

ON THE INTERNAL SUM OF PUISEUX MONOIDS

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ABSTRACT. In this paper, we investigate the internal (finite) sum of submonoids of rank-1 torsion-free abelian groups. These submonoids, when not groups, are isomorphic to nontrivial submonoids of the nonnegative cone of \mathbb{Q} , known as Puiseux monoids, and have been actively studied during the last few years. Here we study how the atomicity and arithmetic of Puiseux monoids behave under their internal (finite) sum inside the abelian group \mathbb{Q} . We study the factorization properties of such internal sums, giving priority to Cohn’s notion of atomicity and the classical bounded and finite factorization properties introduced and studied in 1990 by Anderson, Anderson, and Zafrullah in the setting of integral domains, and then generalized by Halter-Koch to commutative monoids. We pay special attention to how each of the considered properties behaves under the internal sum of a Puiseux monoid with a finitely generated Puiseux monoid. Throughout the paper, we also discuss examples showing that our primary results do not hold for submonoids of torsion-free abelian groups with rank larger than 1.

1. Introduction

Let G be an abelian group (additively written). The internal sum of two submonoids S and T of G , denoted by $S + T$, is the smallest submonoid of G containing the set $S \cup T$. The following is an explicit way to write the same internal sum:

$$S + T = \{s + t : s \in S \text{ and } t \in T\}.$$

It is clear that the property of being finitely generated is preserved under taking internal sums. Therefore, it is natural to wonder whether inside the group G (used as a universe), the internal sum of a submonoid of G with a finitely generated submonoid of G preserves a given property. More formally, we have the following question.

QUESTION 1.1. Given an algebraic property \mathfrak{p} and submonoids M and N of G such that M satisfies the property \mathfrak{p} and N is finitely generated, does the internal sum $M + N$ satisfy \mathfrak{p} ?

The answer to Question 1.1 heavily depends on the property \mathfrak{p} and the universe G . The properties we consider here are precisely those studied in the landmark paper [1], where Anderson, Anderson, and Zafrullah introduced the bounded and finite factorization properties in the class of atomic domains and proposed Diagram 1.1 as a

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methodology for a systematic investigation of the deviation of a given integral domain from being a UFD (Diagram 1.1 is adapted for the class of cancellative and commutative monoids, which is larger than the class of integral domains). We also consider two further properties that are closely connected to the property of being atomic and the property of finite factorization: these are strong atomicity and the length-finite factorization property, which we define later.

$$\begin{array}{ccccccc}
 \mathbf{UFM} & \implies & \mathbf{HFM} & & & & \\
 \Downarrow & & \Downarrow & & & & (1.1) \\
 \mathbf{FFM} & \implies & \mathbf{BFM} & \implies & \mathbf{ACCP\ monoid} & \implies & \mathbf{atomic\ monoid}
 \end{array}$$

The primary purpose of this paper is to answer Question 1.1 for the factorization properties in Diagram 1.1 and considering rank-1 torsion-free abelian groups as our universe. It is well known that every rank-1 torsion-free abelian group is isomorphic to a subgroup of \mathbb{Q} (see [6, Section 18]), and this allows us to fix the abelian group \mathbb{Q} as our universe. By virtue of [9, Theorem 3.12], every submonoid of \mathbb{Q} that is not a group is isomorphic to a submonoid of the nonnegative cone of \mathbb{Q} . Submonoids of $\mathbb{Q}_{\geq 0}$ are known as Puiseux monoids, and they have been actively investigated during the last few years, mostly in the setting of atomicity and factorization theory.

It is worth emphasizing that the abelian group \mathbb{Q} is perhaps the simplest universe where the algebraic properties of submonoids that we are interested in are not trivially preserved by taking the internal sum with a finitely generated submonoid. This statement is justified by the following two facts. First, the internal sum of two submonoids of \mathbb{Z} is either a group or a numerical monoid, and so such an internal sum automatically satisfies all the properties in Diagram 1.1 (with the exception of UFM, which is satisfied if and only if the internal sum is a cyclic monoid or a subgroup of \mathbb{Z}). Second, every submonoid of a finite abelian group is a subgroup, and so the internal sum of any two submonoids inside a finite abelian group is a group, and so it is trivially a UFM. The other potential simple universes that one may consider in Question 1.1 are the free abelian groups \mathbb{Z}^n (for $n \geq 2$), and throughout this paper we discuss examples to illustrate that the answers to Question 1.1 for the properties we are interested in here are negative even for submonoids of \mathbb{Z}^2 .

Although we will give a positive answer to Question 1.1 for most of the properties in Diagram 1.1 (except for UFM and HFM), we should mention that there are important algebraic properties for which the corresponding answers to Question 1.1 are negative even inside the universe \mathbb{Q} . Perhaps two of the most important properties are the property of being a UFM (the first property in Diagram 1.1) and the property of being a Krull monoid (perhaps the property most systematically investigated in the literature of arithmetic and factorization theory); indeed, even though $2\mathbb{N}_0$ and $3\mathbb{N}_0$ are UFM (and thus, Krull monoids), their internal sum, $\mathbb{N}_0 \setminus \{1\}$, is not even Krull (the Puiseux monoids that are Krull monoids are precisely the cyclic submonoids of \mathbb{Q} [15, Corollary 6.7]). It was also recently noted in [8, Example 3.4] that the internal sum of Puiseux monoids satisfying the ascending chain condition on principal ideals (ACCP) may not satisfy the ACCP.

Let M be a submonoid of an (additive) abelian group. Following Cohn [4], we say that M is *atomic* if every non-invertible element of M can be written as a sum of atoms (i.e., irreducible elements). Also, M is called *strongly atomic* if M is atomic and

any two elements of M have a maximal common divisor. In Section 3, we prove that inside any rank-1 torsion-free abelian group, the internal sum of an atomic (resp., a strongly atomic) monoid and a finitely generated monoid is an atomic (resp., a strongly atomic) monoid. This is the main result of Section 3, and we illustrate that this result cannot be generalized to submonoids of torsion-free abelian groups with higher rank. We also show that, even if we take a rank-1 torsion-free abelian group as our universe, the internal sum of atomic submonoids may be far from being atomic; indeed, it may contain no atoms (without being a group).

Now assume that the submonoid M is atomic. Following Anderson, Anderson, and Zafrullah [1] and Halter-Koch [18], we say that M is a *finite factorization monoid* (FFM) if every non-invertible element of M has a finitely many factorizations (i.e., can be written as a sum of atoms in essentially finitely many ways) while we say that M is a *bounded factorization monoid* (BFM) if for each non-invertible element of M there exists a positive upper bound such that the number of atoms in any factorization of such element (counting repetitions) is below that bound. Following Geroldinger and Zhong [11], we say that M is a *length-finite factorization monoid* (LFFM) if for every positive integer ℓ , any non-invertible element of M has only finitely many factorizations with exactly ℓ atoms (counting repetitions). In Section 4, we prove that inside any rank-1 torsion free abelian group, the internal sum of an FFM (resp., a BFM, an LFFM) and a finitely generated monoid is again an FFM (resp., a BFM, an LFFM). We also illustrate that our theorem cannot be generalized to abelian groups with higher rank, and we provide examples of two Puiseux monoids that are BFMs whose internal sum is not even atomic, as well as an example of two rank-2 monoids that are FFMs whose internal sum is not atomic.

2. Background

As it is customary, we let \mathbb{Z} , \mathbb{Q} , and \mathbb{R} denote the set of integers, rational numbers, and real numbers, respectively. In addition, we let \mathbb{P} , \mathbb{N} , and \mathbb{N}_0 denote the set of primes, positive integers, and nonnegative integers, respectively. For $b, c \in \mathbb{Z}$, we denote the discrete closed interval from b to c by $\llbracket b, c \rrbracket$; that is

$$\llbracket b, c \rrbracket := \{n \in \mathbb{Z} : b \leq n \leq c\}.$$

If $q \in \mathbb{Q} \setminus \{0\}$, then $\mathbf{n}(q)$ and $\mathbf{d}(q)$ are, respectively, the unique $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ such that $q = \frac{n}{d}$ and $\gcd(n, d) = 1$. For a nonzero $n \in \mathbb{Z}$ and $p \in \mathbb{P}$, we let $v_p(n)$ denote the p -adic valuation of n , $v_p(n) = \max\{m \in \mathbb{N}_0 : p^m \mid n\}$. Then, for $p \in \mathbb{P}$, the p -adic valuation map $v_p: \mathbb{Q} \setminus \{0\} \rightarrow \mathbb{Z}$ is the function defined by the assignment $q \mapsto v_p(\mathbf{n}(q)) - v_p(\mathbf{d}(q))$ for each $q \in \mathbb{Q} \setminus \{0\}$.

Given that all semigroups we consider here are cancellative and commutative, throughout this paper, a *monoid*¹ is a semigroup with an identity element. Also, unless otherwise specified, monoids here will be denoted additively with identity element denoted by 0 , which we will refer to as the zero element through this paper. We assume that submonoids inherit identity elements and also that monoid homomorphisms respect identity elements. Let M be a monoid with identity element 0 . We let M^\bullet denote the set of nonzero elements of M . The abelian group consisting of all invertible elements of M is denoted by $\mathcal{U}(M)$, and M is called *reduced* provided

¹The most standard definition of a monoid does not assume either cancellation or commutativity.

that the abelian group $\mathcal{U}(M)$ is trivial. One can readily check that the quotient $M/\mathcal{U}(M)$ is a reduced monoid, and we denote this quotient by M_{red} .

For $b, c \in M$, we say that c divides b in M if we can write $b = c + d$ for some $d \in M$, in which case we write $c \mid_M b$ and use $b - c$ to denote d . Two elements $b, c \in M$ are called *associates* if $b \mid_M c$ and $c \mid_M b$. A *maximal common divisor* of a nonempty subset S of M is a common divisor $d \in M$ such that the only common divisors of the set $\{s - d : s \in S\}$ are the invertible elements of M . We say that M is *2-MCD* if any two elements of M have a maximal common divisor.

A non-invertible element $a \in M$ is called an *atom* provided that, for all $b, c \in M$, the equality $a = b + c$ implies that either $b \in \mathcal{U}(M)$ or $c \in \mathcal{U}(M)$. The set consisting of all atoms of M is denoted by $\mathcal{A}(M)$. If $\mathcal{A}(M)$ is empty, then the monoid M is called *antimatter*. The additive monoid $\mathbb{Q}_{\geq 0}$ is a simple example of an antimatter monoid; indeed, every rational $q \in \mathbb{Q}_{>0}$ is the sum of two copies of $\frac{q}{2}$ in $\mathbb{Q}_{\geq 0}$, thus $\mathbb{Q}_{\geq 0}$ contains no atoms. An element $q \in M$ is called *atomic* if either $q \in \mathcal{U}(M)$ or q can be written as a sum of finitely many atoms of M . Following [4], we say that the monoid M is *atomic* if every element of M is atomic. One can readily check that M is atomic if and only if M_{red} is atomic. Following [1], we say that M is a *strongly atomic* monoid if M is simultaneously atomic and 2-MCD. See [5] for a recent survey on atomicity in the class of integral domains and [12] for a survey on factorization theory in the class of commutative monoids. For a generous background on factorization theory in atomic monoids and domains, see [10].

Now suppose that M is an atomic monoid. For $q \in M \setminus \mathcal{U}(M)$ and $a_1, \dots, a_\ell \in \mathcal{A}(M)$ such that $q = a_1 + \dots + a_\ell$, we call the formal sum of (possibly repeated) atoms $a_1 + \dots + a_\ell$ a *factorization* of q . Two factorizations of the same element are considered the same up to permutations of their atoms and replacements of some of their atoms by some of their corresponding associates. We further assume that each element of $\mathcal{U}(M)$ has a unique factorization, namely, 0. For each $q \in M$, we let $Z_M(q)$ denote the set of all factorizations of q in M , and we drop the subscript M from $Z_M(q)$ when we see no risk of ambiguity (in particular, $Z_M(u) := \{0\}$ for any $u \in \mathcal{U}(M)$). We say that M is a *finite factorization monoid* (FFM) if every element of M has only finitely many factorizations, while we say that M is a *unique factorization monoid* (UFM) if every element of M has a unique factorization.

For any a_1, \dots, a_ℓ , the factorization $z := a_1 + \dots + a_\ell$ is said to have *length* ℓ , and we often write the length of z as $|z|$. For each $q \in M$, we set

$$\mathbf{L}_M(q) := \{|z| : z \in Z(q)\}.$$

As for sets of factorizations, we drop the subscript M from $\mathbf{L}_M(q)$ when there seems to be no risk of ambiguity. The monoid M is called a *bounded factorization monoid* (BFM) if $\mathbf{L}(q)$ is finite for every $q \in M$. It follows directly from the definitions that every FFM is a BFM. See [2] for a survey on the bounded and finite factorization property in the class of integral domains. Another weaker notion of the finite factorization is the length-finite factorization property, which we proceed to introduce. For each $q \in M$ and $\ell \in \mathbb{N}$, we set

$$Z_\ell(q) := \{z \in Z(q) : |z| = \ell\}.$$

Then we say that M is a *length-finite factorization monoid* (LFFM) provided that for each $q \in M$ the set $Z_\ell(q)$ is finite for every $\ell \in \mathbb{N}$. It follows directly from the corresponding definitions that every FFM is an LFFM. On the other hand, it is known

that the notions of BFM and LFFM are not comparable (we will discuss examples illustrating these observations in Section 4). See [7] for a survey on sets of lengths.

3. Atomicity

In this section, we study how the property of being atomic behaves under the internal sum in the class of Puiseux monoids. Let us begin by proving that atomicity and strong atomicity are both preserved under the internal sum with a finitely generated Puiseux monoid.

THEOREM 3.1. *Let M and N be Puiseux monoids such that N is finitely generated. Then the following statements hold.*

1. *If M is atomic, then $M + N$ is atomic.*
2. *If M is strongly atomic, then $M + N$ is strongly atomic.*

Proof. Since N is finitely generated, we can assume that N is a cyclic Puiseux monoid and then extend to any finitely generated monoid inductively. Let r be a positive rational, and set $S := M + \mathbb{N}_0 r$. If $r \in M$, then $S = M$ and so S is atomic (resp., strongly atomic) provided that M is atomic (resp., strongly atomic). Therefore, we assume that $r \notin M$.

(1) Suppose that M is atomic. Since $\mathcal{A}(M)$ generates M , it follows that $\mathcal{A}(M) \cup \{r\}$ is a generating set of S . Thus, it remains to show that every element in $A := \mathcal{A}(M) \cup \{r\}$ is atomic in S . Because r is the minimum nonzero element of $\mathbb{N}_0 r$, the fact that $r \notin M$ guarantees that $r \in \mathcal{A}(S)$, and so r is atomic. To argue that every atom of M is atomic in S , fix $a \in \mathcal{A}(M)$. If a is an atom in S , then there is nothing to do. Therefore we assume that a is not an atom in S . Thus, r properly divides a in S . Let m be the largest positive integer such that $mr \mid_S a$. Then $a - mr \in M$, and so the fact that M is atomic ensures the existence of $a_1, \dots, a_k \in \mathcal{A}(M)$ such that $a = (a_1 + \dots + a_k) + mr$. It follows now from the maximality of m that $r \nmid_S a_i$ for any $i \in \llbracket 1, k \rrbracket$; hence, a_1, \dots, a_k remain atoms in S . This, along with the fact that r is atomic in S , guarantees that a is atomic in S . Hence the Puiseux monoid S is atomic.

(2) Now suppose that M is strongly atomic. As M is atomic, it follows from part (1) that S is also atomic. Thus, it is enough to prove that S is a 2-MCD monoid, that is, any two elements of S have a maximal common divisor. As before, the fact that $r \notin M$ ensures that $r \in \mathcal{A}(S)$. For each $c \in S$, we let m_c denote the largest nonnegative integer such that $m_c r \mid_S c$, which must exist as r is positive. Using this notation, each $c \in S$ can be written as $c = b + m_c r$ for some $b \in S$ such that $r \nmid_S b$.

CLAIM. For any $x, y \in S$ with $m_x = 0$ the set $\{x, y\}$ has a maximal common divisor in S .

PROOF OF CLAIM. We proceed by induction on m_y . If $m_y = 0$, then all the divisors of x and y belong to M , so the fact that M is a 2-MCD guarantees that $\{x, y\}$ has a maximal common divisor in M , and thus in S as well. Now suppose that $m_y > 0$, and suppose that the statement of the claim holds for any subset $\{x', y'\}$ of S with $m_{x'} = 0$ and $m_{y'} < m_y$. Write $y = y' + m_y r$ for some $y' \in S$ such that $r \nmid_S y'$. Then each divisor of y' in S is also a divisor of y' in M . Let d_1 be a maximal common divisor of x and y' in M , which must exist because M is a 2-MCD monoid. If $x - d_1$ and

$y - d_1$ have no non-invertible (nonzero in the context of Puiseux monoids) common divisor in S , then d_1 is a maximal common divisor of x and y in S , and we are done. Otherwise, let d_2 be a nonzero common divisor of $x - d_1$ and $y - d_1$ in S . We know that $d_2 \nmid_M y' - d_1$; thus, because each divisor of y' in S must belong to M , we must have $d_2 \nmid_S y' - d_1$. This means that $m_{y-d_1-d_2} < m_y$, since

$$y - d_1 - d_2 - m_y r = y' - d_1 - d_2 \notin S.$$

It follows now from our induction hypothesis that the set $\{x - d_1 - d_2, y - d_1 - d_2\}$ has a maximal common divisor in S , implying that $\{x, y\}$ has a maximal common divisor in S . Hence the claim is established.

Now consider any x, y in S with $m_x > 0$. Without loss of generality, we can assume that $m_x \leq m_y$. Because $m_{x-m_x r} = 0$, it follows from our established claim that the set $\{x - m_x r, y - m_x r\}$ has a maximal common divisor in S . Thus, $\{x, y\}$ also has a maximal common divisor in S . Hence, we conclude that the Puiseux monoid S is strongly atomic. \square

However, if we consider the abelian group \mathbb{Z}^2 as our universe, it is not true that the internal sum of a strongly atomic submonoid of \mathbb{Z}^2 and a finitely generated submonoid of \mathbb{Z}^2 is an atomic monoid. The following example sheds some light upon this observation.

EXAMPLE 3.2. We describe two atomic submonoids M_1 and M_2 of the free abelian group \mathbb{Z}^2 whose internal sum in \mathbb{Z}^2 is not atomic, even though M_1 is finitely generated. Let M_1 be the monoid consisting of all the lattice points in the first quadrant of \mathbb{R}^2 , that is, $M_1 := \mathbb{N}_0 \times \mathbb{N}_0$. Observe that M_1 is the free commutative monoid of rank 2 and, therefore, M_1 is atomic with $\mathcal{A}(M_1) = \{e_1, e_2\}$, where $e_1 := (1, 0)$ and $e_2 := (0, 1)$. In particular, M_1 is generated by the finite set $\{e_1, e_2\}$. Now suppose that M_2 is the submonoid of \mathbb{Z}^2 whose set of nonzero elements are all the lattice points strictly above the x -axis, that is $M_2 := (0, 0) \cup \mathbb{Z} \times \mathbb{N}$. One can readily check that $\mathcal{A}(M_2) = \{(n, 1) : n \in \mathbb{Z}\}$, which immediately implies that M_2 is also atomic. In fact, if we take any two $u_1 := (x_1, y_1), u_2 := (x_2, y_2) \in M_2$ and assume without loss of generality that $y_1 < y_2$, then u_1 is a maximal common divisor of u_1 and u_2 , because it is clear that $u_1 \mid_{M_2} u_2$ and $u_1 - u_1 = 0$ do not have reducible divisors. This means M_2 is strongly atomic. Now let M be the internal sum in \mathbb{Z}^2 of M_1 and M_2 ; that is, $M := M_1 + M_2$. It is clear that $M_1 \cup M_2 \subseteq M$. On the other hand, for each nonzero $v := (x, y) \in M_2$, the fact that $y \in \mathbb{Z}_{\geq 1}$ ensures that the inclusion $M_1 + v \subseteq M_2$ holds. Therefore

$$M = M_1 \cup \left(\bigcup_{v \in M_2^\bullet} (M_1 + v) \right) \subseteq M_1 \cup M_2.$$

Hence $M = M_1 \cup M_2$, from which we see that M is the nonnegative cone of \mathbb{Z}^2 under the lexicographical order with priority on the second coordinate. As a result, the only atom of M is the minimum of M^\bullet , which means that $\mathcal{A}(M) = \{e_1\}$. As a consequence, M is not atomic: indeed, the set of atomic elements of M is $\{(m, 0) : m \in \mathbb{N}_0\}$, which is strict subset of M .

We conclude this section, showing that the internal sum inside \mathbb{Q} of two atomic Puiseux monoids may not be atomic. Indeed, we will provide a more drastic construction: two atomic Puiseux monoids whose internal sum is antimatter.

EXAMPLE 3.3. Let us construct two atomic Puiseux monoids M_1 and M_2 such that $M_1 + M_2$ is antimatter. Let p_n denote the n -th odd prime, and set $a_n := \frac{1}{2^n p_n}$ for every $n \in \mathbb{N}$. Now set $S_1 := \{a_n : n \in \mathbb{N}\}$, and let M_1 be the Puiseux monoid generated by S_1 . It is routine to argue that M_1 is atomic with $\mathcal{A}(M_1) = S_1$ (the Puiseux monoid M_1 is the main ingredient in Grams’s construction of the first atomic domain not satisfying the ACCP, and due to this historical fact M_1 is often called Grams’ monoid).

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be an injective function such that $p_{f(n)} > 2^n p_n$ for all $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, we construct an infinite sequence $(b_i)_{i \geq 1}$ such that

$$b_1 := a_n - a_{f(n)} > \frac{1}{2}a_n \quad \text{and} \quad b_{i+1} := b_i - \frac{1}{2^{c_i}},$$

where $c_i \in \mathbb{N}$ is chosen such that $b_{i+1} > \frac{1}{2}a_n$. We will refer to the elements in $(b_i)_{i \geq 1}$ as n -atoms. Let S_2 be the union of the sets of n -atoms over all n , and let M_2 be the Puiseux monoid generated by S_2 .

CLAIM. M_2 is atomic with $\mathcal{A}(M_2) = S_2$.

PROOF OF CLAIM. Suppose by way of contradiction that there exists an element in S_2 that can be expressed as the finite sum of other elements in S_2 . Suppose that this element is an n -atom, b_k , for some $n \in \mathbb{N}$. Then

$$b_k = \sum_{s \in S'} s$$

for some multiset $S' \subset S_2 \setminus \{b_k\}$ with $|S'| < \infty$.

Because $\nu_{p_n}(b_k) < 0$, the set S' must also contain elements with negative p_n -adic valuation. The only such elements in S_2 are either n -atoms or m -atoms for the at most one value of m such that $f(m) = n$. We split the rest of the argument into two cases.

Case i: If S' contains the n -atom b_i , note that we can have at most one n -atom in S' , since $b_i + b_j \geq 2b_j > b_1 \geq b_k$ for all $j \geq i$. We can write

$$b_k - b_i = \sum_{s \in S' \setminus \{b_i\}} s,$$

where $b_k - b_i$ is a sum of powers of $\frac{1}{2}$. Furthermore, $b_k - b_i < b_i \leq b_1 < \frac{1}{2}$.

Let $\ell \in \mathbb{N}$ be such that S' contains ℓ -atoms but does not contain $f(\ell)$ -atoms; this is possible because S' is finite. Since the denominator of $b_k - b_i$ is not divisible by any odd primes, there must exist at least $p_{f(\ell)}$ ℓ -atoms in S' . Since all ℓ -atoms are greater than $\frac{1}{2}a_\ell$, the total sum of these ℓ -atoms in S' is greater than $p_{f(\ell)} \cdot \frac{1}{2}a_\ell = \frac{1}{2} \left(\frac{p_{f(\ell)}}{2^\ell p_\ell} \right) > \frac{1}{2}$ (recall that $p_{f(\ell)} > 2^\ell p_\ell$). Thus

$$\sum_{s \in S' \setminus \{b_i\}} s > \frac{1}{2} > b_k - b_i,$$

a contradiction.

Case ii: Otherwise, we must have at least $p_n - 1$ m -atoms in S' . Note that

$$\sum_{s \in S'} s = b_k \leq b_1 = a_n - a_{f(n)} \leq a_n < \frac{1}{2}.$$

However, since all m -atoms are greater than $\frac{1}{2}a_m$, the sum of these $p_n - 1$ m -atoms in S' is already greater than $\frac{1}{2} \left(\frac{p_n - 1}{2^m p_m} \right) \geq \frac{1}{2}$. This dictates that

$$\sum_{s \in S'} s > \frac{1}{2},$$

a contradiction.

Since both cases yield a contradiction, all elements of S_2 must be atoms. Hence M_2 is atomic, and our claim is established.

We can finally argue that the Puiseux monoid $M := M_1 + M_2$ is antimatter despite being the internal sum of two atomic Puiseux monoids. To do so, first observe that any $a_n \in S_1$, which is an atom of M_1 , is divisible in M by any n -atom and therefore cannot be an atom of M . Also, any n -atom $b_k \in S_2$, which is an atom of M_2 , is divisible in M by b_{k+1} , which means it cannot be an atom of M . Thus, none of the generators of M is irreducible, which clearly implies that the Puiseux monoid M is antimatter.

4. The Bounded and Finite Factorization Properties

As the following theorem indicates, inside the class of Puiseux monoids the properties of being an FFM, a BFM, or an LFFM are preserved under the internal sum with a finitely generated monoid (in Example 4.4 we show that the same properties are not preserved under the internal sum with a finitely generated monoid in the class consisting of all submonoids of \mathbb{Z}^2).

THEOREM 4.1. *Let M and N be Puiseux monoids such that N is finitely generated, then the following statements hold.*

1. *If M is an FFM, then $M + N$ is an FFM.*
2. *If M is a BFM, then $M + N$ is a BFM.*
3. *If M is an LFFM, then $M + N$ is an LFFM.*

Proof. Since N is finitely generated, it suffices to consider the case when N is a cyclic monoid. To do so, let r be a positive rational and consider the Puiseux monoid $S := M + \mathbb{N}_0 r$. In light of Theorem 3.1, the monoid S is atomic if M is either an FFM, BFM, or LFFM. Observe that if $r \in M$, then $S = M$ and so S is an FFM (resp., a BFM or an LFFM) if and only if M is an FFM (resp., a BFM or an LFFM). Therefore we assume that $r \notin M$.

(1) Suppose that M is an FFM, and let us argue that S is an FFM. As S is atomic, it suffices to show that every element of S has only finitely many factorizations. Fix a nonzero $s \in S$. Since $r \notin M$, the fact that r is the minimum of $\mathbb{N}r$ implies that $r \in \mathcal{A}(S)$. Now notice that there are only finitely many $c \in \mathbb{N}_0$ such that $s - cr \in M$: let them be c_1, \dots, c_n . Since every factorization of s in S has the form $(a_1 + \dots + a_m) + c_k r$, it follows that

$$Z_S(s) = \bigcup_{k=1}^n (Z_M(s - c_k r) + c_k r). \quad (4.1)$$

Now the fact that M is an FFM ensures that $|\mathbf{Z}_M(s - c_k r)| \leq \infty$ for every $k \in \llbracket 1, n \rrbracket$. As a consequence, $\mathbf{Z}_S(s)$ is the finite union of finite sets, and therefore is itself finite. Hence we conclude that the Puiseux monoid S is an FFM.

(2) Now suppose that M is a BFM, and let us show that S is also a BFM. To do so, fix a nonzero $s \in S$. As in the proof of part (1), we can see that $r \in \mathcal{A}(S)$ and we can check that there are only finitely many $c \in \mathbb{N}_0$ such that $s - cr \in M$, which we denote by c_1, \dots, c_n . Observe that the equality (4.1) still holds, and it implies that $\mathbf{L}_S(s) = \bigcup_{k=1}^n (\mathbf{L}_M(s - c_k r) + c_k)$. Now the fact that M is a BFM ensures that $|\mathbf{L}_M(s - c_k r)| \leq \infty$ for every $k \in \llbracket 1, n \rrbracket$, and this implies that the set $\mathbf{L}_S(s)$ is finite. Hence the Puiseux monoid S is a BFM.

(3) Finally, suppose that M is an LFFM. Assume, for the sake of contradiction, that there exist $\ell \in \mathbb{N}$ and an element $q \in S$ with infinitely many factorizations of length ℓ . As in the previous two parts, $r \in \mathcal{A}(S)$. Since M is an LFFM, the element r must appear in infinitely many of these length- ℓ factorizations. However, since $r > 0$, we know that r can only divide q a finite number of times, say $f \in \mathbb{N}_0$. As a result, there must be an $i \in \mathbb{N}$ with $i \leq f$ and $0 < i < \ell$ such that infinitely many length- ℓ factorizations have exactly i occurrences of the factor r and the other factors are all in M . This implies that $q - ir \in M$ has infinitely many factorizations of length $\ell - i$, which is a contradiction. \square

In order to argue that each statement in Theorem 4.1 is sharp, we proceed to construct two Puiseux monoids that are FFMs whose internal sum is not even an LFFM, and then we construct two Puiseux monoids that are BFMs whose internal sum is not even atomic.

EXAMPLE 4.2. For each $n \in \mathbb{N}$, we let p_n denote the n -th prime number in the set $\mathbb{P}_{\geq 5}$. Now we consider the sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ whose terms are the positive rationals defined as follows:

$$a_n := \frac{p_n - 1}{p_n} \text{ and } b_n := \frac{p_n + 1}{p_n}$$

for every $n \in \mathbb{N}$. Now consider the Puiseux monoids M_A and M_B respectively generated by the sets $A := \{a_n : n \in \mathbb{N}\}$ and $B := \{b_n : n \in \mathbb{N}\}$. It is routine to verify that $\mathcal{A}(M_A) = A$ and $\mathcal{A}(M_B) = B$, and so M_A and M_B are atomic monoids.

Let us show that both M_A and M_B are FFMs. Because the sequence $(a_n)_{n \geq 1}$ is an increasing sequence, it follows from [16, Theorem 5.6] that M_A is an FFM. As M_B is an atomic reduced monoid, it follows from [2, Proposition 3.6] that M_B is an FFM if and only if every nonzero element of M_B is divisible by only finitely many atoms. To argue this equivalent condition, let $q \in M_B$ be a nonzero element. Take $N \in \mathbb{N}$ such that $p_n \nmid \mathbf{d}(q)$ and $p_n > q$ for any $n \geq N$.

CLAIM. $b_n \nmid_{M_b} q$ for any $n \geq N$.

PROOF OF CLAIM. Suppose, by way of contradiction, that $b_k \mid_{M_b} q$ for some $k \geq N$. In this case, the fact that M_B is atomic with $\mathcal{A}(M_B) = \{b_n : n \in \mathbb{N}\}$ allows us to write

$$q = \sum_{n=1}^{\ell} c_n \frac{p_n + 1}{p_n} \tag{4.2}$$

for some coefficients $c_1, \dots, c_{\ell} \in \mathbb{N}_0$ such that $k \leq \ell$ and $c_k > 0$. Since $k \geq N$, it follows that $p_k \nmid \mathbf{d}(q)$. Therefore the p_k -adic valuation of q is nonnegative and so,

after applying the p_k -adic valuation map to both sides of (4.2), we obtain that $p_k \mid c_k$. Thus, $q \geq \frac{c_k}{p_k}(p_k + 1) \geq p_k + 1$, which contradicts the fact that $k \geq N$. Hence the claim is established.

From the established claim, we obtain that q is only divisible by some of the atoms b_1, \dots, b_{N-1} , and so the Puiseux monoid M_B is also an FFM.

Finally, consider the Puiseux monoid $M := M_A + M_B$. It is clear that $\mathcal{A}(M) \subseteq \mathcal{A}(M_A) \cup \mathcal{A}(M_B) = \{a_n, b_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, the fact that $p_n \in \mathbb{P}_{\geq 5}$ implies that $\frac{3}{4} \leq a_n \leq 1$ and $1 \leq b_n \leq \frac{5}{4}$. This, along with the fact that all the atoms of M belong to $\{a_n, b_n : n \in \mathbb{N}\}$, guarantees that

$$\mathcal{A}(M) = \{a_n, b_n : n \in \mathbb{N}\} = \left\{ \frac{p_n \pm 1}{p_n} : n \in \mathbb{N} \right\}.$$

Also, as 0 is not a limit point of $\mathcal{A}(M)$, it follows from [16, Proposition 4.5] that M is a BFM. In particular, M is atomic. On the other hand, the equality

$$2 = \frac{p_n - 1}{p_n} + \frac{p_n + 1}{p_n} = a_n + b_n$$

holds for every $n \in \mathbb{N}$, which implies that 2 has infinitely many length-2 factorizations in M . As a consequence, the monoid M is not an FFM. Hence M_A and M_B are two FFMs whose internal sum is not an FFM.

Now we illustrate that the internal sum of two Puiseux monoids that are BFMs may not be a BFM. This example is motivated by [8, Example 3.4], where the authors show that the internal sum of two Puiseux monoids satisfying the ascending chain condition on principal ideals may not satisfy the same condition.

EXAMPLE 4.3. We will produce two Puiseux monoids M_1 and M_2 that are BFMs such that their sum $M_1 + M_2$ is not even atomic. Set $M_1 := \{0\} \cup \mathbb{Q}_{\geq 1}$. Because 0 is not a limit point of M_1^\bullet , it follows from [16, Proposition 4.5] that M_1 is a BFM. Now consider the Puiseux monoid M_2 defined as follows:

$$M_2 := \left\langle \frac{p_n + 1}{p_n^2} : n \in \mathbb{N} \right\rangle,$$

where p_n is the n -th prime. It is routine to check that $\mathcal{A}(M_2) = \{ \frac{p_n + 1}{p_n^2} : n \in \mathbb{N} \}$, which immediately implies that M_2 is an atomic monoid. Moreover, we can actually prove that M_2 is a FFM by mimicking the argument used to show that M_B is a FFM in Example 4.2. Thus, M_2 is a BFM.

Now let M be the internal sum of M_1 and M_2 inside \mathbb{Q} ; that is, $M = M_1 + M_2$. We claim that M is not atomic. First, note that because $\min M_1^\bullet = 1 > \sup \mathcal{A}(M_2)$, no nonzero element of M can divide any of the atoms of M_2 in M , which implies that $\mathcal{A}(M_2) \subseteq \mathcal{A}(M)$. On the other hand, we observe that if we take $q \in M_1$ with $q > 1$, the fact that $\lim_{n \rightarrow \infty} \frac{p_n + 1}{p_n^2} = 0$ allows us to take $n \in \mathbb{N}$ large enough so that $\frac{p_n + 1}{p_n^2} < q - 1$, in which case the equality

$$q = \left(q - \frac{p_n + 1}{p_n^2} \right) + \frac{p_n + 1}{p_n^2} \in M^\bullet + M^\bullet$$

ensures that $q \notin \mathcal{A}(M)$. Hence $\mathcal{A}(M) \subseteq \{1\} \cup \{ \frac{p_n + 1}{p_n^2} : n \in \mathbb{N} \}$. This in turn implies that the element $\frac{9}{8} \in M$ is not atomic because if $r \in \langle \{1\} \cup \{ \frac{p_n + 1}{p_n^2} : n \in \mathbb{N} \} \rangle$, then the denominator of r cannot be divisible by the cube of any prime. As a result, the

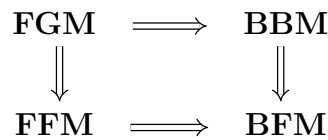
Puiseux monoid $M_1 + M_2$ is not atomic (and so not a BFM), even though M_1 and M_2 are BFMs.

For torsion-free abelian groups of rank larger than 1, we can find submonoids that are FFMs whose internal sum is not even atomic.

EXAMPLE 4.4. We describe two submonoids M_1 and M_2 of the free abelian group \mathbb{Z}^2 , both FFMs, whose internal sum in \mathbb{Z}^2 is not even atomic. Let M_1 be the monoid consisting of all the lattice points on the x -axis to the right of the origin; that is, $M_1 := \mathbb{N}_0 \times \{0\}$. Observe that M_1 is the free commutative monoid of rank 1 and, therefore, M_1 is an FFM with $\mathcal{A}(M_1) = \{e_1\}$, where $e_1 = (1, 0)$. Now suppose that M_2 is the submonoid of \mathbb{Z}^2 whose set of nonzero elements are all the lattice points to the left of the y -axis and strictly above the x -axis, that is $M_2 := (0, 0) \cup (\mathbb{Z} \setminus \mathbb{N}) \times \mathbb{N}$. One can check that $\mathcal{A}(M_2) = \{(-n, 1) : n \in \mathbb{N}_0\}$, implying that M_2 is also atomic. In fact, if we take any nonzero $u := (-x, y) \in M_2$, the only atoms of M_2 that can divide u are $(-n, 1)$ for natural numbers $n \leq x$. This means that M_2 is a reduced atomic monoid where every nonzero element is divisible by only finitely many atoms. Thus, it follows from [18, Theorem 2] that M_2 is an FFM. Now let M be the internal sum in \mathbb{Z}^2 of M_1 and M_2 , with $M := M_1 + M_2$. As in Example 3.2, we now find that M is the nonnegative cone of \mathbb{Z}^2 under the lexicographical order (on the second coordinate), and so it is not even atomic.

5. Final Remarks on the Main Results

In our main results, Theorems 3.1 and 4.1, we have chosen finitely generated monoids because they are some of the most standard and basic monoids studied in monoid and semigroup theory. One may wish to extend both theorems by replacing finitely generated monoids by a larger and natural class of monoids. It follows from [10, Proposition 2.7.8] that every finitely generated monoid is an FFM, and so a BFM. On the other hand, it follows from [16, Proposition 4.5] that every Puiseux monoid M is a BFM provided that 0 is not a limit point of M^\bullet . We call a Puiseux monoid M a *bounded below monoid* (BBM) if 0 is not a limit point of M^\bullet . Therefore every Puiseux monoid that is a BBM is also a BFM, and so we obtain the following diagram of implications in the class of Puiseux monoids:



where FGM stands for “finitely generated monoid”. Observe that the Puiseux monoids M_A and M_B in Example 4.2 are both FFMs and BBMs but they are not finitely generated. In addition, the Puiseux monoid M_2 in Example 4.3 is an FFM (and so a BFM) that is not a BBM, while the Puiseux monoid $\{0\} \cup \mathbb{Q}_{\geq 1}$ is a BBM that is not an FFM (and so not a LFFM) because $|\mathbf{Z}(3)| = \infty$ (indeed, since the set of atoms of $\{0\} \cup \mathbb{Q}_{\geq 1}$ is $\mathbb{Q} \cap [1, 2)$, it follows that for each $n \geq 3$ the equality $3 = (\frac{3}{2} - \frac{1}{n}) + (\frac{3}{2} + \frac{1}{n})$ yields a length-2 factorization of 3).

We conclude with the following remark on some potential natural ways to strengthen the main theorems we have established in this paper.

REMARK 5.1. We cannot strengthen Theorems 3.1 and 4.1 by replacing the condition that N is finitely generated by the condition that N is a BBM or by the condition that N is an FFM.

- Checking this for Theorem 3.1 amounts to observing that the Puiseux monoid M_2 in Example 4.3 is an FFM that is not a BBM (in particular, M_2 is strongly atomic) while the Puiseux monoid $\{0\} \cup \mathbb{Q}_{\geq 1}$ is a BBM that is not an FFM (in particular, $\{0\} \cup \mathbb{Q}_{\geq 1}$ is strongly atomic), but their internal sum is not even atomic, as argued in Example 4.3.
- The previous observation also guarantees that we cannot replace the condition that N is finitely generated by the condition that N is either a BBM or an FFM in parts (2) and (3) of Theorem 4.1. The same observation ensures that we cannot replace the condition that N is finitely generated by the condition that N is a BBM in part (1) of the same theorem. Finally, in light of Example 4.2, where we exhibited two Puiseux monoids that are FFMs whose internal sum is not an FFM, we can conclude that we cannot replace the condition that N is finitely generated by the condition that N is an FFM to strengthen part (1) of Theorem 4.1.

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