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On the Spherical Trigonometry of Jo Hui-sun II

조희순의 구면삼각법 2

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This paper is a continuation of the authors' previous work of the same title, which focused on Jo Hui-sun's *Sanhag Seubyu*. We analyze the development of Joseon mathematics at the conclusion of this process of adopting Western mathematics. Joseon began the research on the *Shixian Calendar* (時憲歷) in the mid 17$^{\text{th}}$ century. This resulted in the mathematical development in the 19$^{\text{th}}$ century Joseon which incudes Jo Hui-sun's *Sanhag Seubyu*. Notably, *Sanhag Seubyu* demonstrates that wide spread was there a unique strategy developed in Joseon to adopt and apply advanced mathematics beyond the traditional level. This paper explores the content of Chapters 4, 5, and 6 among the seven chapters of *Sanhag Seubyu*, which reveals the unique characteristic of Joseon mathematics.

Keywords: Jo Hui-sun, Sanhag Seub-yu, spherical trigonometry, logarithm, geometry, Joseon mathematics; 조희순, 산학습유(算學拾遺), 구면삼각법, 대수(對數)산, 기하학, 조선의 수학.

MSC: 01A25, 97G60 ZDM: A30, G60

1 Introduction

This paper is the second in a series of studies on Jo Hui-sun's *Sanhag Seubyu* (算學拾遺, 1869). In the first paper [12], the authors analyzed Jo Hui-sun's explanation of solving spherical triangles, which corresponds to Chapters 2 and 3 of the book. As previously explained, Chapters 1 and 7 have already been studied ([8, 9]). This paper, therefore, focuses on Chapters 4, 5, and 6.

These three chapters summarize three essential computational methods required for solving spherical triangles, as discussed in earlier chapters. Chapter 4 explains how to efficiently use logarithmic calculations for multiplication and division of numbers up to seven decimal places. Chapter 5 expands on the twelve trigonometric formulas for spherical triangles in *Celiang Quanyi* (測量全義) [2], adding six new formulas and proving all of them. Chapter 6 explains methods for finding angles

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from trigonometric values and vice versa, using power series polynomials and solving for their roots. These chapters consolidate the mathematical tools most needed for practical calculations in astronomy, especially spherical trigonometry.

The introduction of the *Shixian Calendar* (時憲歷) in the mid-17th century brought significant changes to Joseon astronomy. The *Shixian Calendar* required advanced mathematical methods that traditional Joseon mathematics struggled to address, especially after the decline in mathematical capabilities following two major wars. Although there appear to have been ongoing efforts to rebuild mathematical expertise starting in the 17th century, these efforts lacked sufficient momentum. This was partly because Joseon failed to fully comprehend the changes made by Western mathematics introduced into China. While the presence of Western mathematics was probably recognized within the Gwansanggam (Royal Observatory), it does not seem to have reached higher-level officials. This is evidenced by the fact that even by the early 18th century, government official mathematician like Hong Jeong-ha (洪正夏) had not fully grasped its content.

The first instance of recognition of Western mathematics by Joseon officials appears in the early 18th century mathematical texts of Choe Seok-jeong (崔錫鼎) and Jo Tae-gu (趙泰耉). By then, much of Western mathematics had already been introduced to Joseon, but the knowledge displayed in these works remained fragmentary. Even in Hong Dae-yong's (洪大容) *Juhae Suyong* (籌解需用) from the mid-18th century and Seo Ho-su's (徐浩修) *Su-ri Jeong-on Bo-hae* (數理精蘊補解) from the late 18th century, there is no mention of spherical trigonometry, the core mathematics of the *Shixian Calendar*. By the late 18th century, spherical trigonometric calculations and applications of *Lixiang Kaocheng* (曆象考成) were probably being conducted properly within the Gwansanggam. However, scholars largely overlooked its importance. This suggests that while the mathematics of the *Shixian Calendar* was utilized in practical contexts, it was not theoretically studied or developed, marking a significant departure from the early Joseon period, where astronomical research and mathematical advancements were led by scholars. This situation began to change only in the early 19th century with Hong Gil-ju's (洪吉周) research.

This implies that while the institutional framework of Joseon remained intact, the connection between the middle-class technicians of the Gwansanggam and the scholar-officials had weakened or even broken by the mid-Joseon period. Nevertheless, Hong Gil-ju bridged this gap by identifying and addressing the necessary mathematical challenges. This divide was eventually closed in the mid-19th century when Nam Byeong-gil (南秉吉) and Nam Byeong-cheol (南秉哲) took charge of the Gwansanggam. Their efforts culminated in the emergence of distinguished mathematicians like Lee Sang-hyeog (李尙爀) and Jo Hui-sun.

This paper presents the findings from studying Jo Hui-sun's mathematics, emphasizing that the most pressing issues for Joseon scholars were overcoming the lack of resources and improving computational efficiency.

For historical details on Jo Hui-sun, we refer to the three previously mentioned papers and other existing studies [10, 13]. Relevant studies will also be referenced for the state of late 19th century Joseon mathematics and the influence of late Qing mathematics on Jo Hui-sun's work [6, 7, 11]. The authors thank Professor Lee Yongbok for bringing their attention to the mathematics in Gyuilgo written by the Joseon mathematician Lee Sang-hyeog.

All Chinese sources cited in this paper are from *Zhongguo Kexue Jishu Dianji Tonghui Shuxuejuan* (中國科學技術典籍通彙 數學卷) [1] and *Siku Quanshu* (四庫全書) [2]. Korean sources are drawn from *Hanguk Gwahakgisulsa Jaryo Daegye Suhakpyeon* (韓國科學技術史資料大系 數學篇) [3], *The Annals of the Joseon Dynasty* (朝鮮王朝實錄) [4], and *Seungjeongwon Ilgi* (承政院日記) [5].

2 Applications of logarithm

The title of Chapter 4 is *Hosamgak-hyeong Yong Dae-su San* (弧三角形用對數算), which translates to "Use of Logarithms for Spherical Triangles." This chapter explains how logarithmic calculations can simplify the multiplication and division of large numbers that arise when solving spherical triangles. It demonstrates that applying logarithms to these operations and converting multiplication and division into addition and subtraction makes the computations more convenient. Jo Hui-sun applied this method to the calculations in the *Zongjiao Method*[1)] (總較法) for solving spherical triangles and in finding angles from trigonometric values or vice versa. He showed that lotarithm replaced multiplications and squares or square root calculations with simpler operations like additions or halving and doubling. In this chapter, he unpacks the steps of the *Zongjiao Method*, illustrating the core principles of each with simple diagrams. The *Zongjiao Method* itself will be further explored in the following subsection.

At that time, trigonometric values were typically obtained using the *Table of Eight Segments*[2)] (八線表), where the values represented segment lengths based on a circle with a radius of 10^5 or 10^7. Solving spherical triangles thus required performing multiplication and division with 5- or 7-digit numbers. Even though Joseon mathematicians used counting rods (算籌) which were faster than manual calculation, such tasks were still labor-intensive. Therefore, converting the numbers in the *Table*

1) In the algorithm of this method one utilizes the *zonghu* (總弧, arc sum) and the *jiaohu* (arc difference, 較弧). This possibly gave rise to the term *Zongjiao Method*.

2) The trigonometric table.

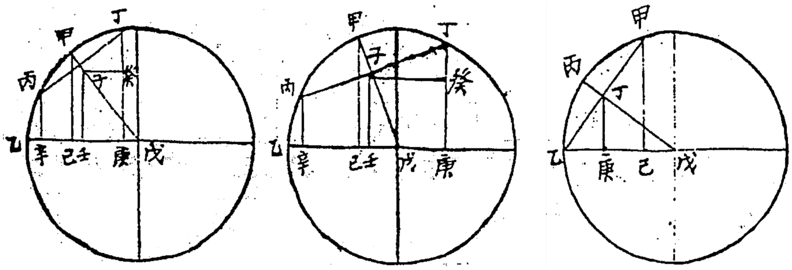

Figure 1. Left and center: Jo Hui-sun distinguishes the cases when the angle sum is acute and obtuse. Right: He also proved and used the half angle formula.

of Eight Segments to their logarithmic values and replacing multiplication and division with addition and subtraction proved to be highly efficient.[3] This required the use of logarithmic tables.

At the beginning of this chapter, Jo Hui-sun states:

> When calculating right spherical triangles, no method is more convenient than using logarithms. [. . .] However, the various ratios used in the *Zongjiao Method* has never been calculated using logarithms in the past. To address this, I have newly supplemented this method. I hope that all calculations of arcs in spherical triangles can now be performed through addition and subtraction.[4]

He then explains how to use logarithms to calculate *zhongshu* (中數) and *shishu* (矢數). Here, *sisu* appears to be a transcription error for *shijiao* (矢較) by the copyist.[5] This terminology arises in the *Zongjiao Method*. He explains the geometric differences between cases when the sum of two angles is obtuse and cases when it is acute, using diagrams to distinguish them. (See Figure 1 left and center.) This represents a more advanced logical structure compared to Hong Gil-ju's approach.[6]

First let us correspond the vertices of the spherical triangle 甲乙丙 to the roman characters A, B, and C, and their respective opposite sides (arcs) to a, b, and c.

Jo Hui-sun described the method for calculating the *zhongshu* (中數) as follows: In the left two diagrams of Figure 1, he overlapped arcs 甲乙 (c) and 甲丙 (b) at the common point 甲 and drew an additional arc 甲丁 of the same length as 甲丙 on the

3) In the West, the use of logarithmic scales for addition and subtraction persisted until the mid-20th century, when electronic calculators replaced them.

4) [Original text] 凡算正弧三角形莫便於對數. (中略) 惟摠較法所用諸率 古無徑求對數之法. 今爲撰補. 庶弧三角弧一切可以加減入術云.

5) The manuscript contains many transcription errors. The scribe who copied this text must have been unfamiliar with mathematical terminology, as evidenced by errors in mathematical terms. For example, the term right triangle (正弧) is incorrectly transcribed as arc 乙 (乙弧) in some instances.

6) In Hong Gil-ju's case, these two cases were either not distinguished or treated as entirely separate problems in his explanations.

opposite side of 甲. In this illustration, he explained that the *yuxian*[7] (餘弦) of the sum of the two arcs 乙丁 is 庚戊, and the *yuxian* of the difference of the two arcs 乙丙 is 辛戊. If 乙丁 is greater than a right angle, the sum of these two *yuxian*'s must be calculated, whereas if it is less than a right angle, their difference must be found. In both cases, the result 庚壬 is half of 庚辛, which is the *zhongshu*.

Additionally, he noted that since the two right spherical triangles 甲己戊 and 子癸丁 are similar, the following relationship holds:

$$\text{半徑} : \text{甲乙 正弦} = \text{甲丙 正弦} : \text{中數}, \quad \text{that is,} \quad 1 : \sin c = \sin b : (\text{zhongshu}).$$

He explained that this is because 甲己 $=$ 甲乙 正弦 and 子丁 $=$ 甲丙 正弦. In this way, he effectively proved the formula that converts the sum of *yuxian*'s into the product of sines.[8] For similarity, he used the term *tongshi xing* (同式形). He then demonstrated that by converting the *zhongshu* in the *Zongjiao Method* into the product of two sines, logarithms can be used to simplify the multiplication into an addition of logarithms.

In the equation above, the *zhongshu* is represented in modern notation, including its sign, as:

$$\frac{1}{2}\big(\cos(b-c) - \cos(b+c)\big).$$

Next, Jo explained the method for calculating the *shijiao* (矢較): He defined it as the difference between the versed sines of the arc (c) opposite the anble being solved and the *shijiao* $(c-b)$. The difference in versed sines corresponds to the difference in their cosine values, and thus, calculating the *shijiao* is equivalent to finding the *zhongshu* for these two arcs. Therefore, the method described earlier for calculating the *zhongshu* can be directly applied.

Then Jo explained the problem of calculating an angle from its versine (v) or cosine value, and vice versa, which arises in the final step of the *Zongjiao Method*. This corresponds to solving the following equation for θ in modern mathematics:

$$v = \operatorname{versin}\theta = 1 - \cos\theta = 2\sin^2\frac{\theta}{2}.$$

He stated, "From $v/2$, find the logarithm, add it to the logarithm of the radius, and then divide the sum by two; the result is the sine of half the desired angle."[9]

In other words, Jo accurately understood the process of solving general spherical triangles using the *Zongjiao Method*. He applied the formula that converts the sum of the *yuxian*'s of the *zonghu* (總弧, arc sum) and the *jiaohu* (arc difference, 較弧) into a product of sines. By using logarithms to transform this multiplication into addition,

7) Cosine in its absolute value.

8) In modern terms, this corresponds to converting the difference of two cosines into the product of two sines. In the diagram, the sum of two positive complements corresponds to the difference between a positive and a negative value in modern trigonometry.

9) [Original text] 半矢假數 與半徑假數 相加折半 爲半弧正弦假數.

he simplified the calculation. Similarly, after calculating each ratio, he derived the sine of half the desired angle and then applied the formula:

$$\operatorname{versin}\theta = 2\sin^2\frac{\theta}{2},$$

to transform the result into the square of the sine of half the angle, multiplied by two. Finally, instead of calculating the square root directly, he determined the logarithm and divided it by two to complete the process.

2.1 Hong Gil-ju's Proof of the *Zongjiao Method*

Jo Hui-sun introduced the *Zongjiao Method* calculation in the final problem of Chapter 3, Problem 3. However, the explanation provided there includes only the minimum necessary details for computation. A more detailed explanation of the *Zongjiao Method*, including the theory of spherical triangles and calculation methods, can be found in the introductory sections of *Lixiang Kaocheng*.

In Joseon, the *Zongjiao Method* was first introduced in Hong Gil-ju's *Hogag Yeonrye* (弧角演例), although the specific terms such as *Zongjiao Method*, *zhongshu*, or *shijiao* do not appear there. It seems that Hong Gil-ju, along with most Joseon scholars, focused their research on the spherical trigonometry in *Lixiang Kaocheng*.

Here, we follow Hong Gil-ju's solution to examine the calculation process and his proof. In *Hogag Yeonrye*, the problem of 'finding the three angles of a spherical triangle when its three sides (arcs) are given' appears as the final three problems among the 60 problems of general spherical triangles. Its dual problem—finding the three sides when the three angles are given—constitutes the first three problems among the 60.

Among the last three problems, the first sub-problem of Problem 20 (二十之一) involves determining angle 甲 of a spherical triangle 甲乙丙 from the lengths of its three sides. Hong Gil-ju assumes the case where the arc 甲乙 is smaller than 甲丙 ($\widehat{AB} < \widehat{AC}$). For the opposite case, the figure can simply be reversed. Note that the arrangement of the angles in Hong Gil-ju's diagram differs from that in Jo Hui-sun's.

The left diagram in Figure 2 reproduces Hong Gil-ju's original diagram, while the right diagram translates the points into Roman letters. The triangle in question is $\triangle ABC$, where the arcs opposite each angle are denoted as a, b, and c, respectively. The construction process of this diagram is as follows:

The circle centered at point A with radius equal to the arc $\widehat{AC}$ passes through points D and E. Thus, the arcs $\widehat{AC}$, $\widehat{AD}$, and $\widehat{AE}$ are all of equal length. Therefore, the sum $\widehat{AB}+\widehat{AC}$ equals $\widehat{BD}$, and their difference $\widehat{AC}-\widehat{AB}$ equals $\widehat{BE}$. These are the *zonghu* and *jiaohu* , respectively.

Let F denote the foot of the perpendicular dropped from O, the center of the sphere, to the line OB. Additionally, let P be the intersection of the diameter DE with the

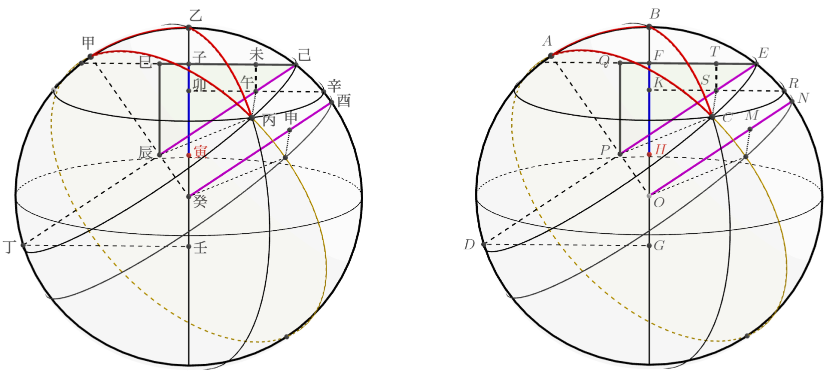

Figure 2. Left: Hong Gil-ju's diagram of the first sub-problem of Problem 20 of solving general spherical triangle. Right: Same diagram with points converted to Roman characters. Hong used this diagram to prove the *Zongjiao Method* of finding an angle from the three side arcs given.

segment AO, and Q the foot of the perpendicular dropped from P onto the line EF. Finally, let S be the foot of the perpendicular dropped from C onto DE, and T the foot of the perpendicular dropped from S onto EF. Then, triangles $\triangle EPQ$ and $\triangle EST$ are similar.

The great circle (the equator) corresponding to the point A as a pole passes through point N, and therefore the versine of angle A is equal to the segment MN. When the radius is set to 1, this value is given by:

$$\frac{\overline{MN}}{\overline{ON}} = \frac{\overline{SE}}{\overline{PE}} = \frac{\overline{ST}}{\overline{PQ}} = \frac{\overline{KF}}{\overline{HF}},$$

where H is the midpoint of segment FG. Here, $\overline{HF}$ is referred to as the *zhongshu* (中數), and $\overline{KF}$ as the *shijiao* (矢較). Once these values are determined, one can consult the trigonometric tables to find the value of angle $\angle A$.

Now, since

$$\overline{FG} = \overline{OG} + \overline{OF} = \cos\widehat{BD} + \cos\widehat{BE},$$

it follows that $\overline{FG}$ is the sum of the cosine values of the *zonghu* (總弧) and *jiaohu* (較弧). Half of this sum gives the value of $\overline{HF}$.

On the other hand:

$$\overline{KF} = \overline{BK} - \overline{BF} = \text{versin}\,\widehat{BC} - \text{versin}\,\widehat{BE},$$

which means that $\overline{KF}$ is the difference between the cosine value of the arc opposite angle A and that of the *jiaohu*. With this, all required values can be calculated.

In Jo Hui-sun's *Sanhag Seubyu*, only the formulas are presented without proofs, but he distinguishes between the cases where the angle sum is acute and where it is obtuse.

2.2 Lee Sang-hyeog and the *Zongjiao Method*

Another source demonstrating the widespread use of the *Zongjiao Method* in $19^{e}xtth$-century Joseon is Lee Sang-hyeog's work, *Gyuilgo*. Although Lee Sang-hyeog meticulously organized all the mathematics he studied, he only briefly addressed spherical trigonometry in the appendix of *Sansul Gwangan* (算術管見), where he explained only Napier's Analogies (不分線三率法). However, in his astronomical treatise *Gyuilgo*, the *Zongjiao Method* is explicitly employed to calculate the sun's altitude for each season and time.

In this astronomical problem, which exemplifies the application of spherical trigonometry, point B of Figure 2 represents the local zenith, A is the celestial north pole, and the horizontal equator represents the horizon. The tilted equator passing through point N represents the true equator, and the great circle containing arc AB represents the celestial meridian. At a specific date and time, the sun is located at point C, a vertex of the spherical triangle $\triangle ABC$, then the angle A corresponds to the hour angle, indicating the current time.

Meanwhile, arc AB represents the co-latitude of the location, and arc AC represents the co-declination, where the declination is the angular distance of the sun from the equator on that date.

Solving this spherical triangle involves calculating arc BC, the altitude of the sun above the horizon at point C, using the known values of arcs AB, AC, and angle A. Lee Sang-hyeog explained that this problem could be solved using the similar method described above.

3 Trigonometric Relationships of Eight Segments (八線相當)

The content of Chapter 5 consists of a series of formulas for spherical trigonometry.

At the beginning of this chapter, Jo Hui-sun references the Chinese text *Celiang Quanyi* (測量全義). This work, comprising ten volumes, is included in the *Chongzhen Lixue* (崇禎曆書). It was compiled in 1631 (the 4th year of Chongzhen) under the supervision of Xu Guangqi (徐光啓), with contributions from the Italian Jesuit Giacomo Rho (罗雅谷) and the German Jesuit Johann Adam Schall von Bell (湯若望). The book is an astronomical and mathematical treatise covering topics such as area, volume, plane and spherical trigonometry, and surveying instruments. Spherical geometry is addressed in Volumes 7 through 9.

Jo Hui-sun briefly noted: "*Celiang Quanyi* contains 12 proportional formulas for the eight segments of a right spherical triangle. I have added six more formulas to supplement it." The 12 formulas in *Celiang Quanyi* are for the case of a right spherical triangle where angle 甲 is a right angle, and 甲丙 and 乙丙 are given. The problem is

Table 1. This table lists the trigonometric formulas for right spherical triangles(八線相當) in Sanhag Seubyu. The numbering S is for Sanhag Seubyu and Q is for Celiang Qianyi.

S	Q		S	Q		S	Q	
1	1	$\frac{\sin c}{1} = \frac{\sin a}{\sin A}$	7	7	$\frac{\cos c}{\cot c} = \frac{\sin a}{\sin A}$	13	11	$\frac{\sec a}{\tan a} = \frac{\csc c}{\sin A}$
2	4	$\frac{1}{\csc c} = \frac{\sin a}{\sin A}$	8	8	$\frac{\cot c}{\cos c} = \frac{\csc a}{\csc A}$	14	10	$\frac{\tan a}{\sec a} = \frac{\sin c}{\csc A}$
4	5	$\frac{\csc c}{1} = \frac{\csc a}{\csc A}$	9		$\frac{\sec c}{\tan c} = \frac{\csc a}{\csc A}$	15	12	$\frac{\sin a}{\tan c} = \frac{\cos c}{\csc A}$
3	3	$\frac{\csc a}{1} = \frac{\csc c}{\sin A}$	10		$\frac{\tan c}{\sec c} = \frac{\sin a}{\sin A}$	16		$\frac{\csc a}{\sec c} = \frac{\cot c}{\sin A}$
5	2	$\frac{1}{\sin c} = \frac{\csc a}{\csc A}$	11		$\frac{\cos a}{\cot a} = \frac{\sin c}{\csc A}$	17	9	$\frac{\sin c}{\tan a} = \frac{\cos a}{\sin A}$
6	6	$\frac{\sin a}{1} = \frac{\sin c}{\csc A}$	12		$\frac{\cot a}{\cos a} = \frac{\csc c}{\sin A}$	18		$\frac{\csc c}{\cot a} = \frac{\sec a}{\csc A}$

to find angle 乙, with the given sides set to 30° and 11° 31′, respectively. The radius is denoted as 全 (whole), with a value of 10^5. However, the justification for each formula is not provided in *Celiang Quanyi*.

Jo Hui-sun listed all 18 formulas with numbered labels. In his setup, the triangle is defined as having a right angle at 丙, with 甲乙 and 乙丙 given, and the task is to find angle 甲. Unlike *Celiang Quanyi*, no specific numerical values are used, and proofs are provided for each formula. He based the derivation of the additional formulas on the three fundamental proportional equations introduced in Chapter 2, *Jeongho Yagbeob* (正弧約法). (See [12])

As in previous discussions, we correspond 甲乙丙 to $\triangle ABC$, with 甲乙 and 乙丙 denoted as c and a, respectively. The formulas recorded in *Sanhag Seubyu* are listed in order in Table 1.

4 Hyeonsi Cheobbeob

Chapter 6 titled *Hyeongsi Cheobbeob* (弦矢捷法) introduces methods for calculating the trigonometric values (弦矢) of an angle or an arc, and determining the angle or arc from given trigonometric values. Jo Hui-sun explains this technique using a single numerical example. At the beginning of the chapter, he mentions that this method was introduced in China by Du Demei (杜德美, Pierre Jartoux) in the *Chourenzhen* (籌人傳). While acknowledging that this formula enables highly accurate calculations, he notes that its explanation of how to multiply or divide successive powers of the radius when applying Taylor polynomials is insufficient. Thus, he provides detailed steps for the calculation.

The problem involves determining the angle or arc given its sine value. However, Jo Hui-sun immediately subtracts the given value from R (the radius) and refers to the result as the *zhengshi* (正矢, versed sine). This implies that the given value must be the cosine value (*yuxian*). With this adjustment, the rest of the description is consistent. The given equation is:

$$\text{Yuxian} = R\cos\theta = 7,071,068,$$

where $R = 10^7$.

He calculates the *zhengshi* as $R - 7,071,068 = 2,928,932$ and applies the Taylor polynomial for the versed sine. Defining $x = \theta^2$, he scales the polynomial coefficients to integers by multiplying by the least common multiple of the denominators, $40,320$. For the constant term, he divides the *zhengshi* by R to ensure the value is less than 1 before multiplying by $40,320$. He describes the resulting fractional values with a term *xiaoyu* (小餘, small remainder).

The resulting equation is:

$$40,320\frac{(\text{Jeongsi})}{R} = 40,320\left[\frac{1}{2}x - \frac{1}{24}x^2 + \frac{1}{720}x^3 - \frac{1}{40,320}x^4\right].$$

Simplifying this equation, he obtains:

$$11,809.453824 = 20,160x - 1,680x^2 + 56x^3 - x^4.$$

He notes the constant term as: "一萬一千八百〇九 小餘 四五三八二四." Finally, he concludes by stating: "Solve the cubic equation to find x, and then calculate the square root to obtain the angle."

In smaller print following this section, he elaborates on Du's polynomial expansion for calculating the cosine (*yuxian*) and explains that the above method is based on this formula. He describes the factorials in the Taylor coefficients as successive products of 2, 3, ... 7, using the phrase "相挨兩兩相乘" (multiply sequentially in pairs).

Jo Hui-sun's contribution was primarily to introduce Du Demei's polynomial for the versed sine and to emphasize that when solving the equation, the constant term must use the value of the complement divided by the radius rather than the complement itself. He does not elaborate further on solving the equation, as it seems to have been an accepted practice by this time. However, this method does not appear to have been used previously in China.

5 Conclusion

Jo Hui-sun, a military official during the late Joseon period, appears to have been a remarkably versatile scholar. He served three consecutive terms as the magistrate of Jeju, a position typically held for less than a year, and was so well-regarded for his care for the people that the residents of Jeju erected several monuments in his honor. According to records in the *Seungjeongwon Ilgi* (承政院日記), while serving as the magistrate of Juksan, Jo Hui-sun "donated grain for disaster relief, implemented lenient policies, restored order in a suffering region, and was consequently reappointed." It was further noted that "this region, among the most impoverished in the capital area, was restored thanks to the efforts of this excellent official."[10)]

Despite his reputation as a skilled and busy administrator, Jo Hui-sun not only found time to study mathematics but also demonstrated a level of expertise in the field comparable to Lee Sang-hyeog, a contemporary widely regarded as a genius in the field. Jo Hui-sun's mathematical acumen is evident in his concise and penetrating explanations, which reveal his structural understanding of the subject's core principles.

The mid-19$^{\text{th}}$ century marked a period in which the theoretical foundation for the *Shixian Calendar* was fully consolidated in Joseon. Nam Byeong-gil, Nam Byeong-cheol, and Lee Sang-hyeog systematically organized the mathematical knowledge introduced from China. While Nam Byeong-gil and Nam Byeong-cheol were accomplished astronomers, they may have faced limitations in practical computations, a gap filled by Lee Sang-hyeog. This contribution can be seen as giving Joseon mathematics a significant boost. Lee Sang-hyeog is the only well-known mathematician of the time and this raises a question about whether this remarkable progress depended solely on his individual brilliance. However, Jo Hui-sun's small book, *Sanhag Seubyu*, dispels such doubts. As a busy official who studied mathematics in his spare time, Jo Hui-sun not only mastered the mathematics required for the *Shixian Calendar* but also demonstrated achievements and capabilities on par with Lee Sang-hyeog. This indicates that other Joseon scholars of the time could, if necessary and indeed did, achieve similar levels of accomplishment.

In the final chapter of *Sanhag Seubyu*, *Saji Sanryak* (四之筭略), Jo Hui-sun writes: "The methods of *Tianyuan* have already reached a profound and sophisticated level, and their extension to *Siyuan* makes them even more intricate and grand, surpassing Western *Chageunbang Method* (差根方法)." He further notes that "while *Yangyuan* includes *Chageunbang*, the latter is limited in dealing with *Yangyuan*." Here, *Yangyuan* refers to a method involving two unknowns, a subset of the *Siyuan* method. This statement implies that Western methods for solving systems of two equations with two unknowns were seen as inefficient compared to *Siyuan*. Early in their collaboration, Nam Byeong-gil and Lee Sang-hyeog wrote *Muihae* (無異解) where they argued there is no distinction between the two approaches. Later, after studying *Siyuan*, they abandoned *Chageunbang* in favor of *Tianyuan* and *Siyuan*. Jo Hui-sun reached the same conclusion.

Jo Hui-sun also emphasized the importance of generalizing mathematical methods. He noted that mathematical problems often relied on the Pythagorean triplet

10) [Original text] 此邑 卽畿內極弊之局 得此良吏 蘇弊復完云.

(3, 4, 5), which could lead to doubts about whether the results were coincidental. He stressed the necessity of using problems with different ratios to ensure the general applicability of the methods. This focus on generalization demonstrates a structural and modern perspective.

As discussed in the previous paper, mathematics in late Joseon developed in close connection with the *Shixian Calendar*. The topic of spherical trigonometry, which had been rarely addressed in earlier Joseon mathematics, gained prominence among scholars for nearly a century starting in the late 18$^{\text{th}}$ century. This field reached its first fruition with Hong Gil-ju, was fully organized during the 30 to 50 years thereafter, and was finally consolidated by scholars such as Lee Sang-hyeog and Jo Hui-sun. This development diverged from Chinese mathematics. While methods such as the *Suho Method* (垂弧法), *Zongjiao Method* (總較法), and *Napier's Analogy* (不分線三率法) were utilized, the underlying computations relied heavily on counting rods (算帶). This reliance gave rise to techniques such as *Hyeonsi Cheobbeob*, which utilized *Tianyuan* instead of trigonometric tables. Although the *Shuli Jingyun* (數理精蘊) served as the foundation of late Joseon mathematics, practical computation methods diverged significantly, reflecting the unique circumstances of Joseon. These methods, emphasized by both Jo Hui-sun and Lee Sang-hyeog, must be regarded as distinctive features of Joseon mathematics.

The return to *Tianyuan* by Joseon scholars, despite their complete understanding of the *Shixian Calendar*, probably stemmed from the inefficiency of Western *Chageunbang*. In Joseon, where counting rods were widely used, there was no need to rely on laborious manual calculations. This was especially true for the Gwansanggam staff, who often performed large-scale computations. Furthermore, general scholars seemed to have had limited access to trigonometric tables, necessitating alternative methods like *Hyeonsi Cheobbeob*. Efforts to address resource limitations and enhance computational efficiency culminated in these methods, creating a system uniquely suited to Joseon's conditions.

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