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MEASURE-VALUED SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS ON HILBERT SPACES DRIVEN BY LÉVY MEASURE AND THEIR OPTIMAL CONTROL

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Abstract. In this paper we consider a general class of nonlinear stochastic differential equations on Hilbert spaces determined by nonstandard infinitesimal generators (drift, diffusion, jump-kernel) and driven by Lévy process (measure). The infinitesimal generators are assumed to be only continuous and bounded on bounded sets. Under such relaxed assumptions, these equations do not have solutions in the usual sense (classical, strong, mild and weak). We prove existence of measure-valued solutions and consider several control problems (including control of the range of vector measures) and prove existence of partially observed optimal feedback controls. This paper is an extension of our previous studies on similar problems for deterministic as well as stochastic differential equations driven by cylindrical Brownian motion.

1. Introduction

Deterministic as well as Stochastic differential equations have been extensively studied in the literature using Lipschitz (or locally Lipschitz) and at most linear growth properties. It is known that for finite dimensional deterministic systems, simple continuity of the vector field is sufficient to prove existence of local solutions which may blow up in finite time. In the case of infinite dimensional systems, even this is no longer true. The vector field may be continuous, yet the system has no solution. There are counter examples as presented by Dieudonne [12] and Godunov [15].

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The system we consider in this paper is described by the following stochastic differential equation

$$(1) \qquad dx = Axdt + F(t,x)dt + G(t,x)dW + \int_{E_\delta} \mathcal{H}(t,x,v)q(dv \times dt),$$
$$t \in I = [0,T], x(0) = x_0$$

on a Hilbert space E where $\{A, F, G, \mathcal{H}\}$ are the infinitesimal generators defining the system equation (1) driven by the Wiener process and the compensated Poisson random measure q. These processes are assumed to be stochastically independent. More details are given below.

Let H be a separable real Hilbert space and let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ be a complete filtered probability space where $\{\mathcal{F}_t, t \geq 0\}$ is an increasing family of sub-sigma algebras of the sigma algebra $\mathcal{F}$, assumed to be continuous from the right and having limits from the left. Let $W \equiv \{W(t), t \geq 0\}$ be an $\mathcal{F}_t$-adapted H valued cylindrical Brownian motion, and $p(d\xi \times dt)$ be a random measure defined on the sigma algebra of subsets of the set $E_\delta \times I$ where $E_\delta \equiv E \setminus B_\delta$ with B_δ denoting the open ball in E of radius $\delta > 0$ and centered at the origin. The measure p is said to be a Poisson random measure or a counting measure if for each time interval $\Delta \subset I$ and any Borel set $S \subset E_\delta$, the probability that there are exactly n jumps of sizes (or with range) confined in the set S is given by

$$P\{p(S \times \Delta) = n\} = \frac{(\Lambda(S)\lambda(\Delta))^n}{n!} \exp\{-\Lambda(S)\lambda(\Delta)\},$$

where $\lambda(dt) \equiv dt$ denotes the Lebesgue measure on I and Λ denotes the Lévy (jump) measure on the sigma algebra of Borel subsets of the set E_δ. The term $\Lambda(S)$ (the Lévy measure of the set S) denotes the mean rate of jumps of all sizes confined in the set S. We note that the measure Λ can be chosen according to the specific needs of applications. Define the random measure

$$q(S \times \Delta) \equiv p(S \times \Delta) - \Lambda(S)\lambda(\Delta)$$

with mean zero and variance $\Lambda(S)\lambda(\Delta)$. This is the measure used in equation (1). The random measure q is called the compensated Poisson random measure.

Throughout the rest of the paper, it is assumed that for each $t \geq 0$, $\mathcal{F}_t(\subset \mathcal{F})$ is the smallest σ-algebra with respect to which both the processes $\{W(s), s \leq t\}$ and $\{q(B,s), B \in Bor.(E_\delta), 0 \leq s \leq t\}$ are measurable (or $\mathcal{F}_t$-adapted). This fact is essential to justify the martingale representation given by the expression (18).

2. Background materials

Let X be any regular topological space and $BC(X)$ the Banach space of bounded continuous real valued functions endowed with supnorm topology. It is well known that the topological dual of this space is given by $\mathcal{M}_{rba}(X)$, the space of regular bounded finitely additive measures. The space $\mathcal{M}_{rba}(X)$, endowed with total variation norm, is a Banach space.

Lemma 2.1 ([14]). *The topological dual $BC(X)^*$ is isometrically isomorphic to $\mathcal{M}_{rba}(X)$ in the sense that for any $\ell \in BC(X)^*$ there exists a unique element $\mu \in \mathcal{M}_{rba}(X)$ such that*

$$\ell(\varphi) = \int_X \varphi(x)\mu(dx), \text{ for all } \varphi \in BC(X)$$

and that $|\ell| = |\mu|_v$.

We are interested in the space of regular bounded finitely additive probability measures on X which is denoted by $\mathcal{P}_{rba}(X)$. Clearly this is a subset of $\mathcal{M}_{rba}(X)$. Further on, we need the concept of measure valued functions $\mu : I \equiv [0,T] \longrightarrow \mathcal{M}_{rba}(X)$. It is well known that the spaces $BC(X)$ and $\mathcal{M}_{rba}(X)$ do not satisfy the RNP (Radon-Nikodym property). Hence the dual of $L_1(I, BC(X))$ is not given by $L_\infty(I, \mathcal{M}_{rba}(X))$. However, by virtue of the theory of lifting [19, Tulcea and Tulcea, Theorem 7, p.94], the (topological) dual of $L_1(I, BC(X))$ is given by $L^w_\infty(I, \mathcal{M}_{rba}(X))$ which consists of weak star measurable $\mathcal{M}_{rba}(X)$ valued functions. This is furnished with the weak star (w^*) topology. Any continuous linear functional ℓ on $L_1(I, BC(X))$ has the representation

$$\ell(\varphi) = \int_{I\times X} \varphi(t,x)\mu_t(dx)dt$$

for some $\mu \in L^w_\infty(I, \mathcal{M}_{rba}(X))$. This follows directly from Lemma 2.1.

In this paper we are interested in measure valued stochastic processes. These are $\mathcal{F}_t$-adapted w^*-measurable $\mathcal{M}_{rba}(X)$-valued random processes. Consider the Banach space $L_1(I\times\Omega, BC(X))$ of $dt\times dP$ integrable $BC(X)$ valued random processes with the norm topology given by

$$\begin{aligned}\| \varphi \|_{L_1(I\times\Omega, BC(E))} &= \int_{I\times\Omega} \sup\{|\varphi(t,\omega,\xi)|, \xi \in X\} dtdP \\ &= \int_{I\times\Omega} \| \varphi \|_{BC(X)} \, dtdP.\end{aligned}$$

Again it follows from the theory of lifting [19, Tulcea and Tulcea] that the topological dual of this space is given by $L^w_\infty(I\times\Omega, \mathcal{M}_{rba}(X))$ which consists of w^*-measurable $\mathcal{F}_t$-adapted $\mathcal{M}_{rba}(X)$ valued random processes. In other words, $L_1(I\times\Omega, BC(X))^* \cong L^w_\infty(I\times\Omega, \mathcal{M}_{rba}(X))$. Thus, for any $\ell \in (L_1(I\times\Omega, BC(X)))^*$, there exists a unique $\mu \in L^w_\infty(I\times\Omega, \mathcal{M}_{rba}(X))$ such that

$$\ell(\varphi) = \int_{I\times\Omega} \mu_{t,\omega}(\varphi)dtdP \equiv \int_{I\times\Omega\times X} \varphi(t,\omega,x)\mu_{t,\omega}(dx)dtdP. \tag{2}$$

In this work, we use $X = E$ where E is a Hilbert space which is clearly a complete metric space. It is known that every metric space is a Tychonoff space and that Stone-Ĉheck compactification of any Tychonoff space is a compact Hausdorff space. Clearly, E is a Tychonoff space and thus its Stone-Ĉech compactification denoted by $\beta E \equiv E^+$ is a compact Hausdorff space containing a dense subspace which is homeomorphic to E. Since homeomorphic spaces are topologically equivalent, we have $E \subset E^+$ and it is dense in E^+ [20], [3]. Throughout the rest of the paper, we use this Hausdorff space E^+ and consider the topological space $L_1(I \times \Omega, BC(E^+))$ with its topological dual $L^w_\infty(I \times \Omega, \mathcal{M}_{rba}(E^+))$. Since E^+ is a compact Hausdorff space, the regular finitely additive measures have extensions to regular countably additive measures on the sigma algebra of Borel subsets of the set E^+. Thus $\mathcal{M}_{rba}(E^+)$ can be considered as $\mathcal{M}_{rca}(E^+)$, the space of regular countably additive measures. See Dunford & Schwartz [14]. However, we continue to use the original notation in order to remind us that these measures originate from finitely additive measures.

Consider the linear space $L^w_\infty(I\times\Omega, \mathcal{M}_{rba}(E^+))$ endowed with the weak-star topology and let $\mathcal{B}_1$ denote it's closed unit ball. By virtue of Alaoglu's theorem, the set $\mathcal{B}_1$ is weak-star compact. We are interested in the family of probability measure valued stochastic processes denoted by $M^w_\infty(I \times \Omega, \mathcal{P}_{rba}(E^+))$. This is a subset of $L^w_\infty(I \times \Omega, \mathcal{M}_{rba}(E^+))$ with elements taking values in the interval $[0, 1]$ and hence $M^w_\infty(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset \mathcal{B}_1$. Thus every sequence (or net) $\{\mu^n\} \in M^w_\infty(I \times \Omega, \mathcal{P}_{rba}(E^+))$ has a subsequence (subnet) that converges in the weak-star topology to an element μ of $\mathcal{B}_1$. We show that, in fact, the limit belongs to the set $M^w_\infty(I\times\Omega, \mathcal{P}_{rba}(E^+))$. Note that for every nonnegative $\varphi \in L_1(I \times \Omega, BC(E^+))$,

$$\langle \mu^n, \varphi\rangle = \int_{I\times\Omega} \mu^n(\varphi) dtdP \geq 0, \ \forall\ n \in \mathbb{N}$$

and hence the limit $\langle \mu, \varphi\rangle \geq 0$ preserving positivity. Similarly, one can test this with non positive $\varphi \leq 0$. Thus the set $M^w_\infty(I\times\Omega, \mathcal{P}_{rba}(E^+))$ is a weak-star closed subset of the weak star compact set $\mathcal{B}_1$ and hence it is also weak-star compact. Since we forego the standard assumptions (such as Lipschitz and linear growth) on the drift, diffusion and the jump kernel, generally assumed in the literature to prove existence of mild solutions, we need to introduce the following infinitesimal generators. Let $\{E, H\}$ denote the pair of Hilbert spaces mentioned above where E is the state space and H is the space where the Wiener process W takes values from. Let $C(E)$ denote the class of real valued continuous functions defined on the Hilbert space E, not necessarily bounded, and $BC(E) \subset C(E)$ the class of bounded continuous functions endowed with the standard sup norm topology. Clearly, endowed with this topology, $BC(E)$ is a Banach space. Similarly $BC(I \times E) \subset C(I \times E)$. Let Φ denote the class of test functions as defined below

$$\Phi \equiv \{\varphi \in BC(E) : D\varphi \in C(E, E), D^2\varphi \in C(E, \mathcal{L}(E))\},$$

where $D\varphi$ and $D^2\varphi$ denote the first and second Fréchet derivatives of φ. Let $\mathcal{B}_\delta$ denote the Borel algebra of subsets of the set E_δ and consider the (Lévy) measure space $(E_\delta, \mathcal{B}_\delta, \Lambda)$ and let $H_\delta \equiv L_2(E_\delta, \Lambda)$ denote the Hilbert space of real valued functions on E_δ which are square integrable with respect to the Lévy measure Λ. We introduce the following operators $\{\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2\}$ arising in the study of measure valued solutions as follows:

$$\mathcal{D}(\mathcal{A}) = \{\varphi \in \Phi : (\mathcal{A}\varphi) \in C(I \times E)\},$$

where

$$(3) \qquad (\mathcal{A}\varphi)(t,\xi) \equiv (1/2)Tr((D^2\varphi(\xi))G(t,\xi)G^*(t,\xi)) + (1/2)\int_{E_\delta} \langle (D^2\varphi(\xi))\mathcal{H}(t,\xi,v), \mathcal{H}(t,\xi,v)\rangle_E \Lambda(dv), (t,\xi) \in I \times E,$$

and

$$\mathcal{D}(\mathcal{B}) = \{\varphi \in \Phi : D\varphi(\xi) \in \mathcal{D}(A^*), (\mathcal{B}\varphi) \in C(I \times E)\}$$

with $\mathcal{D}(A^*)$ denoting the domain of the adjoint of the operator A and

$$(4) \qquad (\mathcal{B}\varphi)(t,\xi) \equiv \langle A^* D\varphi(\xi), \xi\rangle_E + \langle D\varphi(\xi), F(t,\xi)\rangle_E, (t,\xi) \in I \times E.$$

The operators $\{\mathcal{C}_1, \mathcal{C}_2\}$ are continuous functions on $I \times E$ with values in H and H_δ respectively as described below,

$$(5) \qquad (\mathcal{C}_1\varphi)(t,\xi) \equiv G^*(t,\xi)D\varphi(\xi), \ (\mathcal{C}_1\varphi) \in C(I \times E, H) \text{ and}$$

$$(6) \qquad (\mathcal{C}_2\varphi)(t,\xi) \equiv \langle D\varphi(\xi), \mathcal{H}(t,\xi,\cdot)\rangle_E, \ (\mathcal{C}_2\varphi)(\cdot) \in C(I \times E, H_\delta),$$

where H_δ is the Hilbert space as introduced above.

3. Existence of measure-valued solutions

It is well known that if A is the infinitesimal generator of a C_0-semigroup on E and the nonlinear operators (drift, diffusion and the jump kernel) are Lipschitz in the state variable $x \in E$ and have at most linear growth with Lipschitz and growth coefficients in $L_2^+(I)$, then the system (1) has a unique $\mathcal{F}_t$-adapted path wise solution $x \in B_\infty^a(I,E) \subset L_\infty(I, L_2(\Omega, E))$ satisfying

$$\| x \|_{B_\infty^a(I,E)} \equiv \sup\{\left(\mathbb{E}|x(t)|_E^2\right)^{1/2}, t \in I\} < \infty.$$

Here $B_\infty^a(I,E)$ denotes the space of $\mathcal{F}_t$-adapted E-valued stochastic processes having finite second moments. Endowed with the norm topology as defined above, it is a closed subspace of the Banach space $L_\infty(I, L_2(\Omega, E))$ and hence a Banach space. Further, the solution has discontinuities of no more than that of the first kind. It is known [8,12,15] that in infinite dimensional spaces if the vector field is merely continuous, the system has no solution. This is very well illustrated by counter examples given by Dieudonne [12] and Godunov [15]. However, if the notion of solution is extended beyond the classical ones such as (strong, mild, weak) solutions, it has been proved under some very general assumptions that measure valued solutions do exist [1,2,4–8]. See also the references therein. Some of these papers consider stochastic differential equation driven only by Brownian motion. In this paper, we consider stochastic systems driven both by Brownian motion and Leśvy process in particular Poisson random measure.

We introduce the definition of measure valued solution as follows.

Definition 3.1. The stochastic system given by equation (1) is said to have a measure-valued solution, if for each E-valued random element x_0 with the probability law $\mu_0 \in \mathcal{P}_{rba}(E)$, there exists a $\mu \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$ such that, for every $\varphi \in D(\mathcal{A}) \cap D(\mathcal{B})$ having compact support, the following identity holds P-almost surely

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(\mathcal{A}\varphi)ds + \int_0^t \mu_s(\mathcal{B}\varphi)ds$$
$$+ \int_0^t \langle \mu_s(\mathcal{C}_1\varphi), dW(s)\rangle + \int_0^t \int_{E_\delta} \mu_s(\mathcal{C}_2\varphi)(v)q(dv \times ds), t \in I. \tag{7}$$

Now we introduce the basic assumptions.

Basic assumptions: The operators $\{A, F, G, \mathcal{H}\}$ satisfy the following assumptions:

(A1): The operator A is the infinitesimal generator of a C_0-semigroup $\{S(t), t \geq 0\}$ with values in the space of bounded linear operators $\mathcal{L}(E)$. For a given finite interval $I \equiv [0, T]$, there exists a number $M \geq 1$ such that

$$\sup\{\| S(t) \|_{\mathcal{L}(E)}, t \in I\} \leq M.$$

(A2): (i) The drift operator F, mapping $I \times E$ to E, is continuous and bounded on bounded subsets of $I \times E$.

(ii) There exists a sequence $\{F_n\}$, uniformly bounded on bounded subsets of $I \times E$, such that $F_n(t, \xi) \to F(t, \xi)$ uniformly on compact subsets of $I \times E$ and there exists a sequence of positive real numbers $\{\alpha_n\}$, possibly $\alpha_n \to \infty$, such that

$$|F_n(t, \xi)|_E^2 \leq \alpha_n^2(1 + |\xi|_E^2),\ \xi \in E,$$
$$|F_n(t, \xi) - F_n(t, \eta)|_E^2 \leq \alpha_n^2|\xi - \eta|_E^2,\ \xi, \eta \in E.$$

(A3): (i) The diffusion operator G, mapping $I \times E$ to $\mathcal{L}_2(H, E)$ (the space of Hilbert-Schmidt operators from H to E), is continuous and bounded on bounded subsets of $I \times E$.

(ii) There exists a sequence $\{G_n\}$, uniformly bounded on bounded subsets $I \times E$, such that $G_n(t, \xi) \to G(t, \xi)$ in $\mathcal{L}_2(H, E)$ uniformly on compact subsets of $I \times E$. And there exists a sequence of positive real numbers $\{\beta_n\}$, possibly $\beta_n \to \infty$, such that

$$\| G_n(t, \xi) \|_{\mathcal{L}_2(H,E)}^2 \leq \beta_n^2(1 + |\xi|_E^2),\ \xi \in E$$
$$\| G_n(t, \xi) - G_n(t, \eta) \|_{\mathcal{L}_2(H,E)}^2 \leq \beta_n^2|\xi - \eta|_E^2,\ \xi, \eta \in E.$$

(A4): (i) The jump kernel $\mathcal{H}$, mapping $I \times E \times E_\delta$ to E, is continuous and bounded on bounded subsets of $I \times E \times E_\delta$.

(ii) There exists a sequence $\{\mathcal{H}_n\}$ uniformly bounded on bounded subsets of $I \times E$ for each $v \in E_\delta$ and that for Λ-almost all $v \in E_\delta$, $\mathcal{H}_n(t, \xi, v) \to \mathcal{H}(t, \xi, v)$

uniformly on compact subsets of $I \times E$. Further, there exists a sequence of positive real numbers γ_n, possibly $\gamma_n \to \infty$, such that

$$\int_{E_\delta} |\mathcal{H}_n(t,\xi,v)|_E^2 \Lambda(dv) \leq \gamma_n^2 [1 + |\xi|_E^2], (t,\xi) \in I \times E,$$

$$\int_{E_\delta} |\mathcal{H}_n(t,\xi,v) - \mathcal{H}_n(t,\eta,v)|_E^2 \Lambda(dv) \leq \gamma_n^2 |\xi - \eta|_E^2, (t,\xi),(t,\eta) \ \in I \times E.$$

Before we consider the question of existence of measure-valued solutions, we present some necessary terminologies and preparatory results. Let $\rho(A)$ denote the resolvent set of the operator A and $R(\lambda, A), \lambda \in \rho(A)$, the resolvent of A. Based on the preceding assumptions we can prove the following Lemma.

Lemma 3.2. *Consider the system described by the stochastic differential equation* (1) *and suppose the family of operators* $\{A, F, G, \mathcal{H}\}$ *satisfy the assumptions* (A1), (A2)(ii), (A3)(ii), (A4)(ii) *and let* $A_n = nAR(n,A), n \in \rho(A) \cap \mathbb{N}$ ($\mathbb{N}$ = *set of positive integers) denote the Yosida approximation of* A. *Then for each* $\mathcal{F}_0$*-measurable initial state* $x_0 \in L_2(\Omega, E)$ *with* $x_{0,n} = nR(n,A)x_0$, *the approximating system*

$$dx = A_n x dt + F_n(t,x)dt + G_n(t,x)dW + \int_{E_\delta} \mathcal{H}_n(t,x,v)q(dv \times dt), t \in I,$$

$$x(0) = x_{0,n} \tag{8}$$

has a unique mild solution $x_n \in B_\infty^a(I,E) \equiv B_\infty(I, L_2^a(\Omega,E))$.

Proof. The proof is standard. We present a brief outline. Let $\{S_n(t), t \geq 0\}$ denote the semigroup corresponding to the infinitesimal generator A_n. It follows from semigroup theory that $\{S_n(t), t \in I\}$ is uniformly continuous on compact subsets of $[0,\infty)$ and that it converges in the strong operator topology to $\{S(t), t \in I\}$. Using the semigroup $S_n(t), t \in I$, differential equation (8) can be reformulated as an integral equation as follows,

$$\begin{aligned} x(t) = S_n(t)x_{0,n} &+ \int_0^t S_n(t-s)F_n(s,x(s))ds \\ &+ \int_0^t S_n(t-s)G_n(s,x(s))dW(s) \\ &+ \int_0^t \int_{E_\delta} S_n(t-s)\mathcal{H}_n(s,x(s),v)q(dv \times ds), t \in I. \end{aligned} \tag{9}$$

The solution of this equation (if one exists) is called the mild solution of equation (8). Introducing the (nonlinear) operator Γ as

$$(\Gamma x)(t) \equiv S_n(t)x_{0,n} + \int_0^t S_n(t-s)F_n(s,x(s))ds + \int_0^t S_n(t-s)G_n(s,x(s))dW(s) + \int_0^t \int_{E_\delta} S_n(t-s)\mathcal{H}_n(s,x(s),v)q(dv \times ds), t \in I, \tag{10}$$

the integral equation (9) can be restated as a fixed point problem $x = \Gamma x$. By virtue of the growth properties of the approximating set of operators $\{F_n, G_n, \mathcal{H}_n\}$ and the bound of the semigroup $S(t), t \in I$, as stated in the assumptions (A1), (A2)(ii), A(3)(ii) and (A4)(ii), one can easily verify that Γ maps $B_\infty^a(I,E)$ into itself. Thus it suffices to verify the existence of a fixed point of the operator Γ in $B_\infty^a(I,E)$. Using the Lipschitz properties as stated in the above assumptions, one can verify the existence of an integer m such that the m-th iterate of Γ denoted by Γ^m is a contraction. Hence $x = \Gamma^m x$ has a unique solution $x_n \in B_\infty^a(I,E)$ with $x_n(0) = x_{0,n}$. Then it follows readily that x_n is also the unique fixed point of Γ. This completes our brief outline. □

We are now prepared to consider the question of existence of measure valued solutions of equation (1). Our main result of this section is presented below.

Theorem 3.3. *Consider the system* (1) *and let the operators* $\{A, F, G, \mathcal{H}\}$ *satisfy the assumptions* (A1), (A2), (A3), (A4). *Then for every initial state* x_0 *having the probability law* $\mu_0 \in \mathcal{P}_{rba}$, *there exists a* $\mu \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$ *satisfying the identity* (7).

Proof. By virtue of Lemma 3.2, the approximating system (8) has a unique solution $x_n \in B_\infty^a(I,E)$. Consider the sequence of measure valued random processes $\{\delta_{x_n(t)}(d\xi) \equiv \mu_t^n(d\xi), t \in I\}$ defined by the sequence of Dirac measures along the path process $\{x_n\}$. Let $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B}) \subset \Phi$ having compact support in E, and for each $n \in \mathbb{N} \cap \rho(A)$, consider the random process $\varphi(x_n(t)), t \in I$. Then using the Ito-differential rule one can verify that

$$d\varphi(x_n(t)) = (\mathcal{A}_n\varphi)(t, x_n(t))dt + (\mathcal{B}_n\varphi)(t, x_n(t))dt + \langle (\mathcal{C}_{1,n}\varphi)(t, x_n(t)), dW(t)\rangle_H + \int_{E_\delta} (\mathcal{C}_{2,n}\varphi)(t, x_n(t))(v)q(dv \times dt) \tag{11}$$

where the operators $\{\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_{1,n}, \mathcal{C}_{2,n}\}$ are the approximations of the operators $\{\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2\}$ as described below. The operator $\mathcal{A}_n$ is obtained from the expression (3) for $\mathcal{A}$ by replacing the pair $\{G, \mathcal{H}\}$ with the corresponding approximating pair $\{G_n, \mathcal{H}_n\}$. The operator $\mathcal{B}_n$ is obtained from the expression (4) for $\mathcal{B}$ by replacing the pair $\{A, F\}$ with the approximating pair $\{A_n, F_n\}$. Similarly, the operators $\{\mathcal{C}_{1,n}, \mathcal{C}_{2,n}\}$ are given by the expressions (5) and (6) respectively with the pair $\{G, \mathcal{H}\}$ replaced by the pair $\{G_n, \mathcal{H}_n\}$. For any $\nu \in \mathcal{M}_{rba}(E^+)$

and $\psi \in BC(E^+)$, we use the notation $\nu(\psi) \equiv \int_{E^+} \psi(\xi)\nu(d\xi)$. It follows from this representation that equation (11) is equivalent to the following identity,

$$\begin{aligned} &\mu_t^n(\varphi) \\ &= \mu_{0,n}(\varphi) + \int_0^t \mu_s^n(\mathcal{A}_n\varphi)ds + \int_0^t \mu_s^n(\mathcal{B}_n\varphi)ds \\ &+ \int_0^t \langle \mu_s^n(\mathcal{C}_{1,n}\varphi), dW(s)\rangle_H + \int_0^t \int_{E_\delta} \mu_s^n(\mathcal{C}_{2,n}\varphi)(v)q(dv \times ds), t \in I. \end{aligned} \tag{12}$$

Considering the sequence of measure valued random processes $\{\mu^n\}$ as defined above, we note that for any $\phi \in L_1(I \times \Omega, BC(E^+))$ the functional

$$\ell_n(\phi) \equiv \int_{I\times\Omega\times E^+} \phi(t,\omega,\xi)\mu_{t,w}^n(d\xi)dtdP = \mathbb{E}\int_{I\times E^+} \phi(t,\xi)\mu_t^n(d\xi)dt$$

is well defined and that

$$|\ell_n(\phi)| \le \mathbb{E}\int_I \| \phi \|_{BC(E^+)} \, dt = \| \phi \|_{L_1(I\times\Omega, BC(E^+))} \quad \text{for all } n \in \mathbb{N}.$$

Hence the sequence $\{\ell_n\}$ is contained in the unit ball of the space of continuous linear functionals on $L_1(I \times \Omega, BC(E^+))$ denoted by $(L_1(I \times \Omega, BC(E^+))^*$. Thus it follows from the characterization of continuous linear functionals on $L_1(I \times \Omega, BC(E^+))$ that there exists a $\mu \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$ such that, along a generalized subsequence (relabeled as the original sequence),

$$\mu^n \xrightarrow{w^*} \mu \quad \text{in } M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+)). \tag{13}$$

We prove that μ satisfies equation (7). Considering the first term on the right-hand side of equation (12), note that the sequence of operators $nR(n, A)$ converges in the strong operator topology to the identity operator in E. Hence for any bounded continuous function φ, the sequence $\varphi(x_{0,n})$ converges to $\varphi(x_0)$ in probability. Thus $\mu_{0,n} \xrightarrow{w^*} \mu_0 \in \mathcal{P}_{rba}(E^+)$ as $n \to \infty$ in the sense that $\mu_{0,n}(\varphi) \longrightarrow \mu_0(\varphi)$ for each $\varphi \in BC(E^+)$. Next, we consider the second term of equation (12) and recall the expression (3) for the operator $\mathcal{A}$ and note that for each $n \in \mathbb{N} \cap \rho(A)$ the operator $\mathcal{A}_n$ is given by

$$\begin{aligned} &(\mathcal{A}_n\varphi)(t,\xi) \\ &\equiv (1/2)Tr(G_n^*(t,\xi)(D^2\varphi(\xi))G_n(t,\xi)) \\ &+ (1/2)\int_{E_\delta} \langle (D^2\varphi(\xi))\mathcal{H}_n(t,\xi,v), \mathcal{H}_n(t,\xi,v)\rangle_E \Lambda(dv), (t,\xi) \in I \times E, \end{aligned} \tag{14}$$

for any $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ having compact support. By virtue of assumptions (A3)(i) and the first part of (A3)(ii), the sequence G_n converges to G in the topology of $\mathcal{L}_2(H, E)$ uniformly on compact subsets of $I \times E$ and it is also uniformly bounded on bounded sets. Thus the first component is bounded as an element of $L_1(I, BC(E^+)) \subset L_1(I \times \Omega, BC(E^+))$. Similarly, by virtue of assumptions (A4)(i) and the first part of (A4)(ii), the sequence $\mathcal{H}_n$ converges to $\mathcal{H}$ uniformly on compact subsets of $I \times E$. Thus the second component is also bounded as an element of $L_1(I, BC(E^+)) \subset L_1(I \times \Omega, BC(E^+))$. Hence it follows from Lebesgue bounded convergence theorem

$$\mathcal{A}_n\varphi \xrightarrow{s} \mathcal{A}\varphi \text{ in } L_1(I, BC(E^+)) \subset L_1(I \times \Omega, BC(E^+)). \tag{15}$$

Next, we consider the third term of equation (12) and recall the expression (4) for the operator $\mathcal{B}$ and note that for each $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ and $n \in \mathbb{N} \cap \rho(A)$ the operator $\mathcal{B}_n$ is given by

$$(\mathcal{B}_n\varphi)(t,\xi) \equiv \langle A_n^* D\varphi(\xi), \xi\rangle + \langle D\varphi(\xi), F_n(t,\xi)\rangle_E, (t,\xi) \in I \times E.$$

Since $D\varphi \in D(A^*)$ and A_n^* converges to A^* in the strong operator topology on its domain and F_n converges to F uniformly on compact subsets of $I \times E$ and φ has compact support, we conclude that $\mathcal{B}_n\varphi$ is bounded on $I \times E$ and since I is a finite interval it belongs to $L_1(I, BC(E^+)) \subset L_1(I \times \Omega, BC(E^+))$. Again it follows from Lebesgue bounded convergence theorem that

$$\mathcal{B}_n\varphi \xrightarrow{s} \mathcal{B}\varphi \text{ in } L_1(I \times \Omega, BC(E^+)). \tag{16}$$

Next, we consider the fourth and fifth terms of equation (12) and recall the expression (5) and (6) giving the operators $\mathcal{C}_1$ and $\mathcal{C}_2$ and note that for each $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ having compact support and $n \in \mathbb{N} \cap \rho(A)$, the operators $\mathcal{C}_{1,n}$ and $\mathcal{C}_{2,n}$ are given by

$$(\mathcal{C}_{1,n}\varphi)(t,\xi) \equiv G_n^*(t,\xi)D\varphi(\xi),\ (\mathcal{C}_{1,n}\varphi) \in BC(I \times E, H) \text{ and}$$
$$(\mathcal{C}_{2,n}\varphi)(t,\xi)(\cdot) \equiv \langle D\varphi(\xi), \mathcal{H}_n(t,\xi,\cdot)\rangle_E,\ (\mathcal{C}_{2,n}\varphi)(\cdot) \in BC(I \times E, H_\delta).$$

Since φ has compact support and G_n converges to G uniformly on compact sets, it is clear that $(\mathcal{C}_{1,n}\varphi)$ converges to $(\mathcal{C}_1\varphi)$ strongly in the Hilbert space H on the set $I \times E$. Following similar argument, we conclude that $(\mathcal{C}_{2,n}\varphi)$ converges to $(\mathcal{C}_2\varphi)$ strongly in the Hilbert space $L_2(E_\delta, \Lambda) \equiv H_\delta$ on the set $I \times E$. Note that the sum of the last two terms on the righthand side of equation (12) is an $\mathcal{F}_t$-martingale given by the expression

$$\begin{aligned} m_t^n \equiv &\int_0^t \langle \mu_s^n(\mathcal{C}_{1,n}\varphi), dW(s)\rangle_H \\ &+ \int_0^t \int_{E_\delta} \mu_s^n(\mathcal{C}_{2,n}\varphi)(v) q(dv \times ds), t \in I. \end{aligned} \tag{17}$$

Let z be an $\mathcal{F}$-measurable random variable in $L_2(\Omega)$. Without loss of generality we may assume that it has zero mean. Then the conditional expectation of z given by $z_t \equiv \mathbb{E}\{z|\mathcal{F}_t\}$ is an $\mathcal{F}_t$-martingale (with zero mean) and it follows from martingale representation theory that there exists a pair $\eta \in L_2^a(I,H), \beta \in L_2^a(I, H_\delta) = L_2^a(I \times E_\delta, dt \times \Lambda(dv))$ such that

$$z_t = \int_0^t \langle \eta(s), dW(s)\rangle + \int_0^t \int_{E_\delta} \beta(s,v) q(dv \times ds), t \in I. \tag{18}$$

Computing the scalar product of the martingales given by the expressions (17) and (18) and using Fubini's theorem we obtain

$$\mathbb{E}\{z\ m_t^n\} = \mathbb{E}\{z_t\ m_t^n\} = \mathbb{E}\int_0^t \mu_s^n(\langle \eta(s), G_n^*(s,\xi)D\varphi(\xi)\rangle_H)ds$$

$$+\,\mathbb{E}\int_0^t \mu_s^n\Big(\int_{E_\delta} \beta(s,v)\langle D\varphi(\xi), \mathcal{H}_n(s,\xi,v)\rangle_E \Lambda(dv)\Big)ds, t\in I. \tag{19}$$

Since φ has compact support in E and $\eta \in L_2^a(I,H), \beta\in L_2^a(I,H_\delta)$, it follows from the properties of the sequence $G_n, \mathcal{H}_n$ as stated in (A3) and (A4) that

$$\langle \eta, G_n^* D\varphi\rangle_H \in L_1(I\times\Omega, BC(E^+))$$

and

$$\int_{E_\delta} \beta(\cdot,v)\langle D\varphi(\xi), \mathcal{H}_n(\cdot,\xi,v)\rangle_E \Lambda(dv) \in L_1(I\times\Omega, BC(E^+)).$$

Further, G_n converges to G uniformly on compact subsets of $I\times E$, and $\mathcal{H}_n$ converges to $\mathcal{H}$ on compact subsets of $I\times E$ for almost all $v\in E_\delta$, and they are uniformly bounded on bounded sets. Hence it follows from dominated convergence theorem that

$$\langle \eta, G_n^* D\varphi\rangle_H \xrightarrow{s} \langle \eta, G^* D\varphi\rangle_H \text{ in } L_1(I\times\Omega, BC(E^+)) \text{ and} \tag{20}$$

$$\int_{E_\delta} \beta(\cdot,v)\langle D\varphi(\cdot), \mathcal{H}_n(\cdot,\cdot,v)\rangle_E \Lambda(dv) \tag{21}$$

$$\xrightarrow{s} \int_{E_\delta} \beta(\cdot,v)\langle D\varphi(\cdot), \mathcal{H}(\cdot,\cdot,v)\rangle_E \Lambda(dv) \text{ in } L_1(I\times\Omega, BC(E^+)).$$

Thus it follows from weak star convergence of μ^n to μ and the strong convergence of the integrands as seen in the expressions (20) and (21) that

$$\lim_{n\to\infty}\mathbb{E}\{z\ m_t^n\} = \lim_{n\to\infty}\mathbb{E}\{z_t\ m_t^n\}$$

$$= \mathbb{E}\int_0^t \mu_s(\langle \eta(s), G^*(s,\cdot)D\varphi(\cdot)\rangle_H)ds$$

$$+\,\mathbb{E}\int_0^t \mu_s\Big(\int_{E_\delta} \beta(s,v)\langle D\varphi(\cdot), \mathcal{H}(s,\cdot,v)\rangle_E \Lambda(dv)\Big)ds, t\in I. \tag{22}$$

Reversing the steps leading to the above expression, we obtain

$$\lim_{n\to\infty}\mathbb{E}\{z\ m_t^n\} = \mathbb{E}\{z\ m_t\} = \mathbb{E}\big\{z\int_0^t \langle \mu_s(\mathcal{C}_1\varphi)), dW(s)\rangle_H\big\}$$

$$+\,\mathbb{E}\big\{z\int_0^t\int_{E_\delta} \mu_s(\mathcal{C}_2\varphi)(v)q(dv\times ds)\big\}. \tag{23}$$

Now multiplying the expression (12) on either side by $z\in L_2(\Omega)$ and recalling that $\mu_{0,n} \xrightarrow{w^*} \mu_0$ in $\mathcal{P}_{rba}(E^+)$ and applying the expectation operation and then letting $n \to \infty$ and using (13), (15) (16) and (23) we arrive at the following expression

$$\begin{aligned} \mathbb{E}\{z\mu_t(\varphi)\} &= \mathbb{E}\{z\mu_0(\varphi)\} + \mathbb{E}\{z\int_0^t \mu_s(\mathcal{A}\varphi)ds\} + \mathbb{E}\{z\int_0^t \mu_s(\mathcal{B}\varphi)ds\} \\ &+ \mathbb{E}\{z\int_0^t \langle \mu_s(\mathcal{C}_1\varphi), dW(s)\rangle_H\} \\ &+ \mathbb{E}\{z\int_0^t \int_{E_\delta} \mu_s(\mathcal{C}_2\varphi)(v)q(dv \times ds)\}, t \in I. \end{aligned} \tag{24}$$

Since this identity holds for every $z \in L_2(\Omega)$, we conclude that, for every $\varphi \in D(\mathcal{A}) \cap D(\mathcal{B})$ having compact support in E,

$$\begin{aligned} \mu_t(\varphi) &= \mu_0(\varphi) + \int_0^t \mu_s(\mathcal{A}\varphi)ds + \int_0^t \mu_s(\mathcal{B}\varphi)ds + \int_0^t \langle \mu_s(\mathcal{C}_1\varphi), dW(s)\rangle_H \\ &+ \int_0^t \int_{E_\delta} \mu_s(\mathcal{C}_2\varphi)(v)q(dv \times ds), \text{ for all } t \in I, P - a.s. \end{aligned} \tag{25}$$

This is precisely the identity (7) proving that $\mu \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$ is a measure valued solution of equation (1) in the sense of Definition 3.1. This completes the proof. □

Remark 3.4. (Extension) In this paper we have used the assumptions on the vector fields $\{F, G, \mathcal{H}\}$ to be continuous and bounded on bounded sets of $I \times E$. Following similar technique as in [5], the results of this paper can be extended further to admit measurable vector fields $\{F, G, \mathcal{H}\}$ under the assumption that they are bounded on bounded sets.

4. Partially observed optimal state feedback control

We consider the following partially observed control system

$$\begin{aligned} dx &= Axdt + F(t,x)dt + K(t, \Pi x)dt + G(t,x)dW, \\ &+ \int_{E_\delta} \mathcal{H}(t,x,v)q(dv \times dt), \; t \in I = [0,T], \quad x(0) = x_0, \end{aligned} \tag{26}$$

where $\Pi : E \longrightarrow E_0 \subset E$ is a projection mapping E to a closed subspace E_0 of E representing the observable subspace. The function K is a continuous map from $I \times E_0$ to E and bounded on bounded sets. Here K is the partially observed state feedback control law from the class of functions as described below. Control theory for infinite dimensional (deterministic) systems on Banach spaces driven by vector measures is well known as seen in [9] and the references therein. Here, we consider stochastic systems driven by Brownian motion with values in H and scaler valued Leѵ́y measures on the Hilbert space E_δ.

Admissible feedback control laws: Consider the Hilbert space E furnished with the weak topology turning it into a locally compact Hausdorff topological space. Let $\mathbb{F}(I \times E_0, E) \equiv E^{I \times E_0}$ denote the space of continuous functions mapping $I \times E_0$ to E endowed with the Tychonoff product topology. For any pair of positive numbers $\{a, b\}$, consider the set $\mathcal{U}_{a,b} \subset \mathbb{F}(I \times E_0, E)$ satisfying the following properties: For all $K \in \mathcal{U}_{a,b}$,

$$(a1) : |K(t, 0)|_E \leq a, \ \forall\, t \in I,$$

$$(a2) : |K(t, \eta_1) - K(t, \eta_2)|_E \leq b|\eta_1 - \eta_2|_{E_0}, \ \forall\, \eta_1, \eta_2 \in E_0, \text{ and } t \in I.$$

For any pair $(t, \eta) \in I \times E_0$, let $\Pi_{t,\eta}$ denote the projection map (or evaluation map) as defined below,

$$\Pi_{t,\eta}(\mathcal{U}_{a,b}) \equiv \{K(t, \eta), K \in \mathcal{U}_{a,b}\} \subset E.$$

It is easy to verify that the set $\mathcal{U}_{a,b}$ is point wise closed in the sense that the point wise limit of any sequence from $\mathcal{U}_{a,b}$ belongs to $\mathcal{U}_{a,b}$, and the projection $\Pi_{t,\eta}(\mathcal{U}_{a,b})$ has compact closure in E (with respect to it's weak topology). Thus the set $\mathcal{U}_{a,b}$ is compact in the point wise topology τ_p [20, Theorem 42.3].

Here we consider some interesting control problems. The objective functional is given by an expression of the form

$$J(K) \equiv \mathbb{E} \int_{I \times E} L(t, \omega, \xi) \mu^K_{t,\omega}(d\xi) dt, \tag{27}$$

where $L : I \times \Omega \times E \longrightarrow R$ is a nonnegative Borel measurable map and the process $\mu^K \in M^w_\infty(I \times \Omega, \mathcal{P}_{rba}(E^+))$ with $\mu^K_0 = \mu_0$, is the measure-valued solution of the feedback control system (26) in the sense that, for every $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ having compact support, μ^K satisfies the following equation,

$$\begin{aligned} \mu_t(\varphi) = \mu_0(\varphi) &+ \int_0^t \mu_s(\mathcal{A}\varphi) ds \\ &+ \int_0^t \mu_s(\mathcal{B}\varphi) ds + \int_0^t \mu_s(\mathcal{K}\varphi) ds + \int_0^t \langle \mu_s(\mathcal{C}_1\varphi), dW(s)\rangle_H \\ &+ \int_0^t \int_{E_\delta} \mu_s(\mathcal{C}_2\varphi)(v) q(dv \times ds), t \in I, P.a.s, \end{aligned} \tag{28}$$

where the operator $\mathcal{K}$ is determined by an element $K \in \mathcal{U}_{a,b}$ and it is given by

$$\begin{aligned} \mathcal{K}\varphi &\equiv \{(\mathcal{K}\varphi)(t, \xi) \\ &\equiv \langle D\varphi(\xi), K(t, \Pi\xi)\rangle_E, (t, \xi) \in I \times E\}, \varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B}). \end{aligned} \tag{29}$$

The problem is to find a control law in $\mathcal{U}_{a,b}$ that extremizes (minimizes or maximizes) the functional (27).

For existence of optimal feedback controls, we need the following result on continuity of the control to solution map as presented in the following theorem.

Theorem 4.1. *Consider the system given by equation* (26) *and suppose the operators* $\{A, F, G, \mathcal{H}\}$ *satisfy the assumptions of Theorem 3.3. Then, for each* $K \in \mathcal{U}_{a,b}$, *equation* (26) *has a measure solution* $\mu^K \in M^w_\infty(I \times \Omega, \mathcal{P}_{rba}(E^+))$ *in the sense that it satisfies the identity* (28). *Further, it is continuously dependent* *on the control law in the sense that as* $K_n \xrightarrow{\tau_p} K_o$ *in* $\mathcal{U}_{a,b}$, $\mu^{K_n} \xrightarrow{w^*} \mu^{K_o}$ *in* $M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$.

Proof. It follows from the properties of the admissible set $\mathcal{U}_{a,b}$ that every element $K \in \mathcal{U}_{a,b}$ satisfies the assumption (A2) of Theorem 3.3. Thus the operators $\{A, F, K, G, \mathcal{H}\}$ satisfy all the assumptions of Theorem 3.3 and hence it follows from this theorem that equation (26) has a measure valued solution $\mu^K \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+))$ in the sense that μ^K satisfies the identity (28). To prove continuity, let $\{K_n\} \in \mathcal{U}_{a,b}$ and suppose $K_n \xrightarrow{\tau_p} K_o$ and let $\{\mu^n, \mu^o\} \subset M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+))$ denote the corresponding solutions of equation (28) respectively. We must verify that (along a subsequence if necessary) $\mu^n \xrightarrow{w^*} \mu^o$. Since the set $M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+))$ is a closed bounded subset of $L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$, it follows from Alaoglu's theorem that it is w^* compact and hence there exists a $\mu^* \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+))$ such that $\mu^n \xrightarrow{w^*} \mu^*$. We show that $\mu^* = \mu^o$. Subtracting term by term the expression given by equation (28) corresponding to the control K_n with the solution μ^n, from the same expression corresponding to the control K_o with solution μ^o, we have

$$\begin{aligned}(\mu_t^o - \mu_t^n)(\varphi) = & \int_0^t (\mu_s^o - \mu_s^n)(\mathcal{A}\varphi)ds + \int_0^t (\mu_s^o - \mu_s^n)(\mathcal{B}\varphi)ds \\ & + \int_0^t (\mu_s^o - \mu_s^n)(\mathcal{K}_o\varphi)ds + \int_0^t \mu_s^n((\mathcal{K}_o - \mathcal{K}_n)\varphi)ds \\ & + \int_0^t \langle(\mu_s^o - \mu_s^n)(\mathcal{C}_1\varphi), dW(s)\rangle_H \\ & + \int_0^t \int_{E_\delta} (\mu_s^o - \mu_s^n)(\mathcal{C}_2\varphi)(v)q(dv \times ds), \text{for all } t \in I, P - a.s. \end{aligned} \tag{30}$$

Consider the fourth term on the righthand side of the above equation given by

$$\int_0^t \mu_s^n((\mathcal{K}_o - \mathcal{K}_n)\varphi)ds, t \in I, n \in \mathbb{N}. \tag{31}$$

Since $K_n \xrightarrow{\tau_p} K_o$ in $\mathcal{U}_{a,b}$ and $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ having compact support, we note that

$$(\mathcal{K}_o - \mathcal{K}_n)\varphi \in L_1(I, BC(E^+)) \subset L_1(I \times \Omega, BC(E^+))$$

and that it converges point wise to zero. Since φ has compact support it follows from the properties of the set $\mathcal{U}_{a,b}$ that it is also bounded from above and so it follows from Lebesgue bounded convergence theorem that it converges to zero in $L_1(I \times \Omega, BC(E^+))$. On the other hand $\mu^n \xrightarrow{w^*} \mu^*$ in $M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$. Hence it is clear that

$$\lim_{n \to \infty} \mathbb{E} \int_0^t \mu_s^n((\mathcal{K}_o - \mathcal{K}_n)\varphi)ds = 0 \text{ for each } t \in I. \tag{32}$$

Define $\nu = \mu^o - \mu^*$ and note that $\nu \in L^w_\infty(I \times \Omega, \mathcal{M}_{rba}(E^+))$. Using these facts and the expression (30) and following the same technique as in the proof of Theorem 3.3, and letting $n \to \infty$ we obtain

$$\nu_t(\varphi) = \int_0^t \nu_s(\mathcal{A}\varphi)ds + \int_0^t \nu_s(\mathcal{B}\varphi)ds + \int_0^t \nu_s(\mathcal{K}_o\varphi)ds$$
$$+ \int_0^t \langle \nu_s(\mathcal{C}_1\varphi), dW(s)\rangle_H + \int_0^t \int_{E_\delta} \nu_s(\mathcal{C}_2\varphi)(v)q(dv \times ds), t \in I, P.a.s. \tag{33}$$

This is a linear homogeneous stochastic functional equation of Volterra type. Hence $\nu = 0$ proving that $\mu^* = \mu^o$. Thus the control to solution map is continuous in the sense as stated in the theorem. □

We are now prepared to consider the following control problem. Let

$$\Gamma(t,\omega), (t,\omega) \in I \times \Omega,$$

be a measurable multi function with values in the class of nonempty closed bounded subsets of E, denoted by $cb(E)$, which is furnished with the Hausdorff metric topology. Problem is to find a control law that maximizes the chance (probability) of following the target set Γ as closely as possible. The appropriate objective functional for this problem is given by

$$\tilde{J}(K) = \mathbb{E}\int_I \mu^K_{t,\omega}(\Gamma(t,\omega))dt. \tag{34}$$

In other words, the objective is to follow the target set described by the nonempty measurable set valued function Γ as closely as possible. This can be realized by maximizing the concentration of mass of the probability measure on the target set. We prove the following theorem.

Theorem 4.2. *Consider the control system given by equation* (28) (*or equivalently the equation* (26)) *with the set of admissible controls $\mathcal{U}_{a,b}$ and the objective functional given by* (34)*. Suppose the assumptions of Theorem 4.1 hold and that Γ is a nonempty measurable multifunction defined on $I \times \Omega$ and taking values from $cb(E)$. Then there exists an optimal control maximizing the functional* (34)*.*

Proof. Since the set $\mathcal{U}_{a,b}$ is compact with respect to the topology τ_p, it suffices to verify that the functional $K \to J(K)$ is upper semicontinuous in this topology. Let us note that the objective functional (34) is equivalent to the following cost functional

$$J(K) = \mathbb{E}\int_I \mu^K_{t,\omega}(E^+ \setminus \Gamma(t,\omega))dt. \tag{35}$$

In other words, maximizing the functional (34) is equivalent to minimizing the functional (35). We prove that this functional is lower semicontinuous. Let $\{K_n\} \subset \mathcal{U}_{a,b}$ and suppose $K_n \xrightarrow{\tau_p} K_o$ and let $\{\mu^n, \mu^o\}$ denote the corresponding solutions of equation (28) (equivalently measure solution of equation (26)). By

Theorem 4.1, $\mu^n \xrightarrow{w^*} \mu^o$ in $M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$. Since the set $E^+ \setminus \Gamma(t,\omega)$ is open for all $(t,\omega) \in I \times \Omega$, it is not difficult to verify [17, Theorem 6.1, p.40] that

$$\mu_{t,\omega}^o(E^+ \setminus \Gamma(t,\omega)) \le \liminf_{n\to\infty} \mu_{t,\omega}^n(E^+ \setminus \Gamma(t,\omega)), \text{for } a.a\ (t,\omega) \in I \times \Omega.$$

Integrating the above expression, we obtain

$$\begin{aligned} \mathbb{E}\int_I \mu_{t,\omega}^o(E^+ \setminus \Gamma(t,\omega))dt &\le \mathbb{E}\int_I \liminf_{n\to\infty} \mu_{t,\omega}^n(E^+ \setminus \Gamma(t,\omega))dt \\ &\le \liminf_{n\to\infty} \mathbb{E}\int_I \mu_{t,\omega}^n(E^+ \setminus \Gamma(t,\omega))dt. \end{aligned} \tag{36}$$

Hence we conclude that $J(K_o) \le \liminf_{n\to\infty} J(K_n)$. This shows that the map $K \to J(K)$ is lower semicontinuous with respect to the τ_p topology. Since $\mathcal{U}_{a,b}$ is τ_p compact, it follows from lower semi continuity of J that it attains its minimum on it. This proves existence of an optimal control. □

Another problem of significant interest is to find a feedback control that minimizes the distance between a desired measure valued process and the measure valued process generated by the system (28) corresponding to feedback controls. This can be formulated using the well known Lévy-Prokhorov metric on the space of probability measures. Let $\mathcal{P}_{rba}(E^+)$ denote the space of probability measures on the Borel sets $Bor.(E^+)$. Let us recall the definition of the Lévy-Prokhorov metric. For any two elements $\mu, \nu \in \mathcal{P}_{rba}(E^+)$, the Lévy-Prokhorov distance is given by

$$\begin{aligned} &d_P(\mu,\nu) \\ &= \inf\{\varepsilon > 0 | \mu(Q) \le \nu(Q_\varepsilon) + \varepsilon; \nu(Q) \le \mu(Q_\varepsilon) + \varepsilon,\ \forall\ Q \in Bor.(E^+)\}, \end{aligned} \tag{37}$$

where Q_ε is the $\varepsilon(> 0)$ neighbourhood of the set Q. We assume that E is a separable Hilbert space and hence a complete separable metric space. Under this assumption, $(\mathcal{P}_{rba}(E^+), d_P)$ is also a complete metric space. It is well known that convergence of measures in the Lévy-Prokhorov metric is equivalent to weak star convergence of measures. For details on this topic see [10,13,18,21].

Now we are prepared to state the problem. Let $\nu \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$ denote the target, a measure valued process, and $\mu^K \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+))$ denote the measure valued process representing the solution of equation (28) corresponding to the control law $K \in \mathcal{U}_{a,b}$. The cost functional for the problem as stated above is given by

$$J(K) \equiv \mathbb{E}\int_I d_P(\mu_t^K, \nu_t)dt = \int_\Omega \int_I d_P(\mu_{t,\omega}^K, \nu_{t,\omega})dtdP. \tag{38}$$

The objective is to prove the existence of a control law $K \in \mathcal{U}_{a,b}$ that minimizes the above cost functional.

Theorem 4.3. *Consider equation* (28) (*representing the system* (26)) *with the set of admissible controls* $\mathcal{U}_{a,b}$ *and the cost functional given by the expression* (38). *Suppose the assumptions of Theorem 4.1 hold. Then there exists an optimal feedback control* $K_o \in \mathcal{U}_{a,b}$ *minimizing the functional* (38).

Proof. We prove that J is continuous in the τ_p topology. Let $\{K_n\} \subset \mathcal{U}_{a,b}$ and suppose $K_n \xrightarrow{\tau_p} K_o \in \mathcal{U}_{a,b}$ and let$\{\mu^n, \mu^o\} \subset M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+))$ denote the corresponding solutions of equation (28). Clearly, it follows from Theorem 4.1 that $\mu^n \xrightarrow{w^*} \mu^o$. Since μ^n converges to μ^o in the weak star topology of $M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+))$ inherited from $L_\infty^w(I \times \Omega, \mathcal{M}_{rba}(E^+))$, there exists a subsequence $\{\mu^{n_k}\}$ of the sequence $\{\mu^n\}$ such that $\mu_{t,\omega}^{n_k} \xrightarrow{w^*} \mu_{t,\omega}^o$ in the relative weak star topology on the space $\mathcal{P}_{rba}(E^+)$ for almost all $(t, \omega) \in I \times \Omega$. As stated above, weak star convergence of probability measures is equivalent to convergence in the Lévy-Prokhorov metric d_P. Hence $d_P(\mu_{t,\omega}^{n_k}, \mu_{t,\omega}^o) \to 0$ for almost all $(t, \omega) \in I \times \Omega$. Recall that $d_P(\mu_1, \mu_2) \leq 2$ for all $\mu^1, \mu^2 \in \mathcal{P}_{rba}(E^+)$. Thus it follows from Lebesgue bounded convergence theorem that

$$\lim_{k \to \infty} \int_{I \times \Omega} d_P(\mu_{t,\omega}^{n_k}, \mu_{t,\omega}^o) \, dtdP = 0. \tag{39}$$

Using this result we prove that J is continuous with respect to τ_p topology on $\mathcal{U}_{a,b}$. Let $K_n \xrightarrow{\tau_p} K_o$ and $\mu^{K_n} \xrightarrow{w*} \mu^{K_o}$. Clearly for all $n \in \mathbb{N}$,

$$J(K_o) = \mathbb{E} \int_I d_P(\mu_{t,\omega}^{K_o}, \nu_{t,\omega}) dt \leq \mathbb{E} \int_I d_P(\mu_{t,\omega}^{K_o}, \mu_{t,\omega}^{K_n}) dt + \mathbb{E} \int_I d_P(\mu_{t,\omega}^{K_n}, \nu_{t,\omega}) dt$$

and hence by virtue of (39), we conclude that

$$J(K_o) \leq \underline{\lim}_{n \to \infty} J(K_n). \tag{40}$$

Similarly, for all $n \in \mathbb{N}$, we have

$$J(K_n) = \mathbb{E} \int_I d_P(\mu_{t,\omega}^{K_n}, \nu_{t,\omega}) dt \leq \mathbb{E} \int_I d_P(\mu_{t,\omega}^{K_n}, \mu_{t,\omega}^{K_o}) dt + \mathbb{E} \int d_P(\mu_{t,\omega}^{K_o}, \nu_{t,\omega}) dt,$$

and we conclude from this inequality that

$$\overline{\lim}_{n \to \infty} J(K_n) \leq J(K_o). \tag{41}$$

Hence it follows from the inequalities (40) and (41) that $J(K_n) \to J(K_o)$ as $n \to \infty$ proving continuity of the functional J in the τ_p topology on $\mathcal{U}_{a,b}$. Since $\mathcal{U}_{a,b}$ is compact in the τ_p topology, J attains both its minimum (and maximum). Hence an optimal feedback control exists. □

Next,we consider an exit time problem. Consider the initial state, a measure $\mu_o \in \mathcal{P}_{rba}(E^+)$ having support given by a closed bounded set $C \subset E$. For any given (finite) positive number r, let $B_r \subset E$ denote the closed ball of radius r centered at the origin. Clearly, we can choose $r > 0$, sufficiently large, so that the set C is contained in the interior of B_r. For any control law $K \in \mathcal{U}_{a,b}$, let $\mu^K \in M_\infty^w(I \times \Omega, \mathcal{P}_{rba}(E^+))$ denote the measure valued solution of the control system (26) or equivalently the solution of equation (28). For any $\varepsilon \in (0,1)$ as small as desired, the set $\{t \in I : \mu^K_{t,\omega}(B_r) \leq 1-\varepsilon\}$ denotes the set of exit times (time instants at which the support of μ_t has nonempty intersection with the set $(E \setminus B_r)$). Then the first exit time corresponding to the control law K is given by

$$\mathcal{T}(K) \equiv \inf\{t \in I : \mu^K_{t,\omega}(B_r) \leq 1-\varepsilon\}.$$

This is an $\mathcal{F}$-measurable random variable possibly taking values in $I \equiv [0,T]$. In case the underlying set is empty, we set $\mathcal{T}(K) = 0$. The problem is to find a control from the admissible set $\mathcal{U}_{a,b}$ such that the expected value of the first exit time is maximum. Hence, the objective functional is given by

$$J(K) \equiv \mathbb{E}\{\mathcal{T}(K)\} \tag{42}$$

and the problem is to find a control law that maximizes it. We prove the following result on existence of optimal control law.

Theorem 4.4. *Consider the control system* (28) *corresponding to the control law $K \in \mathcal{U}_{a,b}$ with the initial state μ_0 having a nonempty closed bounded support $C \subset E$, and the objective functional given by* (42). *Suppose the assumptions of Theorem 4.1 hold. Then there exists a control law in $\mathcal{U}_{ab}$ that maximizes the functional J.*

Proof. We prove that the functional $K \longrightarrow J(K)$ is upper semicontinuous on $\mathcal{U}_{a,b}$. Consider the sequence $\{K^n\} \in \mathcal{U}_{a,b}$ and suppose K^n converges to $K^o \in \mathcal{U}_{a,b}$ in the τ_p topology. Let $\{\mu^n\}$ and μ^o denote the corresponding solutions of equation (28). By virtue of Theorem 4.1, we conclude that $\mu^n \xrightarrow{w^*} \mu^o$ in $M^w_\infty(I \times \Omega, \mathcal{P}_{rba}(E^+))$. Since B_r is a closed set, it follows from a well known result [17, Theorem 6.1, p.40] that $\overline{\lim}\,\mu^n_t(B_r) \leq \mu^o_t(B_r)$ for all $t \in I$ and P-almost all $\omega \in \Omega$. With little reflection one can verify that

$$\{t \in I : \mu^o_t(B_r) \leq 1-\varepsilon\} \subset \{t \in I : \overline{\lim}\,\mu^n_t(B_r) \leq 1-\varepsilon\}.$$

Clearly, it follows from the above inclusion that

$$\inf\{t \in I : \overline{\lim}\,\mu^n_t(B_r) \leq 1-\varepsilon\} \leq \inf\{t \in I : \mu^o_t(B_r) \leq 1-\varepsilon\}. \tag{43}$$

Hence for any $\delta \in (0,\varepsilon)$, there exists an integer n_δ such that $\mu^n_t(B_r) < 1-\varepsilon+\delta$ for all $n > n_\delta$. Therefore,

$$\{t \geq 0 : \mu^n_t(B_r) \leq 1-\varepsilon+\delta\} \supset \{t \geq 0 : \overline{\lim}\,\mu^n_t(B_r) \leq 1-\varepsilon\}$$

for all $n > n_\delta$. Thus for all $n > n_\delta$, we have

$$\inf\{t \geq 0 : \mu^n_t(B_r) \leq 1-\varepsilon+\delta\} \leq \inf\{t \geq 0 : \overline{\lim}\,\mu^n_t(B_r) \leq 1-\varepsilon\}. \tag{44}$$

It follows from the expressions (43) and (44) that, for every $\delta \in (0,\varepsilon)$ and $n > n_\delta$, we have the following inequality

$$\begin{aligned} \mathcal{T}_\delta(K^n) &\equiv \inf\{t \geq 0 : \mu^n_t(B_r) \leq 1-\varepsilon+\delta\} \\ &\leq \inf\{t \geq 0 : \mu^o_t(B_r) \leq 1-\varepsilon\} \equiv \mathcal{T}(K^o). \end{aligned} \tag{45}$$

Hence for every $\delta \in (0,\varepsilon)$, we have $\overline{\lim}\,\mathcal{T}_\delta(K^n) \le \mathcal{T}(K^o)$ P-a.s. This implies that

$$\overline{\lim}\,\mathbb{E}\{\mathcal{T}_\delta(K^n)\} \le \mathbb{E}\{\overline{\lim}\,\mathcal{T}_\delta(K^n)\} \le \mathbb{E}\{\mathcal{T}(K^o)\}. \tag{46}$$

Since this holds for every $\delta \in (0,\varepsilon)$, we conclude that $\overline{\lim}\,\mathbb{E}\{\mathcal{T}(K^n)\} \le \mathbb{E}\{\mathcal{T}(K^o)\}$, and therefore

$$\overline{\lim}\, J(K^n) \le J(K^o), \tag{47}$$

proving upper semicontinuity of J on $\mathcal{U}_{a,b}$ with respect to the τ_p topology. Since $\mathcal{U}_{a,b}$ is compact in the τ_p topology, J attains it's maximum on it. This proves that an optimal feedback control exists. □

Remark 4.5. Here we have considered the questions of existence of optimal feedback controls for several control problems. Using these results one can develop necessary conditions of optimality whereby one can determine the optimal control laws.

5. Range of vector measures and control

In this section we present some properties of the range of measures induced by the solutions of the stochastic control system (26) or equivalently equation (28). Recall, that corresponding to every feedback control $K \in \mathcal{U}_{a,b}$, there exists a measure valued solution $\mu^K \in M_\infty^w(I\times\Omega, \mathcal{P}_{rba}(E^+)) \subset L_\infty^w(I\times\Omega, \mathcal{M}_{rba}(E^+))$. Let Σ_+ denote the algebra of subsets of the set E^+ and $\Sigma_o \equiv \sigma(\Sigma_+)$ the sigma algebra generated by Σ_+. Let $B_1^o(E^+)$ denote the class of Σ_o measurable nonnegative real valued functions defined on E^+ with supremum norm not exceeding one. For each $(t,\omega) \in I \times \Omega$, and $K \in \mathcal{U}_{a,b}$, let

$$\mu_{t,\omega}^K(\Sigma_o) \equiv \{\mu_{t,\omega}^K(\Gamma), \Gamma \in \Sigma_o\} \tag{48}$$

denote the range of the measure corresponding to the associated solution of equation (28). It is well known that any vector measure μ can be decomposed as the sum of two measures one of which is purely atomic and one non-atomic giving $\mu = \hat{\mu} + \mathring{\mu}$. First, we consider the non atomic component denoted by $\mathring{\mu}$ and later the measure-valued solution without decomposition. Since these are countably additive probability measure valued processes, it follows from the well known Lyapunov theorem [11, Corollary 5, p.264] on the range of non-atomic countably additive vector measures taking values in a finite dimensional space that, for each $(t,\omega) \in I \times \Omega$, the set (range) $\mathring{\mu}_{t,\omega}^K(\Sigma_o)$ is a compact convex subset of $[0,1] \subset R$. Hence, letting $L_\infty^+(I \times \Omega)$ denote the class of real valued nonnegative essentially bounded measurable functions, the following set

$$\mathring{\mu}^K(\Sigma_o) = \{g \in L_\infty^+(I \times \Omega) : g(t,\omega) \in \mathring{\mu}_{t,\omega}^K(\Sigma_o)\ \forall\ (t,\omega) \in I \times \Omega\} \tag{49}$$

is well defined. Let $Z \equiv L_1(I \times \Omega)$ with its dual given by $Z^* \equiv L_\infty(I \times \Omega)$ and let $B_1(Z^*)$ denote the closed unit ball in Z^* (centered at the origin) with $B_1^+(Z^*)$ denoting its positive part. Clearly, $\mathring{\mu}^K(\Sigma_o) \subset B_1^+(Z^*) \subset B_1(Z^*)$ and that, for each $K \in \mathcal{U}_{a,b}$, the set $\mathring{\mu}^K(\Sigma_o)$ is a w^*-closed convex subset of Z^*.

Thus, by Alaoglu's theorem [14], it is a w^* compact convex subset of Z^* and it is also contained in the compact set $B_1^+(Z^*)$. Hence it follows from Krein-Milman theorem [14] that it is the w^* closed convex hull of its extreme points, that is, $\mathring{\mu}^K(\Sigma_o) = \overline{co}^*(Ext.(\mathring{\mu}^K(\Sigma_o)))$. If $(\Omega, \mathcal{F}, P)$ is a separable probability space, for example, Ω is a Polish space (separable completely metrizable topological space) and $\mathcal{F}$ is the class of Borel subsets of Ω then $L_1(I\times\Omega) \equiv Z$ is a separable Banach space. (In general a measure space is separable if and only if the sigma algebra is countably generated). Thus the closed unit ball $B_1(Z^*)$ of the space Z^* is metrizable with the metric given by

$$d(z_1^*, z_2^*) = \sum_{n\geq 1}(1/2^n)\frac{|(z_1^*(z_n) - z_2^*(z_n))|}{1+|(z_1^*(z_n) - z_2^*(z_n))|}, \text{ for } z_1^*, z_2^* \in B_1(Z^*) \tag{50}$$

where $\{z_n\} \subset Z$ is a countable set dense in Z. Thus we have a compact metric space $(B_1(Z^*), d)$.

Consider the family of sets $\mathring{\mathcal{R}} \equiv \{\mathring{\mu}^K(\Sigma_o), K \in \mathcal{U}_{a,b}\}$ where each element is the range of the non-atomic component of the measure valued process induced by the system (28) corresponding to $K \in \mathcal{U}_{a,b}$. It is a family of nonempty compact convex sets contained in the compact set $B_1^+(Z^*)$ of the compact metric space $(B_1(Z^*), d)$. Using the metric d as defined above, we can introduce the Hausdorff metric on $\mathring{\mathcal{R}}$ as follows

$$D_H(C_1, C_2) = \max\{\sup\{d(C_1, \eta), \eta \in C_2\}, \sup\{d(\xi, C_2), \xi \in C_1\}\}$$

for $C_1, C_2 \in \mathring{\mathcal{R}}$. It follows from Theorem 4.1, that as $K_n \xrightarrow{\tau_p} K_o$, $\mu^{K_n} \xrightarrow{w^*} \mu^{K_o}$. Thus the corresponding atomic and non-atomic components $\{\hat{\mu}^{K_n}, \mathring{\mu}^{K_n}\}$ converge in the w^*-sense to the respective atomic and non-atomic components $\{\hat{\mu}^{K_o}, \mathring{\mu}^{K_o}\}$ as well. Then it follows from multi-valued analysis [16, Theorem 7.2.1, p.684] that there exists a subsequence $\{\mathring{\mu}^{K_{n_m}}\}$ of the sequence $\{\mathring{\mu}^{K_n}\}$ such that

$$D_H(\mathring{\mu}^{K_{n_m}}(\Sigma_o), \mathring{\mu}^{K_o}(\Sigma_o)) \longrightarrow 0, as\ m \to \infty$$

with $\mathring{\mu}^{K_o}(\Sigma_o) \in \mathring{\mathcal{R}}$.

One interesting control problem is to find a feedback control law K_o from the admissible class $\mathcal{U}_{a,b}$ corresponding to which the expanse of the range $\mathring{\mu}^{K_o}(\Sigma_o)$ is maximum(or minimum). We may use the diameter as the measure of expanse of any element of the family $\mathring{\mathcal{R}}$. Using the metric d as defined above, the diameter of any set $C \in \mathring{\mathcal{R}}$ is given by

$$Dia(C) \equiv sup\{d(x,y), x, y \in C\}.$$

For the control problem, we may choose the objective functional as

$$J(K) \equiv Dia(\mathring{\mu}^K(\Sigma_o)), \text{ for } K \in \mathcal{U}_{a,b}. \tag{51}$$

The problem is to find a control law $K \in \mathcal{U}_{a,b}$ that maximizes(or minimizes) this functional.

Theorem 5.1. *Consider the system* (28) (*equivalently the system* (26)) *with the set of admissible controls* $\mathcal{U}_{a,b}$ *and the objective functional given by the expression* (51). *Suppose the assumptions of Theorem 4.1 hold and that the probability space* $(\Omega, \mathcal{F}, P)$ *is separable. Then there exists an optimal feedback control* $K_o \in \mathcal{U}_{a,b}$ *maximizing* (*minimizing*) *the functional* (51).

Proof. We prove that the functional J given by the expression (51) is continuous with respect to the topology τ_p on $\mathcal{U}_{a,b}$. Let $\{K_n\}$ be any sequence from $\mathcal{U}_{a,b}$ and suppose $K_n \xrightarrow{\tau_p} K_o$ and let $\{\mathring{\mu}^{K_n}(\Sigma_o)\}$ and $\mathring{\mu}^{K_o}(\Sigma_o)$ denote the corresponding elements from the set $\mathring{\mathcal{R}}$. We know that these are compact convex sets and that they are continuous (with respect to the control laws) in the τ_p topology on $\mathcal{U}_{a,b}$ and the Hausdorff metric topology on $\mathring{\mathcal{R}}$ (as discussed above), leading to

$$D_H(\mathring{\mu}^{K_n}(\Sigma_o), \mathring{\mu}^{K_o}(\Sigma_o)) \longrightarrow 0, \text{ as } n \to \infty.$$

For simplicity of notations, we let $\Gamma_n \equiv \mathring{\mu}^{K_n}(\Sigma_o)$ and $\Gamma_o \equiv \mathring{\mu}^{K_o}(\Sigma_o)$ and introduce the following family of sets

$$\Gamma_n^* \equiv \{(x,y) \in \Gamma_n : d(x,y) \geq d(\xi,\eta), \ \forall \ (\xi,\eta) \in \Gamma_n\}, n \in \mathbb{N}$$

and

$$\Gamma_o^* \equiv \{(x,y) \in \Gamma_o : d(x,y) \geq d(\xi,\eta), \ \forall \ (\xi,\eta) \in \Gamma_o\}.$$

Since the family of sets $\{\Gamma_n, \Gamma_o\}$ are compact and the metric (function) $d(\cdot,\cdot)$ is continuous, the family of sets $\{\Gamma_n^*, \Gamma_o^*\}$ are nonempty. These sets consist of pairs of elements of Γ_n and Γ_o respectively that determine their respective diameters. They are nonempty closed subsets of compact sets and hence compact. Since $D_H(\Gamma_n, \Gamma_o) \to 0$, it is clear that $D_H(\Gamma_n^*, \Gamma_0^*) \to 0$ also. In other words, every pair $(x_n, y_n) \in \Gamma_n^*$ converges to a pair $(x_o, y_o) \in \Gamma_o^*$ in the sense that $d(x_n, y_n) \to d(x_o, y_o)$. This follows from the fact that

$$|d(x_n, y_n) - d(x_o, y_o)| \leq d(x_n, x_o) + d(y_n, y_o),$$

and hence, as $n \to \infty$, $d(x_n, y_n) \longrightarrow d(x_o, y_o)$. Thus, as $n \to \infty$,

$$Dia(\Gamma_n) = Dia(\Gamma_n^*) \longrightarrow Dia(\Gamma_o^*) = Dia(\Gamma_o).$$

Hence we conclude that $J(K_n) \to J(K_o)$ as $K_n \xrightarrow{\tau_\pi} K_o$. Since the set $\mathcal{U}_{a,b}$ is compact in the τ_p topology and J is continuous with respect to the same topology, we conclude that J attains its maximum (and minimum) on $\mathcal{U}_{a,b}$. This completes the proof. □

Remark 5.2. It is interesting to note that the end points of the line segments that determine the diameter of a compact convex set are contained in the set of its extreme points. This is justified by the facts that the sets $\{\Gamma_n\}$ and Γ_o are compact and convex and the function $(x,y) \longrightarrow d(x,y)$ is convex and continuous and hence it attains its maximum at the extreme points of the sets $\{\Gamma_n\}$ and Γ_o.

Remark 5.3. Theorem 5.1 is based on the decomposition of a measure into the sum of a non-atomic and an atomic component. The objective functional given by the expression (51) uses only the non-atomic component for which the Lyapunov theorem (on the range of vector measures) holds. However, the objective is to find a control that maximizes the expanse (or size) of the range of the measure-valued process itself determined by the solution of equation (28), not its non-atomic component only. In the general case, Lyapunov theorem does not hold. So, we revise and replace the objective functional (51) by the following relaxed objective functional

$$\tilde{J}(K) \equiv Dia(\tilde{\mu}^K(\Sigma_o)) \text{ for } K \in \mathcal{U}_{a,b}, \tag{52}$$

where

$$\tilde{\mu}^K(\Sigma_o) \equiv \{g \in B_1^+(Z^*) : g(t,\omega) \in \overline{co}(\mu^K_{t,\omega}(\Sigma_o)) \forall (t,\omega) \in I \times \Omega\}.$$

This is a closed (convex) subset of the compact metric space $(B_1(Z^*), d)$ and hence compact (convex). This follows readily from the fact that

$$I \times \Omega \ni (t,\omega) \longrightarrow \overline{co}(\mu^K_{t,\omega}(\Sigma_o))$$

is a measurable multi function with nonempty convex compact values in $[0,1]$. So, it follows from the well known Kuratowski–Ryll-Nardzewski selection theorem [16, Theorem 2.1, p.154], that it has a nonempty set of measurable selections.

Hence, following similar steps as in Theorem 5.1, one can prove existence of optimal feedback control maximizing (minimizing) the functional (52).

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