliminary results. In Section 3, we first give some characterizations of the kernel of C-FP_n-injective modules, and then prove that, if S is a left n-coherent ring, the class of G_C-FP_n-injective left R-modules is closed under extensions, cokernels of monomorphism and direct summands. The stability of the category of G_C-FP_n-injective modules is also investigated. In Section 4, we mainly discuss the Foxby equivalence of G_C-FP_n-injective left R-modules when ${}_SC_R$ is a faithfully semidualizing bimodule and S a left n-coherent ring.

2. Notions and definitions

In this section, we recall some definitions and give some notions needed in the sequel.

2.1. Throughout this paper, R and S are fixed associative rings with unities and all R- or S-modules are understood to be unital left R- or S-modules. Right R- or S-modules are identified with left modules over the opposite rings R^{op} or S^{op}. Let Mod R be the category of R-modules. We use the term "subcategory" to mean a "full and additive subcategory that is closed under isomorphisms".

2.2. Let $\mathcal{X}$ be a subcategory of Mod R. Set

$$^{\perp}\mathcal{X} = \{M \in \mathrm{Mod}R \,|\, \mathrm{Ext}_R^{\geq 1}(M, X) = 0 \text{ for all } X \in \mathcal{X}\} \text{ and}$$
$$^{\perp_1}\mathcal{X} = \{M \in \mathrm{Mod}R \,|\, \mathrm{Ext}_R^{1}(M, X) = 0 \text{ for all } X \in \mathcal{X}\}.$$

$\mathcal{X}^{\perp}$ and $\mathcal{X}^{\perp_1}$ can be defined dually. For the subcategories $\mathcal{X}$ and $\mathcal{Y}$ of Mod R, we write $\mathcal{X} \perp \mathcal{Y}$ if $\mathrm{Ext}_R^{\geq 1}(X,Y) = 0$ for each $X \in \mathcal{X}$ and each $Y \in \mathcal{Y}$. $\mathcal{X}$ is said to be *closed under extensions* if for every exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

with $M', M'' \in \mathcal{X}$, then $M \in \mathcal{X}$.

2.3. Recall from [5] that a *cotorsion pair* is a pair of classes ($\mathcal{A}$, $\mathcal{B}$) in Mod R such that $\mathcal{A}^{\perp_1} = \mathcal{B}$ and ${}^{\perp_1}\mathcal{B} = \mathcal{A}$. $\mathcal{A} \cap \mathcal{B}$ is said to be the *kernel* of the cotorsion pair ($\mathcal{A}$, $\mathcal{B}$). A cotorsion pair ($\mathcal{A}$, $\mathcal{B}$) is said to be *hereditary* if $\mathcal{A} \perp \mathcal{B}$. A cotorsion pair ($\mathcal{A}$, $\mathcal{B}$) is called *complete* if, for any $M \in \mathrm{Mod}\, R$, there are exact sequences $0 \to B_1 \to A_1 \to M \to 0$ and respectively $0 \to M \to B_2 \to A_2 \to 0$, where $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$.

2.4. M is called *n-finitely presented* (FP_n for short) if there exists an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

in Mod R with each P_i finitely generated projective. Note that 1-finitely presented is exactly finitely presented. ∞-finitely presented is just FP_∞ in [2] and *super finitely presented* in [8].

2.5. An (S,R)-bimodule $C = {}_SC_R$ is *semidualizing* [11] if

(a1) ${}_SC$ is FP_∞,

(a2) C_R is FP_∞,

(b1) the homothety map ${}_SS_S \xrightarrow{S\gamma} \mathrm{Hom}_{R^{op}}(C,C)$ is an isomorphism,

(b2) the homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C,C)$ is an isomorphism,

(c1) $\mathrm{Ext}_S^{\geq 1}(C,C) = 0$,

(c2) $\mathrm{Ext}_{R^{op}}^{\geq 1}(C,C) = 0$.

A semidualizing bimodule ${}_SC_R$ is *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_SN$ and M_R:

(1) If $\mathrm{Hom}_S(C,N) = 0$, then $N = 0$.

(2) If $\mathrm{Hom}_{R^{op}}(C,M) = 0$, then $M = 0$.

In what follows, we always assume that C is a semidualizing (S,R)-bimodule.

2.6. The *Auslander class* $\mathcal{A}_C(R)$ [11] with respect to C consists of all modules M in $\mathrm{Mod}\, R$ satisfying:

(A1) $\mathrm{Tor}^R_{\geq 1}(C,M) = 0$.

(A2) $\mathrm{Ext}_S^{\geq 1}(C, C \otimes_R M) = 0$.

(A3) The natural evaluation homomorphism $\mu_M\colon\ M \longrightarrow \mathrm{Hom}_S(C, C \otimes_R M) = 0$ is an isomorphism of R-modules.

The *Bass class* $\mathcal{B}_C(S)$ [11] with respect to C consists of all modules $N \in \mathrm{Mod}\, S$ satisfying:

(B1) $\mathrm{Ext}_S^{\geq 1}(C,N) = 0$.

(B2) $\mathrm{Tor}^R_{\geq 1}(C, \mathrm{Hom}_S(C,N)) = 0$.

(B3) The natural evaluation homomorphism $\nu_N\colon\ C \otimes_R \mathrm{Hom}_S(C,N) \longrightarrow N$ is an isomorphism of S-modules.

2.7. An R-module M is called *FP_n-injective* [3] in case $\mathrm{Ext}^1_R(P, M) = 0$ for every n-finitely presented R-module P. In what follows, we denote by $\mathcal{FI}^n(R)$ the class of FP_n-injective left R-modules.

Inspired by the definition of the kernel of a cotorsion pair, we call $\mathcal{H}(\mathcal{X}) := {}^{\perp}\mathcal{X} \cap \mathcal{X}$ the kernel of $\mathcal{X}$, and we set

$$\mathcal{FI}^n_C(R) = \{\mathrm{Hom}_S(C, E) \mid E \in \mathcal{FI}^n(S)\},$$
$$\mathcal{H}_C(\mathcal{FI}^n(R)) = \{\mathrm{Hom}_S(C, E) \mid E \in \mathcal{H}(\mathcal{FI}^n(S))\}.$$

Modules in $\mathcal{FI}^n_C(R)$ are called *C-FP_n-injective*.

3. G_C-FP_n-injective modules

In this section, we introduce and study G_C-FP_n-injective modules. It will be shown that the results of G_C-FP_n-injective modules cover relevant results with G_C-FP-injective modules [12] and G_C-weak injective modules [7].

Lemma 3.1 ([16, Proposition 3.3 and Lemma 3.5]). *The Auslander class $\mathcal{A}_C(R)$ contains all C-FP_n-injective left R-modules, and the Bass class $\mathcal{B}_C(S)$ contains all FP_n-injective left S-modules.*

Lemma 3.2 ([11, Theorem 6.4(a),(b)]). *Let M and M' be R-modules, let N and N' be S-modules, and let $i \geq 0$.*

(a) *If $M \in \mathcal{A}_C(R)$ and $\mathrm{Tor}^R_{\geq 1}(C, M') = 0$ (e.g., if $M' \in \mathcal{A}_C(R)$), then*
$$\mathrm{Ext}^i_R(M', M) \cong \mathrm{Ext}^i_S(C \otimes_R M', C \otimes_R M).$$

(b) *If $N \in \mathcal{B}_C(S)$ and $\mathrm{Ext}^{\geq 1}_S(C, N') = 0$ (e.g., if $N' \in \mathcal{B}_C(S)$), then*
$$\mathrm{Ext}^i_S(N, N') \cong \mathrm{Ext}^i_R(\mathrm{Hom}_S(C, N), \mathrm{Hom}_S(C, N')).$$

Lemma 3.3. *$M \in {}^{\perp}\mathcal{FI}^n_C(R)$ if and only if $C \otimes_R M \in {}^{\perp}\mathcal{FI}^n(S)$ and $\mathrm{Tor}^R_{\geq 1}(C, M) = 0$.*

Proof. ($\Rightarrow$) Let $M \in {}^{\perp}\mathcal{FI}^n_C(R)$ and I be a faithfully injective left S-module. Then $\mathrm{Ext}^{\geq 1}_R(M, \mathrm{Hom}_S(C, I)) = 0$ by definition. Since

$$\mathrm{Hom}_S(\mathrm{Tor}^R_i(C, M), I) \cong \mathrm{Ext}^i_R(M, \mathrm{Hom}_S(C, I))$$

for each $i \geq 1$ from [5, Theorem 3.2.1], thus $\mathrm{Tor}^R_{\geq 1}(C, M) = 0$. Suppose $N \in \mathcal{FI}^n(S)$, then $N \in \mathcal{B}_C(S)$ by Lemma 3.1. So

$$\mathrm{Ext}^i_S(C \otimes_R M, N) \cong \mathrm{Ext}^i_S(C \otimes_R M, C \otimes_R \mathrm{Hom}_S(C, N)) \cong \mathrm{Ext}^i_R(M, \mathrm{Hom}_S(C, N))$$

for each $i \geq 1$ by Lemma 3.2(a). Note that $\mathrm{Hom}_S(C, N) \in \mathcal{FI}^n_C(R)$, thus $\mathrm{Ext}^{\geq 1}_R(M, \mathrm{Hom}_S(C, N)) = 0$, and hence $\mathrm{Ext}^{\geq 1}_S(C \otimes_R M, N) = 0$, which means that $C \otimes_R M \in {}^{\perp}\mathcal{FI}^n(S)$.

($\Leftarrow$) Let $L \in \mathcal{FI}^n_C(R)$, then there exists $E \in \mathcal{FI}^n(S)$ such that $L = \mathrm{Hom}_S(C, E)$. Since $L \in \mathcal{A}_C(R)$ by Lemma 3.1, and $C \otimes_R L \cong E \in \mathcal{FI}^n(S)$, it follows from Lemma 3.2(a) that $\mathrm{Ext}^i_R(M, L) \cong \mathrm{Ext}^i_S(C \otimes_R M, C \otimes_R L) = 0$ for each $i \geq 1$. So $M \in {}^{\perp}\mathcal{FI}^n_C(R)$. $\square$

Proposition 3.4. *The following are equivalent for* $M \in \operatorname{Mod} R$:

(1) $M \in \mathcal{H}(\mathcal{FI}^n_C(R))$.
(2) $M \in \mathcal{H}_C(\mathcal{FI}^n(R))$.
(3) $C \otimes_R M \in \mathcal{H}(\mathcal{FI}^n(S))$.

In particular, $\mathcal{H}_C(\mathcal{FI}^n(R)) \perp \mathcal{FI}^n_C(R)$.

Proof. (1) $\Rightarrow$ (2) Because $M \in \mathcal{FI}^n_C(R)$, $M = \operatorname{Hom}_S(C, W)$ with $W \in \mathcal{FI}^n(S)$. It is sufficient to show that W is in ${}^{\perp}\mathcal{FI}^n(S)$. Let $N \in \mathcal{FI}^n(S)$, hence $W, N \in \mathcal{B}_C(S)$ by Lemma 3.1. It follows from Lemma 3.2(b) that $\operatorname{Ext}^i_S(W, N) \cong \operatorname{Ext}^i_R(\operatorname{Hom}(C, W), \operatorname{Hom}(C, N)) = \operatorname{Ext}^i_R(M, \operatorname{Hom}_S(C, N))$ for each $i \geq 1$. Since $M \in {}^{\perp}\mathcal{FI}^n_C(R)$, $\operatorname{Ext}^{\geq 1}_R(M, \operatorname{Hom}_S(C, N)) = 0$, which implies $\operatorname{Ext}^{\geq 1}_S(W, N) = 0$. We obtain the assertion.

(2) $\Rightarrow$ (3) Let $M \in \mathcal{H}_C(\mathcal{FI}^n(R))$. Then $M = \operatorname{Hom}_S(C, W)$ with $W \in \mathcal{H}(\mathcal{FI}^n(S))$. By Lemma 3.1, $W \in \mathcal{FI}^n(S) \subseteq \mathcal{B}_C(S)$, and hence $W \cong C \otimes_R \operatorname{Hom}_S(C, W)$. So $C \otimes_R M \cong W \in \mathcal{H}(\mathcal{FI}^n(S))$.

(3) $\Rightarrow$ (1) Assume $C \otimes_R M \in \mathcal{H}(\mathcal{FI}^n(S)) \subseteq \mathcal{FI}^n(S)$. Then $C \otimes_R M \in \mathcal{B}_C(S)$. It follows from [16, Lemma 3.9] that $M \in \mathcal{A}_C(R)$, and thus $\operatorname{Tor}^R_{\geq 1}(C, M) = 0$. Since $C \otimes_R M \in {}^{\perp}\mathcal{FI}^n(S)$, one gets $M \in {}^{\perp}\mathcal{FI}^n_C(R)$ by Lemma 3.3. Moreover, since $M \cong \operatorname{Hom}_S(C, C \otimes_R M)$, then $M \in \mathcal{FI}^n_C(R)$. Therefore, $M \in \mathcal{H}(\mathcal{FI}^n_C(R))$. □

We denote the class of injective R-modules by $\mathcal{I}(\mathcal{R})$ in the following.

Definition 3.5. Let C be a semidualizing bimodule. A *complete* $\mathcal{FI}^n_C\mathcal{I}$*-resolution* is a complex $\mathbb{Y}$ of R-modules

$$\mathbb{Y} = \cdots \xrightarrow{\partial_2} W_1 \xrightarrow{\partial_1} W_0 \xrightarrow{\partial_0} I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} \cdots$$

satisfying the following:

(1) $\mathbb{Y}$ is exact and $\operatorname{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact, and
(2) $W_{i\geq 0} \in \mathcal{FI}^n_C(R)$ and $I^{i\geq 0} \in \mathcal{I}(R)$.

An R-module M is called G_C*-*FP_n*-injective* if there exists a complete $\mathcal{FI}^n_C\mathcal{I}$-resolution $\mathbb{Y}$ such that $M \cong \ker(\partial^0)$. We denote by $\mathcal{G}_C\mathcal{FI}^n(R)$ the class of G_C-FP_n-injective R-modules.

Remark 3.6. (1) An R-module M is G_C-FP_n-injective if and only if $M \in \mathcal{H}_C(\mathcal{FI}^n(R))^{\perp}$ and there is a $\operatorname{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to M \to 0$ with each $W_i \in \mathcal{FI}^n_C(R)$.

(2) Every C-FP_n-injective module and every $\ker \partial_i$ are G_C-FP_n-injective by Proposition 3.4 and (1).

Proposition 3.7. *Given an* R*-module exact sequence* $0 \to A \to B \to D \to 0$. *We have*

(1) *If* $A \in \mathcal{G}_C\mathcal{FI}^n(R)$ *and* $B \in \mathcal{FI}^n_C(R)$, *then* $D \in \mathcal{G}_C\mathcal{FI}^n(R)$.
(2) *If* $A \in \mathcal{FI}^n_C(R)$ *and* $D \in \mathcal{G}_C\mathcal{FI}^n(R)$, *then* $B \in \mathcal{G}_C\mathcal{FI}^n(R)$.

Proof. (1) Since $A \in \mathcal{G}_C\mathcal{FI}^n(R)$, there is a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to A \to 0$, where each $W_i \in \mathcal{FI}^n_C(R)$. By gluing this sequence with the given one, we get a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to B \to D \to 0$. Since $B \in \mathcal{FI}^n_C(R) \subseteq \mathcal{H}_C(\mathcal{FI}^n(R))^{\perp}$ by Proposition 3.4, so is D from the given short exact sequence. Thus we have $D \in \mathcal{G}_C\mathcal{FI}^n(R)$.

(2) Let $D \in \mathcal{G}_C\mathcal{FI}^n(R)$, by Remark 3.6(2), there is an exact sequence $0 \to D' \to W \to D \to 0$ with $W \in \mathcal{FI}^n_C(R)$ and $D' \in \mathcal{G}_C\mathcal{FI}^n(R)$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & D' & = & D' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & Q & \to & W & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & D & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $A, W \in \mathcal{FI}^n_C(R)$, one has $Q \in \mathcal{FI}^n_C(R)$ by [16, Proposition 3.6]. Thus $B \in \mathcal{G}_C\mathcal{FI}^n(R)$ by (1). □

When S is a left n-coherent ring, G_C-FP_n-injective R-module has more nice properties.

Lemma 3.8. *If S is left n-coherent, then ${}^{\perp_1}\mathcal{FI}^n(S) = {}^{\perp}\mathcal{FI}^n(S)$.*

Proof. From [3, Theorem 5.5] we know that $({}^{\perp_1}\mathcal{FI}^n(S), \mathcal{FI}^n(S))$ is a hereditary cotorsion pair. Thus ${}^{\perp_1}\mathcal{FI}^n(S) \perp \mathcal{FI}^n(S)$, and so ${}^{\perp_1}\mathcal{FI}^n(S) \subseteq {}^{\perp}\mathcal{FI}^n(S)$, the desired equality follows. □

Theorem 3.9. *Let S be a left n-coherent ring. Then the following statements are equivalent for an R-module M:*

1. *M is G_C-FP_n-injective;*
2. *There exists a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact exact sequence*
$$\mathbb{Y} = \cdots \to W_1 \to W_0 \to I^0 \to I^1 \to \cdots$$
with each $W_i \in \mathcal{H}_C(\mathcal{FI}^n(R))$ and each $I_i \in \mathcal{I}(R)$ such that $M \cong \ker(I^0 \to I^1)$;
3. *$M \in \mathcal{H}_C(\mathcal{FI}^n(R))^{\perp}$ and there exists a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact exact sequence $\cdots \to W_1 \to W_0 \to M \to 0$ with each $W_i \in \mathcal{H}_C(\mathcal{FI}^n(R))$.*

Proof. (3) ⇒ (2) ⇒ (1) are trivial.

(1) ⇒ (3) By Remark 3.6(2), there is an exact sequence $0 \to M' \to W \to M \to 0$, where $W \in \mathcal{FI}^n_C(R)$ and $M' \in \mathcal{G}_C\mathcal{FI}^n(R)$. Then $W = \mathrm{Hom}_S(C, E_0)$,

in which $E_0 \in \mathcal{FI}^n(S)$. Note that the cotorsion pair $({}^{\perp_1}\mathcal{FI}^n(S), \mathcal{FI}^n(S))$ is complete by [17, Theorem 2.10], hence there exists an exact sequence

$$\alpha : 0 \to E_1 \to T_0 \to E_0 \to 0$$

such that $T_0 \in {}^{\perp_1}\mathcal{FI}^n(S)$ and $E_1 \in \mathcal{FI}^n(S)$. Because $\mathcal{FI}^n(S)$ is closed under extensions, T_0 is FP_n-injective, and hence $T_0 \in \mathcal{H}(\mathcal{FI}^n(S))$ by Lemma 3.8. Since ${}_SC$ is FP_∞, $\mathrm{Ext}^1_S(C, E_1) = 0$, which gives rise to an exact sequence

$$\mathrm{Hom}_S(C, \alpha) : 0 \to \mathrm{Hom}_S(C, E_1) \to \mathrm{Hom}_S(C, T_0) \to W \to 0.$$

Now consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Hom}_S(C, E_1) & = & \mathrm{Hom}_S(C, E_1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Q & \to & \mathrm{Hom}(C, T_0) & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M' & \to & W & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathrm{Hom}_S(C, E_1) \in \mathcal{FI}^n_C(R)$ and $M' \in \mathcal{G}_C\mathcal{FI}^n(R)$, then $Q \in \mathcal{G}_C\mathcal{FI}^n(R)$ by Proposition 3.7(2). This implies that the middle row in the diagram above is $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact. By repeating the above argument to Q, we get an exact sequence $0 \to Q_1 \to \mathrm{Hom}_S(C, T_1) \to Q \to 0$ with $T_1 \in \mathcal{H}(\mathcal{FI}^n(S))$. Continuing the preceding process, one has a $\mathrm{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact exact sequence

$$\cdots \to \mathrm{Hom}_S(C, T_2) \to \mathrm{Hom}_S(C, T_1) \to \mathrm{Hom}_S(C, T_0) \to M \to 0$$

with each $T_i \in \mathcal{H}(\mathcal{FI}^n(S))$, as desired. □

Let $\mathcal{A}$ be an abelian category and $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}$. Recall from [1, Definition 3.11], an object $A \in \mathcal{A}$ is called *weak* $(\mathcal{X}, \mathcal{Y})$*-Gorenstein injective* if $A \in \mathcal{X}^\perp$ and there is an exact sequence $\cdots \to Y_1 \to Y_0 \to A \to 0$ with $Y_i \in \mathcal{Y}$ and $\mathrm{Im}(Y_i \to Y_{i-1}) \in \mathcal{X}^\perp$ for all $i \geq 1$. Denoted by $W\mathcal{GI}(\mathcal{X}, \mathcal{Y})$ the subcategory of weak $(\mathcal{X}, \mathcal{Y})$-Gorenstein injective objects.

Definition 3.10. An R-module M is called *two-degree* G_C*-*FP_n*-injective* if there exists a complete $\mathcal{G}_C\mathcal{FI}^n(R)$-resolution $\mathbb{G}$ of M, which means that $\mathbb{G} = \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ is an exact complex with each G_i and G^i are G_C-FP_n-injective such that $M \cong \ker(G^0 \to G^1)$, and the complex $\mathrm{Hom}_R(Q, \mathbb{G})$ is exact for every $Q \in \mathcal{H}_C(\mathcal{FI}^n(R))$.

We denote by $\mathcal{G}^2_C\mathcal{FI}^n(R)$ the class of two-degree G_C-FP_n-injective R-modules.

Theorem 3.11. *Let S be a left n-coherent ring. Then*

(1) *$\mathcal{G}_C\mathcal{FI}^n(R)$ is closed under extensions, cokernels of monomorphisms, summands and direct products.*
(2) *$\mathcal{G}_C\mathcal{FI}^n(R) = \mathcal{G}_C^2\mathcal{FI}^n(R)$.*

Proof. The closure of direct products follows directly from the definition.

Set $\mathcal{X} = \mathcal{H}_C(\mathcal{FI}^n(R))$. Because $\mathcal{H}_C(\mathcal{FI}^n(R)) \subseteq \mathcal{FI}_C^n(R)$, it follows from Proposition 3.4 that $\mathcal{X} \perp \mathcal{X}$. Since S is left n-coherent, from Theorem 3.9 and the dual result of [1, Lemma 3.10], we have $\mathcal{G}_C\mathcal{FI}^n(R) = W\mathcal{GI}(\mathcal{X}, \mathcal{X})$. Another use of Proposition 3.4, one has $\mathcal{H}_C(\mathcal{FI}^n(R)) = {}^{\perp}\mathcal{FI}_C^n(R) \cap \mathcal{FI}_C^n(R)$. Note that $\mathcal{FI}_C^n(R)$ is closed under extensions by [16, Proposition 3.6], and ${}^{\perp}\mathcal{FI}_C^n(R)$ is clearly closed under extensions. This implies that $\mathcal{X}$ is closed under extensions. Therefore, the assertion follows from the dual result of [1, Theorem 3.30] and [9, Proposition 1.4]. ☐

Corollary 3.12. *Let S be a left n-coherent ring and let $0 \to A \to B \to D \to 0$ be an exact sequence of R-modules with $B, D \in \mathcal{G}_C\mathcal{FI}^n(R)$. Then $A \in \mathcal{G}_C\mathcal{FI}^n(R)$ if and only if $A \in \mathcal{H}_C(\mathcal{FI}^n(R))^{\perp_1}$.*

Proof. The "only if" part is clear. For the "if" part, since $D \in \mathcal{G}_C\mathcal{FI}^n(R)$, there exists an exact sequence $0 \to D' \to W \to D \to 0$ with $W \in \mathcal{H}_C(\mathcal{FI}^n(R))$ and $D' \in \mathcal{G}_C\mathcal{FI}^n(R)$ by Theorem 3.9. Consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & D' & = & D' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & Q & \to & W & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & D & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $D', B \in \mathcal{G}_C\mathcal{FI}^n(R)$, so is Q from Theorem 3.11(1). On the other hand, because $\mathrm{Ext}_R^1(W, A) = 0$, A is a direct summand of Q from the middle row in the diagram above, and so $A \in \mathcal{G}_C\mathcal{FI}^n(R)$ by Theorem 3.11(1) again. ☐

Remark 3.13. Note that 1-coherent ring is coherent and any ring is ∞-coherent. When S is a left n-coherent ring, the results about G_C-FP_n-injective modules extend the corresponding results of G_C-FP-injective modules [12] and G_C-weak-injective modules [7] by Lemma 3.8.

4. Foxby equivalence of G_C-FP_n-injective modules

In this section, we investigate Foxby equivalences relative to C-FP_n-injective and G_C-FP_n-injective modules.

Proposition 4.1. *Let C be a semidualizing bimodule. Then there are equivalences of categories*:

$$\begin{array}{ccc} \mathcal{H}_C(\mathcal{FI}^n(R)) & \underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{H}(\mathcal{FI}^n(S)) \\ \downarrow & & \downarrow \\ \mathcal{A}_C(R) & \underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{B}_C(S) \end{array}$$

Proof. The proof is straightforward by Lemma 3.1, Proposition 3.4 and [11, Proposition 4.1]. □

Lemma 4.2. *Let C be a faithfully semidualizing bimodule and $M \in \mathcal{A}_C(R)$. Then the following are equivalent for any $Y \in \mathcal{B}_C(S)$.*

(1) *the sequence $\mathbb{X} = \cdots \to X_1 \to X_0 \to M \to 0$ in $\mathcal{A}_C(R)$ is $\operatorname{Hom}_R(\operatorname{Hom}_S(C,Y),-)$-exact exact.*
(2) *the sequence $C \otimes_R \mathbb{X} = \cdots \to C \otimes_R X_1 \to C \otimes_R X_0 \to C \otimes_R M \to 0$ in $\mathcal{B}_C(S)$ is $\operatorname{Hom}_S(Y,-)$-exact exact.*

Proof. Firstly, for any $Y \in \mathcal{B}_C(S)$, by Lemma 3.2(b), there are isomorphisms of complex

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_S(C,Y),\mathbb{X}) &\cong \operatorname{Hom}_R(\operatorname{Hom}_S(C,Y),\operatorname{Hom}_S(C,C\otimes_R \mathbb{X})) \\ &\cong \operatorname{Hom}_S(Y,C\otimes_R \mathbb{X}).\end{aligned}$$

So $\mathbb{X}$ is $\operatorname{Hom}_R(\operatorname{Hom}_S(C,Y),-)$-exact if and only if $C\otimes_R\mathbb{X}$ is $\operatorname{Hom}_S(Y,-)$-exact.

Since M and every X_i are in $\mathcal{A}_C(R)$, from [11, Theorem 6.2] we know that each kernel in $\mathbb{X}$ is in $\mathcal{A}_C(R)$. So the exactness of $\mathbb{X}$ implies the exactness of $C \otimes_R \mathbb{X}$. Conversely, since $C \otimes_R M$ and every $C \otimes_R X_i$ are in $\mathcal{B}_C(S)$, then all kernels in the sequence $C \otimes_R \mathbb{X}$ are in $\mathcal{B}_C(S)$ by [11, Corollary 6.3]. Thus the exact sequence $C \otimes_R \mathbb{X}$ is $\operatorname{Hom}_S(C,-)$-exact. Since $\operatorname{Hom}_S(C, C \otimes_R \mathbb{X}) \cong \mathbb{X}$, it means that $\mathbb{X}$ is exact. □

When ${}_SC_R={}_RR_R$, G_C-FP_n-injective modules are called G-FP_n-injective modules, and denoted by $\mathcal{GFI}^n(S)$.

Proposition 4.3. *Let S be a left n-coherent ring and C faithfully semidualizing. If $M \in \mathcal{A}_C(R)$, then M is G_C-FP_n-injective if and only if $C \otimes_R M$ is G-FP_n-injective.*

Proof. Let M be G_C-FP_n-injective. Then there exists a $\operatorname{Hom}_R(\mathcal{H}_C(\mathcal{FI}^n(R)), -)$-exact exact sequence

$$(*)\qquad \mathbb{X} = \cdots \to X_1 = \operatorname{Hom}_S(C,W_1) \to X_0 = \operatorname{Hom}_S(C,W_0) \to M \to 0$$

in $\mathcal{A}_C(R)$ with $W_i \in \mathcal{H}(\mathcal{FI}^n(S))$ by Theorem 3.9 and Proposition 4.1. This, from Lemma 4.2, is equivalent to that there is a $\operatorname{Hom}_S(\mathcal{H}(\mathcal{FI}^n(S)),-)$-exact

exact sequence

$$C \otimes_R \mathbb{X} = \cdots \to C \otimes_R X_1 \to C \otimes_R X_0 \to C \otimes_R M \to 0$$

in $\mathcal{B}_C(S)$ with $C \otimes_R X_i \cong W_i \in \mathcal{H}(\mathcal{FI}^n(S))$.

In addition, for any $Y \in \mathcal{H}(\mathcal{FI}^n(S)) \subseteq \mathcal{B}_C(S)$ and $i \geq 1$, Lemma 3.2(b) yields

$$\begin{aligned}\operatorname{Ext}^i_S(Y, C \otimes_R M) &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_S(C, Y), \operatorname{Hom}_S(C, C \otimes_R M)) \\ &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_S(C, Y), M).\end{aligned}$$

It follows that $C \otimes_R M \in \mathcal{H}(\mathcal{FI}^n(S))^\perp$ if and only if $M \in \mathcal{H}_C(\mathcal{FI}^n(S))^\perp$. Therefore, we obtain the assertion. □

By a similar argument of Lemma 4.2, we get the following result.

Lemma 4.4. *Let C be a faithfully semidualizing bimodule and $N \in \mathcal{B}_C(S)$. Then the following are equivalent for any $Y \in \mathcal{B}_C(S)$:*

(1) *the sequence $\mathbb{Z} = \cdots \to Z_1 \to Z_0 \to N \to 0$ in $\mathcal{B}_C(S)$ is $\operatorname{Hom}_S(Y, -)$-exact exact.*

(2) *the sequence $\operatorname{Hom}_S(C, \mathbb{Z}) = \cdots \to \operatorname{Hom}_S(C, Z_1) \to \operatorname{Hom}_S(C, Z_0) \to \operatorname{Hom}_S(C, N) \to 0$ in $\mathcal{A}_C(R)$ is $\operatorname{Hom}_R(\operatorname{Hom}_S(C, Y), -)$-exact exact.*

Proposition 4.5. *Let S be a left n-coherent ring and C faithfully semidualizing. If $N \in \mathcal{B}_C(S)$, then N is G-FP_n-injective if and only if $\operatorname{Hom}_S(C, N)$ is G_C-FP_n-injective.*

Proof. Let $Y \in \mathcal{B}_C(S)$. For any $i \geq 1$, it follows from Lemma 3.2(b) that

$$\operatorname{Ext}^i_R(\operatorname{Hom}_S(C, Y), \operatorname{Hom}_S(C, N)) \cong \operatorname{Ext}^i_R(Y, N),$$

which yields that $\operatorname{Hom}_S(C, N) \in \mathcal{H}_C(\mathcal{FI}^n(S))^\perp$ if and only if $N \in \mathcal{H}(\mathcal{FI}^n(S))^\perp$. By Lemma 4.4, the conclusion is obtained by a similar argument in the proof of Proposition 4.3. □

Theorem 4.6 (Foxby Equivalence)**.** *Let S be a left n-coherent ring and ${}_SC_R$ faithfully semidualizing. There are equivalences of categories:*

$$\begin{array}{ccc}
\mathcal{H}_C(\mathcal{FI}^n(R)) & \underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathcal{H}(\mathcal{FI}^n(S)) \\
\downarrow & & \downarrow \\
\mathcal{FI}^n_C(R) & \underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathcal{FI}^n(S) \\
\downarrow & & \downarrow \\
\mathcal{G}_C\mathcal{FI}^n(R) \bigcap \mathcal{A}_C(R) & \underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathcal{GFI}^n(S) \bigcap \mathcal{B}_C(S) \\
\downarrow & & \downarrow \\
\mathcal{A}_C(R) & \underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathcal{B}_C(S)
\end{array}$$

Proof. From Propositions 4.1, [16, Proposition 4.1] and the classical Foxby equivalence, it only needs to prove the third equivalence in the diagram above.

Let $M \in \mathcal{G}_C\mathcal{FI}^n(R) \bigcap \mathcal{A}_C(R)$. It is easy to see that $C \otimes_R M \in \mathcal{GFI}^n(S) \bigcap \mathcal{B}_C(S)$ by Proposition 4.3 and [11, Proposition 4.1]. From Proposition 4.5 and [11, Proposition 4.1], we have that the image of the functor $\mathrm{Hom}_S(C, -)$ under $\mathcal{GFI}^n(S) \bigcap \mathcal{B}_C(S)$ is in $\mathcal{G}_C\mathcal{FI}^n(R) \bigcap \mathcal{A}_C(R)$. Finally, if $M \in \mathcal{G}_C\mathcal{FI}^n(R) \bigcap \mathcal{A}_C(R)$ and $N \in \mathcal{GFI}^n(S) \bigcap \mathcal{B}_C(S)$, then by definition we have two natural isomorphisms: $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ and $C \otimes_R \mathrm{Hom}_S(C, N) \cong N$, which complete the proof. □

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