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CHARACTERISTICS OF LORENTZIAN-PARA KENMOTSU SPACETIME MANIFOLDS

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Abstract. The primary objectives of this paper is to study a special type of spacetime known as Lorentzian para-Kenmotsu (LPK) spacetime, characterized by a coefficient α, satisfying curvature conditions $\mathcal{C}(X,Y)\cdot\xi = 0$ and $\mathsf{R}(X,Y)\cdot\mathcal{S} = 0$. We study the conditions under which the generalized $\mathcal{Z}$-tensor becomes almost pseudo-$\mathcal{Z}$-symmetric spacetime. Finally, we provide an example of (LPK) spacetime with a coefficient α that satisfies certain notable results.

1. Introduction

In the context of relativity, space and time are combined into a single four-dimensional continuum called spacetime instead of considered as separate objects. Four coordinates are used to describe each event or location in spacetime, three for the spatial dimensions x, y and z and one for the temporal dimension t. Spacetime is a manifold that is continuous and differentiable. This means that we can define scalars, vectors, 1-forms and in general tensor fields and are able to take derivatives at any point. An elementary amorphous arrangement of points (or events in the case of spacetime) is referred to as a differential manifold. Locally, these points are organised as points in a Euclidian space. Next, we specify a distance concept by adding a metric g, which contains information about what are the distances between points.

The basic difference between the Riemannian and the semi-Riemannian geometry is the presence of a null vector v which satisfies $g(v,v) = 0$, where g is the metric tensor. In Riemannian manifold the signature of a metric g is $(+,+,\ldots,+,+)$ whenever the signature of g is $(-,-,-,\ldots,+,+,+)$ in semi Riemannian geometry. A Lorentzian manifold is a special case of semi-Riemannian manifold. Also the signature of metric g in Lorentzian manifold is $(-,+,+,+,+)$. Lorentzian manifold admits three types of vectors such as timelike, spacelike and null vector according as $g(v,v) < 0$, $g(v,v) > 0$ and

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$g(v, v) = 0$, respectively. Generally a Lorentzian manifold may not admit globally timelike vector field, but in case if it admit a globally timelike vector field then it is called spacetime manifold. A Lorentzian manifold is a special case of a semi-Riemannian manifold. Spacetime manifold refers to a four dimensional connected semi-Riemannian manifold $(\mathcal{M}^4, g)$ along with Lorentzian metric g with signature $(-, +, +, +)$.

The notion of Lorentzian para Sasakian manifold with a coefficient α were first introduced by U. C. De, A. A. Shaikh and A. Sengupta [5] in 2002. In this way, I. Mihai and R. Rosca [10] also introduced the same notion of Lorentzian para Sasakian manifolds independently and gives several results. Later R. Prasad and A. Haseeb [11] studied various properties of α-para Kenmotsu manifolds with semi-symmetric metric connection. For an in-depth examination of spacetime in Lorentzian manifolds, we refer the reader to references [6] and [12]. Following up on previous research, this study explores specific characteristics of (LPK) type spacetime, particularly those associated with the coefficient α.

This paper is structured as follows: Following the preliminaries discussion in Section 2, we investigate ξ-conformally flat (LPK) type spacetime in Section 3. In the next section, we prove that under the Ricci semisymmetric condition $\mathsf{R}(X, Y) \cdot \mathcal{S} = 0$, the manifold becomes an Einstein manifold. Section 5 explores Almost pseudo $\mathcal{Z}$-symmetric spacetime with the Ricci tensor is of Codazzi type. The last section focuses on the study of Ricci-soliton and η-Ricci soliton in (LPK) type spacetime. Ultimately, we provide an example that satisfies the conditions for an (LPK) type spacetime manifold with coefficient α.

2. Preliminaries

Let $\mathcal{M}$ be an n-dimensional Lorentzian metric manifold. If it is endowed with a structure (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a Lorentzian metric on $\mathcal{M}$, satisfying [11], [7]

$$\phi^2 X = X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{1}$$

$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \tag{2}$$

for any $X, Y \in \Gamma(\mathcal{M})$, where $\Gamma(\mathcal{M})$ is the set of all smooth vector fields on $\mathcal{M}$, then manifold $\mathcal{M}$ is called a Lorentzian almost paracontact manifold [9].

In a Lorentzian almost paracontact manifold, the following relations hold [8]:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{3}$$

$$\Phi(X, Y) = \Phi(Y, X), \tag{4}$$

where $\Phi(X, Y) = g(X, \phi Y)$.

LPK manifold with a coefficient α: An n-dimensional differentiable manifold $\mathcal{M}$ with a Lorentzian metric g, (1,1) tensor field ϕ and a vector field ξ

satisfied

$$(\nabla_X \phi)Y = -\alpha\{g(\phi X, Y)\xi + \eta(Y)\phi X\}, \tag{5}$$

for any vector fields X, Y on $\Gamma(\mathcal{M})$.

Putting $Y = \xi$ and using (2) in (5), we get

$$\nabla_X \xi = -\alpha\{X + \eta(X)\xi\}. \tag{6}$$

Taking covariant derivative of equation (2) along vector field Y, we get

$$(\nabla_X \eta)Y = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}. \tag{7}$$

Lemma 2.1. *If K is a tensor field of type $(0,1)$ on a Riemannian manifold $(\mathcal{M}, g)$, then we have relation* [4]

$$(\nabla_X \nabla_Y K)(\mathcal{Z}) - (\nabla_Y \nabla_X K)(\mathcal{Z}) - (\nabla_{[X,Y]} K)\mathcal{Z} = -K(\mathsf{R}(X, Y)\mathcal{Z}), \tag{8}$$

known as Ricci identity.

Definition. If the relation

$$(\nabla_{\mathcal{Z}} \Phi)(X, Y) = -\alpha\Phi(\mathcal{Z}, X)\eta(Y) - \alpha\Phi(\mathcal{Z}, Y)\eta(X) \tag{9}$$

or

$$(\nabla_{\mathcal{Z}} \Phi)(X, Y) = \alpha\Big\{g(X, \mathcal{Z})\eta(Y) + g(Y, \mathcal{Z})\eta(X) + 2\eta(X)\eta(Y)\eta(\mathcal{Z})\Big\} \tag{10}$$

and

$$\Phi(X, Y) = -\frac{1}{\alpha}(\nabla_X \eta)Y \tag{11}$$

hold for a Lorentzian almost paracontact manifold, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function then the manifold is known as an (LPK) manifold with a coefficient α.

Equation (11) is rewritten as

$$(\nabla_Y \eta)\mathcal{Z} = -\alpha\Phi(Y, \mathcal{Z}).$$

Covariantly differentiate above, we get

$$(\nabla_X \nabla_Y \eta)\mathcal{Z} = -\nabla_X \alpha\Phi(Y, \mathcal{Z}) - \alpha(\nabla_X \Phi)(Y, \mathcal{Z}) - \alpha\Phi(\nabla_X Y, \mathcal{Z}). \tag{12}$$

Similarly, we have

$$(\nabla_Y \nabla_X \eta)\mathcal{Z} = -\nabla_Y \alpha\Phi(X, \mathcal{Z}) - \alpha(\nabla_Y \Phi)(X, \mathcal{Z}) - \alpha\Phi(\nabla_Y X, \mathcal{Z}) \tag{13}$$

and

$$(\nabla_{[X,Y]} \eta)\mathcal{Z} = -\alpha\Phi([X, Y], \mathcal{Z}) = -\alpha\Phi(\nabla_X Y, \mathcal{Z}) + \alpha\Phi(\nabla_Y X, \mathcal{Z}). \tag{14}$$

Using equations (12)-(14) and Lemma 2.1, we obtain

$$\eta(\mathsf{R}(X, Y)\mathcal{Z}) = \nabla_X \alpha\Phi(Y, \mathcal{Z}) - \nabla_Y \alpha\Phi(X, \mathcal{Z}) + \alpha(\nabla_X \Phi)(Y, \mathcal{Z}) - \alpha(\nabla_Y \Phi)(X, \mathcal{Z}).$$

Using (9) and (7), above equation turns into the form

$$g(\mathsf{R}(X,Y)\mathcal{Z},\xi) = -(X\alpha)\{g(Y,\mathcal{Z})+\eta(Y)\eta(\mathcal{Z})\} + (Y\alpha)\{g(X,\mathcal{Z})+\eta(X)\eta(\mathcal{Z})\} + \alpha^2\{g(X,\mathcal{Z})\eta(Y)+g(Y,\mathcal{Z})\eta(X)\}. \tag{15}$$

Taking $X = \mathcal{Z} = \{e_i\}$ and contracting above, we get

$$\mathcal{S}(Y,\xi) = (n-2)(Y\alpha) - (\xi\alpha)\eta(Y) + \alpha^2(n-1)\eta(Y). \tag{16}$$

The Riemannian curvature tensor is defined as

$$\mathsf{R}(X,Y)\mathcal{Z} = \nabla_X\nabla_Y\mathcal{Z} - \nabla_Y\nabla_X\mathcal{Z} - \nabla_{[X,Y]}\mathcal{Z}. \tag{17}$$

Put $\mathcal{Z} = \xi$, we have

$$\mathsf{R}(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi. \tag{18}$$

Now,

$$\begin{aligned}\nabla_X\nabla_Y\xi &= \nabla_X\{\alpha(X+\eta(X)\xi)\}\\ &= \alpha\{\nabla_X Y + \alpha g(X,Y) + 2\alpha\eta(X)\eta(Y) + \alpha\eta(Y)X + \eta(\nabla_X Y)\xi\}.\end{aligned}$$

Similarly solving second and third term of (18) and simplifying we get

$$\mathsf{R}(X,Y)\xi = (\alpha^2+\sigma)\{\eta(Y)X - \eta(X)Y\}. \tag{19}$$

Putting $X = \{e_i\}$ and contracting (19), we get

$$\mathcal{S}(Y,\xi) = 3(\alpha^2+\sigma)\eta(Y). \tag{20}$$

and

$$\mathcal{Q}\xi = 3(\alpha^2+\sigma)\xi.$$

Again, we know

$$\eta(\mathsf{R}(X,Y)\mathcal{Z}) = g(\mathsf{R}(X,Y)\mathcal{Z},\xi) = (\alpha^2+\sigma)\{\eta(X)g(Y,\mathcal{Z}) - \eta(Y)g(X,\mathcal{Z})\}. \tag{21}$$

Definition. Conformal curvature tensor $\mathcal{C}$ for an n-dimensional (LPK) manifold is given as [1]

$$\begin{aligned}&\mathcal{C}(X,Y)\mathcal{Z}\\ &= \mathsf{R}(X,Y)\mathcal{Z}\\ &\quad - \frac{1}{n-2}\Big\{\mathcal{S}(Y,\mathcal{Z})X - \mathcal{S}(X,\mathcal{Z})Y + g(Y,\mathcal{Z})\mathcal{Q}X - g(X,\mathcal{Z})\mathcal{Q}Y\Big\}\\ &\quad + \frac{r}{(n-1)(n-2)}\Big\{g(Y,\mathcal{Z})X - g(X,\mathcal{Z})Y\Big\},\end{aligned} \tag{22}$$

for all $X, Y, \mathcal{Z} \in \Gamma(\mathcal{M})$, where r is the scalar curvature of $\mathcal{M}$ and $\mathcal{Q}$ is the Ricci operator corresponding to the Ricci tensor $\mathcal{S}$.

Perfect fluid spacetime: The study of perfect fluid spacetimes has far-reaching implications for our understanding of the universe. In particular, perfect fluid models play a crucial role in general relativity. If the Ricci tensor of an Lorentzian manifold is of the form [1]

$$\mathcal{S}(X,Y) = \alpha_1 g(X,Y) + \alpha_2 \eta(X)\eta(Y), \tag{23}$$

where α_1, α_2 are scalar fields and ξ is a unit timelike vector field, then it is called perfect fluid spacetime.

In a perfect fluid spacetime we have [2]

$$g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = \eta(\xi) = -1.$$

The standard form of Einstein's Field Equations, without the inclusion of the cosmological constant, is given by

$$\mathcal{S}(X,Y) - \frac{r}{2} g(X,Y) = \kappa T(X,Y), \tag{24}$$

for all $X, Y \in \Gamma(\mathcal{M})$ where κ and T are the gravitational constant and energy momentum tensor of type $(0,2)$, respectively.

If the energy momentum tensor T describes an ideal fluid, then we have

$$T(X,Y) = (\mu + p)\eta(X)\eta(Y) + pg(X,Y), \tag{25}$$

where μ and p are the energy density function and isotropic pressure function of the fluid, respectively.

3. ξ-conformally flat (LPK) type spacetime

If a 4-dimensional (LPK) type spacetime is of conformally flat with respect to ξ, then the conformal curvature tensor satisfies $\mathcal{C}(X,Y) = 0$ for all vector fields X and Y on the manifold. Utilizing this condition $\mathcal{C}(X,Y)\xi = 0$, (22) implies

$$\begin{aligned} \mathsf{R}(X,Y)\xi = & \frac{1}{2}\Big\{ \mathcal{S}(Y,\xi)X - \mathcal{S}(X,\xi)Y + g(Y,\xi)\mathcal{Q}X - g(X,\xi)\mathcal{Q}Y \Big\} \\ & + \frac{r}{6}\Big\{ g(Y,\xi)X - g(X,\xi)Y \Big\}. \end{aligned} \tag{26}$$

Using $(2), (19)$ and (20) in (26), we get

$$\left(\frac{-\alpha^2}{2} - \frac{r}{6}\right)\big\{\eta(Y)X - \eta(X)Y\big\} = \frac{1}{2}\big\{\eta(Y)\mathcal{Q}X - \eta(X)\mathcal{Q}Y\big\}. \tag{27}$$

Substituting $Y = \xi$ in forgoing equation, we have

$$\mathcal{Q}X = -(\alpha^2 + \sigma + \frac{r}{3})X - \{4(\alpha^2 + \sigma) + \frac{r}{3}\}\eta(X)\xi. \tag{28}$$

Taking inner product with Y, above yields

$$\mathcal{S}(X,Y) = -(\alpha^2 + \sigma + \frac{r}{3})g(X,Y) - \{4(\alpha^2 + \sigma) + \frac{r}{3}\}\eta(X)\eta(Y), \tag{29}$$

which is of the form perfect fluid spacetime. Thus we state the following.

Theorem 3.1. *If an n-dimensional (LPK) type spacetime manifold is ξ-conformally flat then it becomes perfect fluid spacetime.*

Now since the manifold under consideration is conformally flat, it is ξ-conformally flat also. Then we can state following:

Corollary 3.2. *An conformally flat (LPK) type spacetime is a perfect fluid spacetime.*

4. (LPK) type spacetime satistying $\mathsf{R}(X, Y) \cdot \mathcal{S} = 0$

This section focuses on (LPK) type spacetime satisfying $\mathsf{R}(X, Y) \cdot \mathcal{S} = 0$, then from $(\mathsf{R}(X, Y) \cdot \mathcal{S})(Z, W) = 0$, we have

$$\mathcal{S}(\mathsf{R}(X, Y)\mathcal{Z}, W) + \mathcal{S}(\mathcal{Z}, \mathsf{R}(X, Y)W) = 0.$$

Taking $\mathcal{Z} = \xi$ and using (19), (20), we get

$$\mathcal{S}\{(\alpha^2 + \sigma)(\eta(Y)X - \eta(X)Y), W\} + 3(\alpha^2 + \sigma)\eta(\mathsf{R}(X, Y)W) = 0.$$

Using (21) in the above yields

$$(\alpha^2 + \sigma)\eta(Y)\mathcal{S}(X, W) - (\alpha^2 + \sigma)\eta(X)\mathcal{S}(Y, W)$$
$$+ 3(\alpha^2 + \sigma)\Big\{ - (\alpha^2 + \sigma)(\eta(Y)g(X, W)) + (\alpha^2 + \sigma)\eta(X)g(Y, W)\Big\} = 0.$$

Taking $Y = \xi$ and using (20), we obtain

$$\mathcal{S}(X, W) = 3(\alpha^2 + \sigma)g(X, W). \tag{30}$$

Hence the manifold under consideration is an Einstein manifold.

Consider next that a perfect fluid spacetime obeys Einstein's equation and with its velocity vector field is the manifold's characteristic vector field, then we have Einstein's field equation as

$$\mathcal{S}(X, Y) - \frac{r}{2}g(X, Y) = \kappa T(X, Y) = \kappa[(\mu + p)\eta(X)\eta(Y) + pg(X, Y)]. \tag{31}$$

Put $Y = \xi$ and using (30) in (31), we get

$$\mu = \frac{6(\alpha^2 + \sigma) - r}{2\kappa}. \tag{32}$$

Now put $X = Y = \xi$ and contracting (31), we get

$$p = \frac{-6(\alpha^2 + \sigma) - r}{6\kappa}. \tag{33}$$

Equations (32) and (33) together give

$$(p + \mu) = scalar. \tag{34}$$

Thus we have:

Theorem 4.1. *If (LPK) type spacetime is an Einstein spacetime and satisfying $R(X, Y) \cdot \mathcal{S} = 0$, then $(p + \mu)$= scalar is the state equation.*

Definition. A $(0,2)$ symmetric tensor is a generalized $\mathcal{Z}$-tensor if it satisfies [3]

$$\mathcal{Z}(X,Y) = \mathcal{S}(X,Y) + \beta g(X,Y), \tag{35}$$

where β is an arbitrary scalar function. Contracting (35) over X and Y, we get the scalar γ as

$$\gamma = r + n\beta.$$

A manifold is called almost pseudo-$\mathcal{Z}$-symmetric if the generalized $\mathcal{Z}$ tensor is non zero and fulfilling [3]

$$\begin{aligned} &(\nabla_X \mathcal{Z})(Y,W) \\ &= \eta_1(X) + \eta_2(X)\mathcal{Z}(Y,W) + \eta_1(Y)\mathcal{Z}(X,W) + \eta_1(W)\mathcal{Z}(X,Y), \end{aligned} \tag{36}$$

where $\mathcal{Z}$ is the generalized $\mathcal{Z}$-tensor and is obtained by taking $\beta = -\frac{r}{n}$, where r is the scalar curvature and η_1, η_2 are the associated 1-forms of the manifold.

Definition. A Riemannian manifold is said to Codazzi type of Ricci tensor if its Ricci tensor $\mathcal{S}$ is non zero and satisfies [5]

$$(\nabla_X \mathcal{Z})(Y,W) = (\nabla_Y \mathcal{Z})(X,W).$$

5. Almost pseudo $\mathcal{Z}$-symmetric spacetime with Codazzi type of $\mathcal{Z}$-tensor

Assume that the $\mathcal{Z}$-tensor in almost pseudo $\mathcal{Z}$-symmetric is of Codazzi type i.e.,

$$(\nabla_X \mathcal{Z})(Y,W) = (\nabla_Y \mathcal{Z})(X,W).$$

Using this, (36) gives

$$\begin{aligned} &(\nabla_X \mathcal{Z})(Y,W) - (\nabla_Y \mathcal{Z})(X,W) \\ &= \eta_2(X)\mathcal{Z}(Y,W) - \eta_2(W)\mathcal{Z}(X,Y) + \eta_1(Y)\mathcal{Z}(X,W) - \eta_1(Y)\mathcal{Z}(W,X) \\ &= 0. \end{aligned} \tag{37}$$

Since $\mathcal{Z}$-tensor is symmetric i.e. $\mathcal{Z}(X,W) = \mathcal{Z}(W,X)$, above equation becomes

$$\eta_2(X)\mathcal{Z}(Y,W) - \eta_2(W)\mathcal{Z}(X,Y) = 0. \tag{38}$$

Setting $X = \xi$, we obtain

$$\mathcal{Z}(Y,W) = -\eta_2(W)\mathcal{Z}(\xi,Y). \tag{39}$$

Taking $Y = W = \{e_i\}$ and contracting above, we have

$$\eta_2(X)\gamma = \mathcal{Z}(\xi,Y). \tag{40}$$

Using (40) in (39), we get

$$\mathcal{Z}(Y,W) = -\eta_2(W)\eta_2(Y)\gamma. \tag{41}$$

Equations (35) and (41) give

$$\mathcal{S}(Y,W) = c_1 g(Y,W) + c_2 \eta_2(Y)\eta_2(W), \tag{42}$$

where $c_1 = -\beta$ and $c_2 = -\gamma$, which shows that the spacetime taking into account is a perfect fluid spacetime. Hence we have the following:

Theorem 5.1. *If an almost pseudo $\mathcal{Z}$-symmetric spacetime admits Codazzi type of $\mathcal{Z}$-tensor, then the spacetime is a perfect fluid spacetime.*

6. Ricci soliton and η-Ricci soliton in (LPK) type spacetime

The Ricci flow equation on a smooth manifold $\mathcal{M}$ with Riemannian metric g is given by R. S. Hamilton [7]

$$\frac{\partial}{\partial t}g(t) = -2Ric, \tag{43}$$

where Ric is the Ricci tensor of the metric. The solution of this equation is known as Ricci soliton and is defined by

$$\frac{1}{2}\mathcal{L}_V g - \lambda g = -Ric, \tag{44}$$

where $\lambda \in \mathbb{R}$ and $\mathcal{L}_V$ is the Lie derivative of the manifold. The Ricci soliton is expanding, shrinking and steady according as $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$.

Suppose a perfect fluid spacetime, equipped with a concircular vector field ξ, admits a Ricci soliton. Then, from (44), it follows that

$$\mathcal{L}_V g(X, Y) + 2\mathcal{S}(X, Y) + 2\lambda g(X, Y) = 0. \tag{45}$$

Since $\nabla g = 0, \nabla\lambda g = 0$. Therefore $\mathcal{L}_V g + 2\mathcal{S}$ is parallel, which implies $\mathcal{L}_V g + 2\mathcal{S}$ is constant multiple of g.

i.e. $\mathcal{L}_V g + 2\mathcal{S} = c_3 g$, c_3 is a constant.

So, (45) takes form $(c_3 + 2\lambda)g = 0$, which gives $\lambda = -\frac{c_3}{2}$. Thus we have the following result:

Theorem 6.1. *If a perfect fluid spacetime with concircular vector field admitting Ricci soliton, then the Ricci soliton is expanding or shrinking according as c_3 is negative or positive.*

Next, put $V = \xi$ in (45), we get

$$\mathcal{L}_\xi g(X, Y) + 2\mathcal{S}(X, Y) + 2\lambda g(X, Y) = 0. \tag{46}$$

On simplifying it becomes

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2\mathcal{S}(X, Y) + 2\lambda g(X, Y) = 0. \tag{47}$$

Using (6), (47) becomes

$$\mathcal{S}(X, Y) = (\alpha - \lambda)g(X, Y) + \alpha\eta(X)\eta(Y). \tag{48}$$

Thus the spacetime manifold is an η-Einstein. So we have following result:

Theorem 6.2. *If a perfect fluid spacetime $\mathcal{M}$ with concircular vector field admitting Ricci soliton, them the soliton becomes η-Einstein manifold.*

Taking $X = Y = \{e_i\}$ in (48), we obtain

$$r = 4d_1 - d_2, \tag{49}$$

where $d_1 = (\alpha - \lambda)$ and $d_2 = \alpha$.

Definition. An n-dimensional (LPK) manifold with the coefficient α is said to be η-Einstein manifold if it satisfies the relation

$$\mathcal{S}(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{50}$$

where a and b are the associated scalar functions on the manifold.

Consider an n-dimensional (LPK) manifold with the coefficient α is η-Einstein, then by contracting (50), we get

$$r = an - b. \tag{51}$$

Replace Y with ξ, (50) gives

$$\mathcal{S}(X,\xi) = (a-b)\eta(X). \tag{52}$$

Equation (16) and (52), combined yields

$$(a-b)\eta(Y) = (n-2)(Y\alpha) - (\xi\alpha)\eta(Y) + \alpha^2(n-1)\eta(Y). \tag{53}$$

Setting Y equal to ξ and solving, we find the value of scalar a and b as

$$a = \tfrac{r}{n-1} + (\xi\alpha - \alpha^2) \text{ and } b = \tfrac{r}{n-1} + n(\xi\alpha - \alpha^2).$$

From equation (50), we have

$$\begin{aligned}\mathcal{S}(X,Y) = &\left\{\frac{r}{n-1} + (\xi\alpha - \alpha^2)\right\}g(X,Y)\\ &+ \left\{\frac{r}{n-1} + n(\xi\alpha - \alpha^2)\right\}\eta(X)\eta(Y).\end{aligned} \tag{54}$$

Thus we have the following result:

Theorem 6.3. *If an n-dimensional* (LPK) *manifold with a coefficient* α *is an* η*-Einstein then the scalar fields are given as*

$$a = \left\{\tfrac{r}{n-1} + (\xi\alpha - \alpha^2)\right\} \quad \textit{and} \quad b = \left\{\tfrac{r}{n-1} + n(\xi\alpha - \alpha^2)\right\}, \textit{ respectively.}$$

Example 6.4. Consider a 4-dimensional differentiable manifold

$$\mathcal{M} = \{f(x,y,z,t) \in \mathbb{R}^4 : t > 0\},$$

where (x,y,z,t) are the standard coordinates in the 4-dimensional real space $\mathbb{R}^4$. Let (e_1, e_2, e_3, e_4) be a set of linearly independent vector fields at each point of $\mathcal{M}$ and is defined by

$$e_1 = e^{\alpha t}\frac{\partial}{\partial x}, \quad e_2 = e^{\alpha t}\frac{\partial}{\partial y}, \quad e_3 = e^{\alpha t}\frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}.$$

Define the Lorentzian metric g on $\mathcal{M}$ as:

$$g_{ij} = g(e_i, e_j) = \begin{cases} 1, & \text{if} \quad i = j \leq 3 \\ 0, & \text{if} \quad i \neq j \\ -1, & \text{if} \quad i = j = 4. \end{cases} \tag{55}$$

Let η be the 1-form associated with the Lorentzian metric g by

$$\eta(X) = g(X, e_4),$$

for any $X \in \Gamma(\mathcal{M})$.

Let the $(1,1)$-tensor field ϕ be defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = e_3, \quad \phi(e_4) = 0.$$

Using the linearity properties of ϕ and g, we can easily verify the following relations:

$$\eta(e_4) = -1, \quad \phi^2 X = X + \eta(X)e_4, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(\mathcal{M})$. Thus, for $e_4 = \xi$, the manifold under consideration fulfills all the necessary conditions of Lorentzian paracontact manifold for dimension 4.

The non-vanishing components of the Lie brackets are determined as follows:

$$[e_1, e_4] = -\alpha e_1, \quad [e_2, e_4] = -\alpha e_2, \quad [e_3, e_4] = -\alpha e_3, \quad [e_4, e_4] = 0.$$

The renowned Koszul's formula is stated as

$$\begin{aligned} 2g(\nabla_X Y, \mathcal{Z}) = {} & Xg(Y, \mathcal{Z}) + Yg(\mathcal{Z}, X) - \mathcal{Z}g(X, Y) \\ & - g(X, [Y, \mathcal{Z}]) + g(Y, [\mathcal{Z}, X]) + g(\mathcal{Z}, [X, Y]), \end{aligned}$$

by using Koszul's formula we easily calculate

$$\begin{aligned} &\nabla_{e_1} e_1 = -\alpha e_4, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \nabla_{e_1} e_4 = -\alpha e_1, \\ &\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = \alpha e_4, \quad \nabla_{e_2} e_3 = 0, \nabla_{e_2} e_4 = -\alpha e_2, \\ &\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -\alpha e_4, \quad \nabla_{e_3} e_4 = -\alpha e_3, \\ &\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = 0. \end{aligned} \tag{56}$$

Now let

$$X = \sum_{i=1}^{4} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4. \tag{57}$$

Through the utilization of equations (56) and (57), it can be straightforwardly proven that $\nabla_X \xi = -\alpha\{X + \eta(X)\xi\}$ holds for each $X \in \Gamma(\mathcal{M})$. Hence the Lorentzian paracontact manifold is an (LPK) manifold of dimension 4 with coefficient $\alpha \neq 0$.

By using (17) and (56), the components of the curvature tensor are given as

$$\begin{aligned} &\mathsf{R}(e_1, e_2)e_1 = -\alpha^2 e_2, \quad \mathsf{R}(e_1, e_3)e_1 = -\alpha^2 e_3, \quad \mathsf{R}(e_1, e_4)e_1 = -\alpha^2 e_4, \\ &\mathsf{R}(e_1, e_2)e_2 = \alpha^2 e_1, \quad \mathsf{R}(e2, e3)e_2 = -\alpha^2 e_3, \quad \mathsf{R}(e_2, e_4)e_2 = -\alpha^2 e_4, \\ &\mathsf{R}(e_1, e_3)e_3 = \alpha^2 e_1, \quad \mathsf{R}(e_2, e_3)e_3 = \alpha^2 e_2, \quad \mathsf{R}(e_3, e_4)e_3 = -\alpha^2 e_4, \end{aligned}$$

$$\mathsf{R}(e_1, e_4)e_4 = -\,\alpha^2 e_1, \quad \mathsf{R}(e_2, e_4)e_4 = -\alpha^2 e_2, \quad \mathsf{R}(e_3, e_4)e_4 = -\alpha^2 e_3.$$

The Ricci tensor $\mathcal{S}$ of $\mathcal{M}$ is defined as $\mathcal{S}(X, Y) = \sum_{i=1}^{4} \epsilon_i \mathsf{R}(g(e_i, X)Y, e_i)$, where $\epsilon_i = g(e_i, e_i)$. Thus, the matrix form of the Ricci tensor $\mathcal{S}$ is as follows:

$$S(e_i, e_j) = \begin{bmatrix} 3\alpha^2 & 0 & 0 & 0 \\ 0 & 3\alpha^2 & 0 & 0 \\ 0 & 0 & 3\alpha^2 & 0 \\ 0 & 0 & 0 & -3\alpha^2 \end{bmatrix}, \quad i, j = 1, 2, 3, 4,$$

and the scalar curvature $r = \sum_{i=1}^{4} \epsilon_i \mathcal{S}(e_i, e_i) = 12\alpha^2$, this demonstrates that the (LPK) spacetime of dimension 4 has a constant scalar curvature and hence the equations (6), (7), (21) and (49) are satisfied.

7. Conclusion

The geometry of spacetime has far-reaching implications for our understanding of the universe, from the smallest subatomic particles to the largest cosmic structures. It also plays a crucial role in a variety of practical applications, including the Global Positioning System (GPS) and the study of black holes and cosmology.

In this article we have investigated that if an (LPK) type spacetime is ξ-conformally flat then it becomes a perfect fluid spacetime. Furthermore, it is proved that (LPK) type spacetime admitting the curvature condition $\mathsf{R}(X, Y) \cdot \mathcal{S} = 0$, then it becomes Einstein manifold. In this section we additionally show that if a perfect fluid spacetime obey's Einstein equation, then the state equation is given as $(p + \mu) =$ scalar. Ricci soliton and η-Ricci soliton in (LPK) type spacetime are also studied. In this way we find that a perfect fluid spacetime with concircular vector field admitting Ricci soliton then the soliton is expanding or shrinking according as c_3 is negative or positive.

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