Commun. Korean Math. Soc. **39** (2024), No. 4, pp. 927–945
https://doi.org/10.4134/CKMS.c230248
pISSN: 1225-1763 / eISSN: 2234-3024

A STUDY OF HARMONICITY ON COTANGENT BUNDLE WITH BERGER-TYPE DEFORMED SASAKI METRIC OVER STANDARD KÄHLER MANIFOLD

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Abstract. In this paper, we present some results concerning the harmonicity on the cotangent bundle equipped with the Berger-type deformed Sasaki metric over standard Kähler manifolds. We establish necessary and sufficient conditions under which a covector field is harmonic map or is harmonic covector with respect to the Berger-type deformed Sasaki metric and we construct some examples of harmonic covector fields. We also study the harmonicity of a covector field along a map between Riemannian manifolds, the target manifold being standard Kähler equipped with the Berger-type deformed Sasaki metric on its cotangent bundle. After that, we discuss the harmonicity of the composition of the projection map of the cotangent bundle of a Riemannian manifold with a map from this manifold into another Riemannian manifold, the source manifold being standard Kähler whose cotangent bundle is endowed with the Berger-type deformed Sasaki metric.

1. Introduction

In this field, one of the first works which deal with the cotangent bundles of a manifold as a Riemannian manifold is that of Patterson and Walker [16], who constructed from an affine symmetric connection on a manifold a Riemannian metric on the cotangent bundle, which they call the Riemann extension of the connection. A generalization of this metric had been given by Sekizawa [17] in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemann extensions which is a 2-parameter family of metrics, and which had been intensively studied by many authors. On the other hand, inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, Ağca considered another class of metrics on cotangent bundles of Riemannian manifolds, that she called g-natural metrics [1]. Also, there are studies by other

Received September 20, 2023; Revised February 11, 2024; Accepted February 23, 2024.
2020 *Mathematics Subject Classification.* Primary 53C43, 58E20, 53C55, 53A45.
Key words and phrases. Cotangent bundle, Berger-type deformed Sasaki metric, standard Kähler manifold, harmonic maps.

authors, Ağca and Salimov [2], Yano and Ishihara [19], Gezer and Altunbas [11]. In other direction, Yampolsky [18] proposed the Berger-type deformed Sasaki metric on tangent bundle over a Kählerian (standard Kähler) manifold, which was studied by Altunbas and collaborators in [3]. The study of the Berger-type deformed Sasaki metric on the tangent bundle or on the cotangent bundle are not limited to those mentioned above. We also refer to new studies by Zagane among which we refer [21–25].

In a previous work, [26], we proposed the Berger-type deformed Sasaki metric on the cotangent bundle over standard Kähler manifolds, where we studied some geodesic properties on the cotangent bundle with respect to this metric. In this paper, after the introduction and preliminaries, in section 3, we present the Berger-type deformed Sasaki metric on the cotangent bundle T^*M over a standard Kähler manifold (M^{2m}, J, g) and the Levi-Civita connection (Theorem 3.1). In section 4, we study of the harmonicity with respect to the Berger-type deformed Sasaki metric. First, we investigate the harmonicity of a covector field and we establish the necessary and sufficient conditions under which a covector field is harmonic map or is harmonic covector field (Theorem 4.6, Theorem 4.7 and Theorem 4.10). We also construct some examples of harmonic vector fields (Example 4.11 and Example 4.12). Secondly, we also study the harmonicity of a covector field along a map between Riemannian manifolds, the target manifold being standard Kähler equipped with the Berger-type deformed Sasaki metric on its cotangent bundle (Theorem 4.17 and Theorem 4.18). Finally, we study the harmonicity of the composition of the projection map of the cotangent bundle of a Riemannian manifold with a map from this manifold into another Riemannian manifold, the source manifold being standard Kähler whose cotangent bundle is endowed with the Berger-type deformed Sasaki metric (Theorem 4.20 and Theorem 4.21).

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ be the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{\bar{i}=\overline{m+1,2m}}$ on T^*M, where p_i is the component of covector p in each cotangent space $T^*_x M$, $x \in U$ with respect to the natural coframe $\{dx^i\}$, denoted by $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}$. Let $C^\infty(M)$ (resp., $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp., $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s). Denote by Γ^k_{ij} the Christoffel symbol of g and by ∇ the Levi-Civita connection of g.

The Levi Civita connection ∇ defines a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M \tag{1}$$

of the tangent bundle to T^*M at any $(x,p) \in T^*M$ into vertical subspace

$$V_{(x,p)}T^*M = Ker(d\pi_{(x,p)}) = \{\omega_i \partial_{\bar{i}}|_{(x,p)},\, \omega_i \in \mathbb{R}\},$$

and the horizontal subspace

$$H_{(x,p)}T^*M = \{X^i\partial_i|_{(x,p)} + X^i p_a \Gamma^a_{hi}\partial_{\bar{h}}|_{(x,p)},\, X^i \in \mathbb{R}\}.$$

Note that the map $X \to {}^HX = X^i\partial_i|_{(x,p)} + X^i p_a\Gamma^a_{hi}\partial_{\bar{h}}|_{(x,p)}$ is an isomorphism between the vector spaces T_xM and $H_{(x,p)}T^*M$. Similarly, the map $\omega \to {}^V\omega = \omega_i\partial_{\bar{i}}|_{(x,p)}$ is an isomorphism between the vector spaces T^*_xM and $V_{(x,p)}T^*M$. Obviously, each tangent vector $Z \in T_{(x,p)}T^*M$ can be written in the form $Z = {}^HX + {}^V\omega$, where $X \in T_xM$ and $\omega \in T^*_xM$ are uniquely determined.

Let $X = X^i\partial_i$ and $\omega = \omega_i dx^i$ be local expressions in $(U, x^i)_{i=\overline{1,m}}$, of a vector and covector (1-form) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the horizontal lift ${}^HX \in \Im^1_0(T^*M)$ of $X \in \Im^1_0(M)$ and the vertical lift ${}^V\omega \in \Im^1_0(T^*M)$ of $\omega \in \Im^0_1(M)$ are defined, respectively by

$$ {}^HX = X^i\partial_i + p_h\Gamma^h_{ij}X^j\partial_{\bar{i}}, \tag{2}$$

$$ {}^V\omega = \omega_i\partial_{\bar{i}}, \tag{3}$$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$ (see [19] for more details).

From (2) and (3), we see that ${}^H(\partial_i)$ and ${}^V(dx^i)$ have respectively local expressions of the form

$$\begin{aligned} {}^H(\partial_i) &= \partial_i + p_a\Gamma^a_{hi}\partial_{\bar{h}}, \\ {}^V(dx^i) &= \partial_{\bar{i}}. \end{aligned}$$

The set of vector fields $\{{}^H(\partial_i)\}$ on $\pi^{-1}(U)$ defines a local frame for HT^*M over $\pi^{-1}(U)$ and the set of vector fields $\{{}^V(dx^i)\}$ on $\pi^{-1}(U)$ defines a local frame for VT^*M over $\pi^{-1}(U)$. The set $\{{}^H(\partial_i), {}^V(dx^i)\}$ defines a local frame on T^*M, adapted to the direct sum decomposition (1).

Let (M, g) be a Riemannian manifold, we define the map

$$\begin{array}{ccc} \Im^0_1(M) & \to & \Im^1_0(M) \\ \omega & \mapsto & \widetilde{\omega} \end{array}$$

by $g(\widetilde{\omega}, X) = \omega(X)$, for all $X \in \Im^1_0(M)$. Locally for all $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have $\widetilde{\omega} = g^{ij}\omega_i\partial_j$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T^*_xM by $g^{-1}(\omega, \theta) = g(\widetilde{\omega}, \widetilde{\theta}) = g^{ij}\omega_i\theta_j$. In this case we have $\widetilde{\omega} = g^{-1} \circ \omega$.

3. Berger-type deformed Sasaki metric

Let M^r be an r-dimensional differentiable manifold. An almost complex structure J on M is a $(1, 1)$-tensor field on M such that $J^2 = -I$ (I is the $(1, 1)$-identity tensor field on M). The pair (M^r, J) is called an almost complex manifold. Since every almost complex manifold is even dimensional, we will take $r = 2m$. Also, note that every complex manifold (topological space endowed with a holomorphic atlas) carries a natural almost complex structure [14]. An almost complex structure J on M is integrable if the Nijenhuis tensor

$$N_J(X,Y) = [JX,JY] - J\,[JX,Y] - J\,[X,JY] - [X,Y]$$

vanishes identically on M for all vector fields $X,Y \in \Im^1_0(M)$. Moreover, an almost complex structure J on M is integrable if and only if M admits a symmetric almost complex linear connection [5,14].

On an almost complex manifold (M^{2m},J), a Hermitian metric is a Riemannian metric g on M such that

$$g(JX,Y) = -g(X,JY) \Leftrightarrow g(JX,JY) = g(X,Y),$$

or equivalently [26]

$$g^{-1}(\omega J,\theta) = -g^{-1}(\omega,\theta J) \Leftrightarrow g^{-1}(\omega J,\theta J) = g^{-1}(\omega,\theta), \tag{4}$$

for all $X,Y \in \Im^1_0(M)$ and $\omega,\theta \in \Im^0_1(M)$.

The almost complex manifold (M^{2m},J) having the Hermitian metric g is called an almost Hermitian manifold. Let (M^{2m},J,g) be an almost Hermitian manifold. We define the fundamental or Kähler 2-form Ω on M by

$$\Omega(X,Y) = g(X,JY)$$

for any vector fields X and Y on M. A Hermitian metric g on an almost Hermitian manifold M^{2m} is called a standard Kähler metric if the fundamental 2-form Ω is closed, i.e., $d\Omega = 0$. In the case, the triple (M^{2m},J,g) is called an almost standard Kähler manifold. If the almost complex structure is integrable, then the triple (M^{2m},J,g) is called a standard Kähler manifold. Moreover, the following conditions are equivalent:

(1) $\nabla J = 0$, (∇ is the Levi-Civita connection of g)
(2) $\nabla\Omega = 0$,
(3) $N_J = 0$ and $d\Omega = 0$ [14].

As a result, the almost Hermitian manifold (M^{2m},J,g) is a standard Kähler manifold if and only if $\nabla J = 0$. Using the formula

$$\omega(\nabla_X J) = \nabla_X(\omega J) - (\nabla_X\omega)J$$

for all $X \in \Im^1_0(M)$, $\omega \in \Im^0_1(M)$, we have also the almost Hermitian manifold (M^{2m},J,g) is a standard Kähler manifold if and only if

$$\nabla_X(\omega J) = (\nabla_X\omega)J \tag{5}$$

for all $X \in \Im^1_0(M)$, $\omega \in \Im^0_1(M)$.

Definition. [26] Let (M^{2m},J,g) be an almost Hermitian manifold and T^*M be its tangent bundle. A fiber-wise Berger-type deformation of the Sasaki metric noted ${}^{BS}g$ is defined on T^*M by

$$\begin{aligned} {}^{BS}g({}^HX,{}^HY) &= g(X,Y),\\ {}^{BS}g({}^HX,{}^V\theta) &= 0, \end{aligned}$$

$$^{BS}g(^{V}\omega, {}^{V}\theta) = g^{-1}(\omega,\theta) + \delta^2 g^{-1}(\omega, pJ)g^{-1}(\theta, pJ)$$

for all $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$, where δ is some constant. (For anti-paraKähler manifold, see [21–23]).

In the following, we put $\lambda = 1 + \delta^2\alpha$ and $\alpha = g^{-1}(p,p) = |p|^2$, where $|\cdot|$ denotes the norm with respect to g^{-1}.

Theorem 3.1. [26] Let (M^{2m}, J, g) be a standard Kähler manifold and T^*M its cotangent bundle equipped with the Berger-type deformed Sasaki metric ^{BS}g. Then we have the following formulas.

(i) $^{BS}\nabla_{^HX}{}^HY = {}^H(\nabla_X Y) + \frac{1}{2}{}^V(pR(X,Y))$,

(ii) $^{BS}\nabla_{^HX}{}^V\theta = {}^V(\nabla_X\theta) + \frac{1}{2}\big({}^H(R(\widetilde{p},\widetilde{\theta})X) + \delta^2 g^{-1}(\theta, pJ)\,{}^H(R(J\widetilde{p},\widetilde{p})X)\big)$,

(iii) $^{BS}\nabla_{^V\omega}{}^HY = \frac{1}{2}\big({}^H(R(\widetilde{p},\widetilde{\omega})Y) + \delta^2 g^{-1}(\omega, pJ)\,{}^H(R(J\widetilde{p},\widetilde{p})Y)\big)$,

(iv) $^{BS}\nabla_{^V\omega}{}^V\theta = \delta^2\big(g^{-1}(\omega,pJ)\,{}^V(\theta J) + g^{-1}(\theta,pJ)\,{}^V(\omega J)\big)$
$\qquad - \frac{\delta^4}{\lambda}\big(g^{-1}(\omega,pJ)g^{-1}(\theta,p) + g^{-1}(\omega,p)g^{-1}(\theta,pJ)\big)\,{}^V(pJ)$,

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where ∇ is the Levi-Civita connection of (M^{2m}, J, g) and R is its curvature tensor.

4. Berger-type deformed Sasaki metric and Harmonicity

Let $\Phi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds. The map Φ is said to be harmonic if it is a critical point of the energy functional

$$E(\Phi, K) = \int_K e(\Phi)\, v^g \tag{6}$$

for any compact domain $K \subseteq M$. Here

$$e(\Phi) := \frac{1}{2} Tr_g\, h(d\Phi, d\Phi) \tag{7}$$

is the energy density of Φ and v^g is the Riemannian volume form on M. For any smooth 1-parameter variation $\{\Phi_t\}_{t\in I}$ of Φ with $\Phi_0 = \Phi$ and $V = \frac{d}{dt}\Phi_t\Big|_{t=0}$ [10, 13], we have

$$\frac{d}{dt}E(\Phi_t)\Big|_{t=0} = -\int_K h(\tau(\Phi), V)v^g. \tag{8}$$

Then, Φ is harmonic if it satisfies the associated Euler-Lagrange equations given by the following formula:

$$0 = \tau(\Phi) := Tr_g \nabla d\Phi, \tag{9}$$

where $\tau(\Phi)$ is the tension field of Φ. For more details see [8–10,12,15]. In recent years, this theme has been widely developed even on the tangent bundle and on the cotangent bundle has been done by many authors, see [4,6,25,27–29].

4.1. Harmonicity of a covector field

A covector field $\omega \in \Im_1^0(M)$ on (M^{2m}, J, g) can be regarded as the immersion

$$\begin{array}{rcl} \omega : (M^{2m}, J, g) & \to & (T^*M, {}^{BS}g) \\ x & \mapsto & (x, \omega_x) \end{array}$$

into its cotangent bundle T^*M equipped with the Berger-type deformed Sasaki metric ${}^{BS}g$.

Lemma 4.1 ([20]). *Let (M^m, g) be a Riemannian manifold. If ω is a covector field (1-form) on M and $(x,p) \in T^*M$ such that $\omega_x = p$, then we have*

$$d_x\omega(X_x) = {}^HX_{(x,p)} + {}^V(\nabla_X\omega)_{(x,p)} \tag{10}$$

for any $X \in \Im_0^1(M)$, where ∇ denotes the Levi-Civita connection of (M^m, g).

Lemma 4.2. *Let (M^{2m}, J, g) be a standard Kähler manifold, ω be a covector field on M. Then the following equation is satisfied:*

$$g^{-1}(\bar{\Delta}\omega, \omega) = |\nabla\omega|^2 - \frac{1}{2}\Delta(|\omega|^2), \tag{11}$$

where $\bar{\Delta}\omega = -Tr_g(\nabla_*\nabla_* - \nabla_{\nabla_**})\omega$ is the rough Laplacian of ω and Δ is the ordinary Laplace-Beltrami operator acting on functions.

Proof. Let $\{e_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M, then we have

$$\begin{aligned} g^{-1}(\bar{\Delta}\omega, \omega) &= - g^{-1}(Tr_g(\nabla_*\nabla_* - \nabla_{\nabla_**})\omega, \omega) \\ &= -\sum_{i=1}^{m} \big(g^{-1}(\nabla_{e_i}\nabla_{e_i}\omega, \omega) - g^{-1}(\nabla_{\nabla_{e_i}e_i}\omega, \omega)\big) \\ &= -\sum_{i=1}^{m} \big(e_i(g^{-1}(\nabla_{e_i}\omega, \omega)) - g^{-1}(\nabla_{e_i}\omega, \nabla_{e_i}\omega) \\ &\quad - \frac{1}{2}\nabla_{e_i}e_i(g^{-1}(\omega, \omega))\big) \\ &= -\sum_{i=1}^{m} \big(\frac{1}{2}e_ie_i(|\omega|^2) - |\nabla_{e_i}\omega|^2 - \frac{1}{2}\nabla_{e_i}e_i(|\omega|^2)\big) \\ &= |\nabla\omega|^2 - \frac{1}{2}\Delta(|\omega|^2). \end{aligned}$$

□

Lemma 4.3. *Let (M^{2m}, J, g) be a standard Kähler manifold, ω be a covector field on M. Then the following equation is satisfied:*

$$\bar{\Delta}(f\omega) = f\bar{\Delta}\omega - (\Delta f)\omega - 2\nabla_{grad f}\omega, \tag{12}$$

where f is a smooth function of M and $grad f$ is the gradient of f.

Proof. Let $\{e_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M. Then we have

$$\begin{aligned}
\bar{\Delta}(f\omega) &= -\sum_{i=1}^{m}\big(\nabla_{e_i}\nabla_{e_i}(f\omega) - \nabla_{\nabla_{e_i}e_i}(f\omega)\big)\\
&= -\sum_{i=1}^{m}\big(\nabla_{e_i}(e_i(f)\omega + f\nabla_{e_i}\omega) - \nabla_{e_i}e_i(f)\omega - f\nabla_{\nabla_{e_i}e_i}\omega\big)\\
&= -\sum_{i=1}^{m}\big(e_ie_i(f)\omega + e_i(f)\nabla_{e_i}\omega + e_i(f)\nabla_{e_i}\omega + f\nabla_{e_i}\nabla_{e_i}\omega\\
&\quad - \nabla_{e_i}e_i(f)\omega - f\nabla_{\nabla_{e_i}e_i}\omega\big)\\
&= -\sum_{i=1}^{m}\big((e_ie_i(f) - \nabla_{e_i}e_i(f))\omega + 2\nabla_{e_i(f)e_i}\omega + f(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i})\omega\big)\\
&= f\bar{\Delta}\omega - (\Delta f)\omega - 2\nabla_{grad f}\omega.
\end{aligned}$$

□

Lemma 4.4. *Let (M^{2m}, J, g) be a standard Kähler manifold and $(T^*M, {}^{BS}g)$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric. If ω is covector field on M, then the energy density associated to ω is given by*

$$e(\omega) = m + \frac{1}{2}|\nabla\omega|^2 + \frac{\delta^2}{2}Tr_g\, g^{-1}(\nabla_*\omega, \omega J)^2. \tag{13}$$

Proof. Let $(x,p) \in T^*M$, $\omega \in \Im^0_1(M)$, $\omega_x = p$ and $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M, then, from (7), we have

$$\begin{aligned}
e(\omega)_x &= \frac{1}{2}Tr_g\,{}^{BS}g(d\omega, d\omega)_{(x,p)}\\
&= \frac{1}{2}\sum_{i=1}^{2m}{}^{BS}g(d\omega(e_i), d\omega(e_i))_{(x,p)}.
\end{aligned}$$

Using (10), we obtain

$$\begin{aligned}
e(\omega) &= \frac{1}{2}\sum_{i=1}^{2m}{}^{BS}g({}^He_i + {}^V(\nabla_{e_i}\omega), {}^He_i + {}^V(\nabla_{e_i}\omega))\\
&= \frac{1}{2}\sum_{i=1}^{2m}\big({}^{BS}g({}^He_i, {}^He_i) + {}^{BS}g({}^V(\nabla_{e_i}\omega), {}^V(\nabla_{e_i}\omega))\big)\\
&= \frac{1}{2}\sum_{i=1}^{2m}\big(g(e_i, e_i) + g^{-1}(\nabla_{e_i}\omega, \nabla_{e_i}\omega) + \delta^2 g^{-1}(\nabla_{e_i}\omega, \omega J)^2\big)\\
&= m + \frac{1}{2}|\nabla\omega|^2 + \frac{\delta^2}{2}Tr_g\, g^{-1}(\nabla_*\omega, \omega J)^2.
\end{aligned}$$

□

Theorem 4.5. *Let (M^{2m}, J, g) be a standard Kähler manifold and $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. If*

ω is a covector field on M, then the tension field associated to ω is given by

$$\tau(\omega) = {}^H\Big(Tr_g\big[R(\tilde{\omega},\widetilde{\nabla_*\omega})*+\delta^2 g^{-1}(\nabla_*\omega,\omega J)R(J\widetilde{\omega},\widetilde{\omega})*\big]\Big)$$
$$+{}^V\Big(2\delta^2 Tr_g\big[g^{-1}(\nabla_*\omega,\omega J)\big(\nabla_*(\omega J)-\frac{\delta^2}{\lambda}g^{-1}(\nabla_*\omega,\omega)\omega J\big)\big]-\bar{\Delta}\omega\Big) \tag{14}$$

where $\lambda = 1+\delta^2|\omega|^2$ and $\bar{\Delta}\omega = -Tr_g(\nabla_*\nabla_* - \nabla_{\nabla_**})\omega$.

Proof. Let $(x,p)\in T^*M$, $\omega\in\Im^0_1(M)$, $\omega_x = p$ and $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M, then, from (9), we have

$$\begin{aligned}
\tau(\omega)_x =& Tr_g(\nabla d\omega)_x\\
=& \sum_{i=1}^{2m}\big((\nabla^\omega_{e_i} d\omega(e_i))_x - d_x\omega(\nabla_{e_i}e_i)_x\big)\\
=& \sum_{i=1}^{2m}\big(({}^{BS}\nabla_{d\omega(e_i)}d\omega(e_i))_{(x,p)} - {}^H(\nabla_{e_i}e_i)_{(x,p)} - {}^V(\nabla_{(\nabla_{e_i}e_i)}\omega)_{(x,p)}\big)\\
=& \sum_{i=1}^{2m}\big({}^{BS}\nabla_{({}^He_i+{}^V(\nabla_{e_i}\omega))}({}^He_i+{}^V(\nabla_{e_i}\omega)) - {}^H(\nabla_{e_i}e_i) - {}^V(\nabla_{(\nabla_{e_i}e_i)}\omega)\big)_{(x,p)}\\
=& \sum_{i=1}^{2m}\big({}^{BS}\nabla_{{}^He_i}{}^He_i + {}^{BS}\nabla_{{}^He_i}{}^V(\nabla_{e_i}\omega) + {}^{BS}\nabla_{{}^V(\nabla_{e_i}\omega)}{}^He_i\\
&+ {}^{BS}\nabla_{{}^V(\nabla_{e_i}\omega)}{}^V(\nabla_{e_i}\omega) - {}^H(\nabla_{e_i}e_i) - {}^V(\nabla_{(\nabla_{e_i}e_i)}\omega)\big)_{(x,p)}.
\end{aligned}$$

Using Theorem 3.1 and (5), we obtain

$$\begin{aligned}
\tau(\omega) =& \sum_{i=1}^{2m}\Big({}^H(\nabla_{e_i}e_i) - \frac{1}{2}{}^V(\omega R(e_i,e_i)) + {}^V(\nabla_{e_i}\nabla_{e_i}\omega)\\
&+ \frac{1}{2}{}^H(R(\tilde{\omega},\widetilde{\nabla_{e_i}\omega})e_i) + \frac{\delta^2}{2}g^{-1}(\nabla_{e_i}\omega,\omega J){}^H(R(J\widetilde{\omega},\widetilde{\omega})e_i)\\
&+ \frac{1}{2}{}^H(R(\tilde{\omega},\widetilde{\nabla_{e_i}\omega})e_i) + \frac{\delta^2}{2}g^{-1}(\nabla_{e_i}\omega,\omega J){}^H(R(J\widetilde{\omega},\widetilde{\omega})e_i)\\
&+ \delta^2 g^{-1}(\nabla_{e_i}\omega,\omega J){}^V((\nabla_{e_i}\omega)J) + \delta^2 g^{-1}(\nabla_{e_i}\omega,\omega J){}^V((\nabla_{e_i}\omega)J)\\
&- \frac{\delta^4}{\lambda}g^{-1}(\nabla_{e_i}\omega,\omega J)g^{-1}(\nabla_{e_i}\omega,\omega){}^V(\omega J) - {}^H(\nabla_{e_i}e_i)\\
&- \frac{\delta^4}{\lambda}g^{-1}(\nabla_{e_i}\omega,\omega)g^{-1}(\nabla_{e_i}\omega,\omega J){}^V(\omega J) - {}^V(\nabla_{(\nabla_{e_i}e_i)}\omega)\Big)\\
=& \sum_{i=1}^{2m}\Big({}^H(R(\tilde{\omega},\widetilde{\nabla_{e_i}\omega})e_i) + \delta^2 g^{-1}(\nabla_{e_i}\omega,\omega J){}^H(R(J\widetilde{\omega},\widetilde{\omega})e_i)\\
&+ {}^V(\nabla_{e_i}\nabla_{e_i}\omega) - {}^V(\nabla_{(\nabla_{e_i}e_i)}\omega) + 2\delta^2 g^{-1}(\nabla_{e_i}\omega,\omega J){}^V(\nabla_{e_i}(\omega J))\\
&- \frac{2\delta^4}{\lambda}g^{-1}(\nabla_{e_i}\omega,\omega J)g^{-1}(\nabla_{e_i}\omega,\omega){}^V(\omega J)\Big)
\end{aligned}$$

$$= {}^{H}\big(Tr_g\big[R(\tilde{\omega},\widetilde{\nabla_*\omega})*+\delta^2 g^{-1}(\nabla_*\omega,\omega J)R(J\widetilde{\omega},\widetilde{\omega})*\big]\big)$$
$$+\,{}^{V}\big(2\delta^2 Tr_g\big[g^{-1}(\nabla_*\omega,\omega J)\big(\nabla_*(\omega J)-\frac{\delta^2}{\lambda}g^{-1}(\nabla_*\omega,\omega)\omega J\big)\big]-\bar{\Delta}\omega\big). \qquad \square$$

Theorem 4.6. *Let (M^{2m},J,g) be a standard Kähler manifold and $(T^*M,{}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. If ω is a covector field on M, then ω is a harmonic map if and only if the following conditions are verified*:

$$Tr_g\big[R(\tilde{\omega},\widetilde{\nabla_*\omega})*+\delta^2 g^{-1}(\nabla_*\omega,\omega J)R(J\widetilde{\omega},\widetilde{\omega})*\big]=0,$$

and

$$2\delta^2 Tr_g\big[g^{-1}(\nabla_*\omega,\omega J)\big(\nabla_*(\omega J)-\frac{\delta^2}{\lambda}g^{-1}(\nabla_*\omega,\omega)\omega J\big)\big]=\bar{\Delta}\omega.$$

Proof. The statement is a direct consequence of Theorem 4.5. $\square$

Let (M^{2m},J,g) be a compact oriented standard Kähler manifold, $(T^*M,{}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric and ω a covector field on M. The energy $E(\omega)$ of ω is defined to be the energy of the corresponding map $\omega:(M^{2m},J,g)\to(T^*M,{}^{BS}g)$. More precisely, from (13), we get

$$\begin{aligned}E(\omega)&=\int_M e(\omega)v^g\\&=\int_M\big(m+\frac{1}{2}|\nabla\omega|^2+\frac{\delta^2}{2}Tr_g g^{-1}(\nabla_*\omega,\omega J)^2\big)v^g\\&=m\,Vol(M)+\frac{1}{2}\int_M|\nabla\omega|^2v^g+\frac{\delta^2}{2}\int_M Tr_g g^{-1}(\nabla_*\omega,\omega J)^2v^g.\end{aligned}$$

Definition. Let (M^{2m},J,g) be a standard Kähler manifold, $(T^*M,{}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. A covector field ω on M is called *harmonic covector field* if the corresponding map $\omega:(M^{2m},J,g)\to(T^*M,{}^{BS}g)$ is a critical point for the energy functional E, only considering variations among maps defined by covector fields.

In the following theorem, we determine the first variation of the energy restricted to the space $\Im^0_1(M)$.

Theorem 4.7. *Let (M^{2m},J,g) be a compact oriented standard Kähler manifold, $(T^*M,{}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric, ω a covector field on M and $E:\Im^0_1(M)\to[0,+\infty)$ the energy functional restricted to the space of all covector fields. Then*

$$\begin{aligned}\frac{d}{dt}E(\omega_t)\big|_{t=0}=\int_M g^{-1}\big(&\bar{\Delta}\omega+\delta^2 g^{-1}(\bar{\Delta}\omega,\omega J)\omega J\\&-2\delta^2 Tr_g\big[g^{-1}(\nabla_*\omega,\omega J)\nabla_*(\omega J)\big],\vartheta\big)v^g\end{aligned}\tag{15}$$

for any smooth 1*-parameter variation* $\phi : M \times (-\epsilon, \epsilon) \to T^*M$ *of* ω *through covector fields, i.e.,* $\phi(x,t) = \omega_t(x) \in T^*M$ *for any* $x \in M$ *and any* $|t| < \epsilon$, $(\epsilon > 0)$*. or equivalently* $\omega_t \in \Im^0_1(M)$ *for any* $|t| < \epsilon$*. Also,* $\vartheta \in \Im^0_1(M)$ *is the covector field on* M *given by*

$$\vartheta(x) = \lim_{t\to 0} \frac{1}{t}(\omega_t(x) - \omega(x)) = \frac{d}{dt}\phi_x(t)\big|_{t=0}, \quad x \in M,$$

where $\phi_x(t) = \omega_t(x), (x,t) \in M \times (-\epsilon, \epsilon)$*. (For tangent bundle version, see* [7]*.)*

Proof. Let $\phi : M \times (-\epsilon, \epsilon) \to T^*M$ be a smooth 1-parameter variation of ω, i.e. $\phi(x,t) = \omega_t(x) \in T^*_xM$ for any $(x,t) \in M \times (-\epsilon, \epsilon)$ and $\phi(x,0) = \omega_0(x) = \omega(x)$. From (6), we have

$$E(\omega_t) = \int_M e(\omega_t)v^g,$$

and from (8), we have

$$\frac{d}{dt}E(\omega_t)\Big|_{t=0} = -\int_M {}^{BS}g(\mathcal{V}, \tau(\omega))v^g, \tag{16}$$

where $\mathcal{V}$ is the infinitesimal variation induced by ϕ, i.e.,

$$\mathcal{V}(x) = d_{(x,0)}\phi(0, \frac{d}{dt})\big|_{t=0} = d\phi_x(\frac{d}{dt})\big|_{t=0} = \frac{d}{dt}\omega_t(x)\big|_{t=0} \in T_{\omega(x)}T^*M.$$

Let us set

$$\phi_i(x,t) = x^i(\phi(x,t)) = x^i(x),$$
$$\phi_{i+m}(x,t) = x^{\bar{i}}(\phi(x,t)) = (\omega_t)_i(x), \quad 1 \le i \le m.$$

Then, on one hand, we have

$$\frac{d\phi_i}{dt}(x,0) = 0, \quad 1 \le i \le m.$$

On the other hand if $\vartheta = \vartheta_i dx^i$ then

$$\begin{aligned}\frac{d\phi_{i+m}}{dt}(x,0) &= \lim_{t\to 0}\frac{1}{t}(\phi_{i+m}(x,t) - \phi_{i+m}(x,0))\\ &= \lim_{t\to 0}\frac{1}{t}((\omega_t)_i(x) - \omega_i(x))\\ &= \vartheta_i(x), \quad 1 \le i \le m,\end{aligned}$$

hence

$$\begin{aligned}\mathcal{V}(x) &= \frac{d\phi_i}{dt}(x,0)\partial_i|_{(x,\omega(x))} + \frac{d\phi_{i+m}}{dt}(x,0)\partial_{\bar{i}}|_{(x,\omega(x))}\\ &= \vartheta_i(x)^V(dx^i)|_{(x,\omega(x))}\\ &= {}^V\vartheta_{(x,\omega(x))}.\end{aligned}$$

We may conclude that

$$\mathcal{V} = {}^V\vartheta \circ \omega. \tag{17}$$

Finally, by taking into account (14), (16) and (17), we find

$$\begin{aligned}\frac{d}{dt}E(\omega_t)\Big|_{t=0} &= -\int_M {}^{BS}g({}^V\vartheta, \tau(\omega))v^g \\ &= \int_M g^{-1}\big(\bar{\Delta}\omega + \delta^2 g^{-1}(\bar{\Delta}\omega, \omega J)\omega J \\ &\quad - 2\delta^2 Tr_g\big[g^{-1}(\nabla_*\omega, \omega J)\nabla_*(\omega J)\big], \vartheta\big)v^g.\end{aligned}$$

□

Remark 4.8. Theorem 4.7 holds if (M^{2m}, J, g) is a non-compact standard Kähler manifold. Indeed, if M is non-compact, we can take an open subset D in M whose closure is compact, and take an arbitrary ϑ whose support is contained in D. Theorem 4.7 holds under the form:

$$\begin{aligned}\frac{d}{dt}E(\omega_t)\big|_{t=0} &= \int_D g^{-1}\big(\bar{\Delta}\omega + \delta^2 g^{-1}(\bar{\Delta}\omega, \omega J)\omega J \\ &\quad - 2\delta^2 Tr_g\big[g^{-1}(\nabla_*\omega, \omega J)\nabla_*(\omega J)\big], \vartheta\big)v^g.\end{aligned}$$

Corollary 4.9. *Let (M^{2m}, J, g) be a standard Kähler manifold, $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. A covector field ω on M is a harmonic covector field if and only if*

$$\bar{\Delta}\omega = 2\delta^2 Tr_g\big[g^{-1}(\nabla_*\omega, \omega J)\nabla_*(\omega J)\big] - \delta^2 g^{-1}(\bar{\Delta}\omega, \omega J)\omega J.$$

Note that if ω is parallel, then ω is a harmonic covector field. Conversely we have the following theorem.

Theorem 4.10. *Let (M^{2m}, J, g) be a compact oriented standard Kähler manifold, $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric and ω a covector field on M. Then ω is a harmonic covector field if and only if ω is parallel.*

Proof. We assume that the covector field ω is a harmonic vector field, i.e., critical point of the energy functional E restricted to the space of all covector fields of (M^m, g). We consider the smooth 1-parameter variation $\omega_t = (1+t)\omega$ of ω for any $t \in (-\epsilon, \epsilon), \epsilon > 0$. From (15) we have

$$\begin{aligned}0 &= \frac{d}{dt}E(\omega_t)\Big|_{t=0} \\ &= \int_M g^{-1}(\bar{\Delta}\omega, \omega)v^g + \delta^2\int_M g^{-1}(\bar{\Delta}\omega, \omega J)g^{-1}(\omega J, \omega)v^g \\ &\quad - 2\delta^2\int_M Tr_g\big[g^{-1}(\nabla_*\omega, \omega J)g^{-1}(\nabla_*(\omega J), \omega)\big]v^g.\end{aligned}$$

Using (11), (4) and (5), we find

$$0 = \int_M |\nabla\omega|^2 v^g - \frac{1}{2}\int_M \Delta(|\omega|^2)v^g + 2\delta^2\int_M Tr_g\big[g^{-1}(\nabla_*\omega, \omega J)^2\big]v^g.$$

Applying the divergence theorem, we get

$$\int_M \Delta(|\omega|^2)v^g = 0,$$

hence

$$\int_M |\nabla\omega|^2 v^g + 2\delta^2 \int_M Tr_g\big[g^{-1}(\nabla_*\omega, \omega J)^2\big]v^g = 0.$$

Since the function $|\nabla\omega|^2_{g^{-1}}$ and $Tr_g\big[g^{-1}(\nabla_*\omega, \omega J)^2\big]$ are positive, we easily conclude that $\nabla\omega = 0$. Conversely, we suppose that the covector field ω is parallel. By virtue of Corollary 4.9, ω is a harmonic covector field. □

Example 4.11. Let $(\mathbb{R}^2, g, J)$ be a standard Kähler manifold such that

$$g = e^{2x}dx^2 + e^{2y}dy^2, \ J = \begin{pmatrix} 0 & e^{y-x} \\ -e^{x-y} & 0 \end{pmatrix}.$$

Relatively to the orthonormal frame

$$e_1 = e^{-x}\partial_x, \quad e_2 = e^{-y}\partial_y.$$

we have

$$Je_1 = -e_2, \quad Je_2 = e_1,$$

and

$$\nabla_{e_i}e_j = 0 \text{ for all } i, j = 1, 2.$$

We consider the vector field $\omega = f(x)dx$, where f is a smooth real function depending on the variable x. Using (12) and direct calculations, we find

$$\begin{aligned} \bar{\Delta}\omega &= e^{-2x}(-f'' + 3f' - 2f)dx, \\ g^{-1}(\bar{\Delta}\omega, \omega J) &= Tr_g\big[g^{-1}(\nabla_*\omega, \omega J)\nabla_*(\omega J)\big] = 0. \end{aligned} \tag{18}$$

From Corollary 4.9, and (18), we deduce that $\omega = f(x)dx$ is a harmonic covector field if and only if $\bar{\Delta}\omega = 0$ or equivalently, the function f satisfies the following homogeneous two order differential equation

$$-f'' + 3f' - 2f = 0. \tag{19}$$

The general solution of the differential equation (19) is

$$f = c_1e^x + c_2e^{2x},$$

where c_1 and c_2 are real constants.

Example 4.12. Let $\mathbb{R}^2$ be endowed with the structure standard Kähler (J, g) in polar coordinate defined by

$$g = dr^2 + r^2d\theta^2 \ , \ \ J = \begin{pmatrix} 0 & r \\ -\frac{1}{r} & 0 \end{pmatrix}.$$

The non-null Christoffel symbols of the Riemannian connection are

$$\Gamma^2_{12} = \Gamma^2_{21} = -\frac{1}{r} \ , \ \ \Gamma^1_{22} = -r.$$

We have

$$\nabla_{\partial_r} dr = 0,\ \nabla_{\partial_r} d\theta = -\frac{1}{r} d\theta,\ \nabla_{\partial_\theta} dr = r d\theta,\ \nabla_{\partial_\theta} d\theta = -\frac{1}{r} dr.$$

The covector field $\omega = \cos\theta dr - r\sin\theta d\theta$ is harmonic because ω is parallel, indeed,

$$\nabla_{\partial_r}\omega = \cos\theta\nabla_{\partial_r} dr - \sin\theta d\theta - r\sin\theta\nabla_{\partial_r} d\theta = 0,$$

$$\nabla_{\partial_\theta}\omega = -\sin\theta dr + \cos\theta\nabla_{\partial_\theta} dr - r\cos\theta d\theta - r\sin\theta\nabla_{\partial_\theta} d\theta = 0,$$

i.e., $\nabla\omega = 0$.

Theorem 4.13. *Let (M^{2m}, J, g) be a standard Kähler manifold, $(T^*M, {}^{BS}g)$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric and ω a covector field on M. Then ω is an isometric immersion if and only if ω is parallel.*

Proof. Let Y, Z be vector fields and $\omega_x = p$. From (10), we have

$$\begin{aligned}
{}^{BS}g(d\omega(Y), d\omega(Z)) =& {}^{BS}g({}^HY + {}^V(\nabla_Y\omega), {}^HZ + {}^V(\nabla_Z\omega))\\
=& {}^{BS}g({}^HY, {}^HZ) + {}^{BS}g({}^V(\nabla_Y\omega), {}^V(\nabla_Z\omega))\\
=& g(Y,Z) + g^{-1}(\nabla_Y\omega, \nabla_Z\omega)\\
&+ \delta^2 g^{-1}(\nabla_Y\omega, J\omega)g^{-1}(\nabla_Z\omega, J\omega),
\end{aligned}$$

from which it follows that

$${}^{BS}g(d\omega(Y), d\omega(Z)) = g(Y,Z).$$

Therefore, ω is an isometric immersion if and only if

$$g^{-1}(\nabla_Y\omega, \nabla_Z\omega) + \delta^2 g^{-1}(\nabla_Y\omega, J\omega)g(\nabla_Z\omega, J\omega) = 0,$$

which is equivalent to $\nabla\omega = 0$. □

As a direct consequence of Theorem 4.6 and Theorem 4.13, we get the following.

Theorem 4.14. *Let (M^{2m}, J, g) be a standard Kähler manifold, T^*M its cotangent bundle equipped with the Berger-type deformed Sasaki metric ${}^{BS}g$ and ω a covector field on M. If ω is an isometric immersion, then ω is totally geodesic.*

Corollary 4.15. *Let (M^{2m}, J, g) be a standard Kähler manifold, T^*M its cotangent bundle equipped with the Berger-type deformed Sasaki metric ${}^{BS}g$ and ω a covector field on M. If ω is isometric immersion, then ω is a harmonic map (a harmonic covector field).*

4.2. Harmonicity of covector fields along smooth maps

Lemma 4.16 ([21]). *Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and*

$$\begin{aligned} \sigma : M &\longrightarrow T^*N \\ x &\longmapsto (\xi \circ \phi)(x) = (\phi(x), \xi_{\phi(x)}) \end{aligned}$$

a smooth map, such that ξ is a covector field on N. Then

$$d\sigma(X) = {}^H(d\phi(X)) + {}^V(\nabla^{\phi}_X \sigma) \tag{20}$$

for all $X \in \Im^1_0(M)$, where ∇^{ϕ} is the pull-back connection.

Theorem 4.17. *Let (M^m, g) be a Riemannian manifold, (N^{2n}, J, h) be a standard Kähler manifold and let $(T^*N, {}^{BS}h)$ be the cotangent bundle of N equipped with the Berger-type deformed Sasaki metric. Let $\phi : M \to N$ be a smooth map and*

$$\begin{aligned} \sigma : M &\longrightarrow T^*N \\ x &\longmapsto (\xi \circ \phi)(x) = (\phi(x), \xi_{\phi(x)}) \end{aligned}$$

a smooth map, such that ξ is a covector field on N. The tension field of σ is given by

$$\begin{aligned} \tau(\sigma) =& {}^H\Big(\tau(\phi) + Tr_g\big(R^N(\tilde{\sigma}, \widetilde{\nabla^{\phi}\sigma})d\phi(*) + \delta^2 h^{-1}(\nabla^{\phi}\sigma, \sigma J)R^N(J\widetilde{\sigma}, \widetilde{\sigma})d\phi(*)\big)\Big) \\ &+ {}^V\Big(Tr_g\big((\nabla^{\phi})^2\sigma + 2\delta^2 h^{-1}(\nabla^{\phi}\sigma, \sigma J)(\nabla^{\phi}\sigma)J \\ &- \frac{2\delta^4}{\lambda} h^{-1}(\nabla^{\phi}\sigma, \sigma J)h^{-1}(\nabla^{\phi}\sigma, \sigma)\sigma J\big)\Big), \end{aligned}$$

where $\lambda = 1 + \delta^2|\sigma|^2$ and $(\nabla^{\phi})^2\sigma = \nabla^{\phi}\nabla^{\phi}\sigma - \nabla^{\phi}_{\nabla}\sigma$.

Proof. Let $\{e_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M. Using (20), we have

$$\begin{aligned} &\tau(\sigma) \\ =& \sum_{i=1}^m \Big\{\nabla^{\sigma}_{e_i} d\sigma(e_i) - d\sigma(\nabla_{e_i} e_i)\Big\} \\ =& \sum_{i=1}^m \Big\{{}^{BS}\nabla_{d\sigma(e_i)} d\sigma(e_i) - {}^H(d\phi(\nabla_{e_i} e_i)) - {}^V(\nabla^{\phi}_{\nabla_{e_i} e_i}\sigma)\Big\} \\ =& \sum_{i=1}^m \Big\{{}^{BS}\nabla_{({}^H(d\phi(e_i)) + {}^V(\nabla^{\phi}_{e_i}\sigma))}({}^H(d\phi(e_i)) + {}^V(\nabla^{\phi}_{e_i}\sigma)) \\ &- {}^H(d\phi(\nabla_{e_i} e_i)) - {}^V(\nabla^{\phi}_{\nabla_{e_i} e_i}\sigma)\Big\} \\ =& \sum_{i=1}^m \Big\{{}^{BS}\nabla_{{}^H(d\phi(e_i))} {}^H(d\phi(e_i)) + {}^{BS}\nabla_{{}^H(d\phi(e_i))} {}^V(\nabla^{\phi}_{e_i}\sigma) + {}^{BS}\nabla_{{}^V(\nabla^{\phi}_{e_i}\sigma)} {}^H(d\phi(e_i)) \end{aligned}$$

$$+\ {}^{BS}\nabla_{{}^V(\nabla^{\phi}_{e_i}\sigma)}{}^V(\nabla^{\phi}_{e_i}\sigma) - {}^H(d\phi(\nabla_{e_i}e_i)) - {}^V(\nabla^{\phi}_{\nabla_{e_i}e_i}\sigma)\Big\}.$$

From Theorem 3.1, we obtain

$$\begin{aligned}
\tau(\sigma) =& \sum_{i=1}^{m}\Big({}^H(\nabla^N_{d\phi(e_i)}d\phi(e_i)) + \frac{1}{2}{}^V(\sigma R^N(d\phi(e_i), d\phi(e_i)))\\
&+ {}^H(R^N(\tilde{\sigma}, \widetilde{\nabla^{\phi}_{e_i}\sigma})d\phi(e_i)) + \delta^2 h^{-1}(\nabla^{\phi}_{e_i}\sigma, \sigma J)^H(R^N(J\widetilde{\sigma}, \widetilde{\sigma})d\phi(e_i))\\
&+ 2\delta^2 h^{-1}(\nabla^{\phi}_{e_i}\sigma, \sigma J)^V((\nabla^{\phi}_{e_i}\sigma)J) - {}^H(d\phi(\nabla_{e_i}e_i)) - {}^V(\nabla^{\phi}_{\nabla_{e_i}e_i}\sigma)\\
&- \frac{2\delta^4}{\lambda}h^{-1}(\nabla^{\phi}_{e_i}\sigma, \sigma J)h^{-1}(\nabla^{\phi}_{e_i}\sigma, \sigma)^V(\sigma J) + {}^V(\nabla^N_{d\phi(e_i)}\nabla^{\phi}_{e_i}\sigma)\Big)\\
=& \sum_{i=1}^{m}\Big({}^H(\nabla^{\phi}_{e_i}d\phi(e_i)) - {}^H(d\phi(\nabla_{e_i}e_i)) + {}^H(R^N(\tilde{\sigma}, \widetilde{\nabla^{\phi}_{e_i}\sigma})d\phi(e_i))\\
&+ \delta^2 h^{-1}(\nabla^{\phi}_{e_i}\sigma, \sigma J)^H(R^N(J\widetilde{\sigma}, \widetilde{\sigma})d\phi(e_i)) + {}^V(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\sigma)\\
&- {}^V(\nabla^{\phi}_{\nabla_{e_i}e_i}\sigma) + 2\delta^2 h^{-1}(\nabla^{\phi}_{e_i}\sigma, \sigma J)^V((\nabla^{\phi}_{e_i}\sigma)J)\\
&- \frac{2\delta^4}{\lambda}h^{-1}(\nabla^{\phi}_{e_i}\sigma, \sigma J)h^{-1}(\nabla^{\phi}_{e_i}\sigma, \sigma)^V(\sigma J)\Big)\\
=& {}^H\Big(Tr_g\big(R^N(\tilde{\sigma}, \widetilde{\nabla^{\phi}\sigma})d\phi(*) + \delta^2 h^{-1}(\nabla^{\phi}\sigma, \sigma J)R^N(J\widetilde{\sigma}, \widetilde{\sigma})d\phi(*)\big)\\
&+ \tau(\phi)\Big) + {}^V\Big(Tr_g\big((\nabla^{\phi})^2\sigma + 2\delta^2 h^{-1}(\nabla^{\phi}\sigma, \sigma J)(\nabla^{\phi}\sigma)J\\
&- \frac{2\delta^4}{\lambda}h^{-1}(\nabla^{\phi}\sigma, \sigma J)h^{-1}(\nabla^{\phi}\sigma, \sigma)\sigma J\big)\Big).
\end{aligned}$$

□

From Theorem 4.17 we obtain

Theorem 4.18. *Let (M^m, g) be a Riemannian manifold, (N^{2n}, J, h) be a standard Kähler manifold and let $(T^*N, {}^{BS}h)$ be the cotangent bundle of N equipped with the Berger-type deformed Sasaki metric. Let $\phi : M \to N$ be a smooth map and*

$$\begin{aligned}
\sigma : M &\longrightarrow T^*N\\
x &\longmapsto (\xi \circ \phi)(x) = (\phi(x), \xi_{\phi(x)})
\end{aligned}$$

a smooth map, such that ξ is a covector field on N. Then σ is harmonic if and only if the following conditions are verified:

$$\tau(\phi) = -\mathrm{Tr}_g\big(R^N(\tilde{\sigma}, \widetilde{\nabla^{\phi}\sigma})d\phi(*) + \delta^2 h^{-1}(\nabla^{\phi}\sigma, \sigma J)R^N(J\widetilde{\sigma}, \widetilde{\sigma})d\phi(*)\big),$$

and

$$\mathrm{Tr}_g\big((\nabla^{\phi})^2\sigma + 2\delta^2 h^{-1}(\nabla^{\phi}\sigma, \sigma J)(\nabla^{\phi}\sigma)J - \frac{2\delta^4}{\lambda}h^{-1}(\nabla^{\phi}\sigma, \sigma J)h^{-1}(\nabla^{\phi}\sigma, \sigma)\sigma J\big) = 0.$$

4.3. Harmonicity of the composition of canonical projection on the cotangent bundle with smooth maps

Lemma 4.19. *Let (M^{2m},J,g) be a standard Kähler manifold and T^*M its cotangent bundle equipped with the Berger-type deformed Sasaki metric ${}^{BS}g$. The canonical projection*

$$\begin{array}{rcl}\pi:(T^*M,{}^{BS}g) & \longrightarrow & (M,g)\\ (x,p) & \longmapsto & x\end{array}$$

is harmonic, i.e., $\tau(\pi)=0$.

Proof. Let $\{e_i\}_{i=\overline{1,2m}}$ and $\{\omega^i\}_{i=\overline{1,2m}}$ be a local orthonormal frame, coframe on M, respectively such that $\omega^{2m}=\dfrac{pJ}{|pJ|}=\dfrac{pJ}{|p|}$. Then

$$\{F_i=e_i^H, F_{2m+j}={}^V\omega^j, F_{4m}=\frac{1}{\sqrt{\lambda}}{}^V\omega^{2m}\}_{i=\overline{1,2m},j=\overline{1,2m-1}} \tag{21}$$

is a local frame on T^*M with respect to the Berger-type deformed Sasaki metric.

$$\begin{aligned}\tau(\pi)=&\mathrm{Tr}_{{}^{BS}g}(\nabla d\pi)\\ =&\sum_{i=1}^{2m}\Big(\nabla_{d\pi({}^He_i)}d\pi({}^He_i)-d\pi({}^{BS}\nabla_{{}^He_i}{}^He_i)\Big)\\ &+\sum_{j=1}^{2m-1}\Big(\nabla_{d\pi({}^V\omega^j)}d\pi({}^V\omega^j)-d\pi({}^{BS}\nabla_{{}^V\omega^j}{}^V\omega^j)\Big)\\ &+\nabla_{d\pi(\frac{1}{\sqrt{\lambda}}{}^V\omega^{2m})}d\pi(\frac{1}{\sqrt{\lambda}}{}^V\omega^{2m})-d\pi({}^{BS}\nabla_{(\frac{1}{\sqrt{\lambda}}{}^V\omega^{2m})}(\frac{1}{\sqrt{\lambda}}{}^V\omega^{2m})).\end{aligned}$$

Using $d\pi({}^V\omega^j)=0$ and $d\pi({}^He_i)=e_i\circ\pi$, we have

$$\begin{aligned}\tau(\pi)=&\sum_{i=1}^{2m}\Big((\nabla_{e_i\circ\pi}e_i\circ\pi)-d\pi({}^{BS}\nabla_{{}^He_i}{}^He_i)\Big)-\sum_{j=1}^{2m-1}d\pi({}^{BS}\nabla_{{}^V\omega^j}{}^V\omega^j)\\ &-d\pi\Big(\frac{1}{\sqrt{\lambda}}{}^V\omega^{2m}(\frac{1}{\sqrt{\lambda}}){}^V\omega^{2m}+\frac{1}{\lambda}{}^{BS}\nabla_{{}^V\omega^{2m}}{}^V\omega^{2m}\Big).\end{aligned}$$

From Theorem 3.1, we obtain

$$\begin{aligned}\tau(\pi)=&\sum_{i=1}^{2m}\Big((\nabla_{e_i}e_i)\circ\pi-d\pi(\nabla_{e_i}e_i)^H\Big)\\ =&\sum_{i=1}^{2m}\Big((\nabla_{e_i}e_i)\circ\pi-(\nabla_{e_i}e_i)\circ\pi\Big)=0.\end{aligned}$$

□

Theorem 4.20. *Let (M^{2m},J,g) be a standard Kähler manifold, (N^n,h) be a Riemannian manifold and let $(T^*M,{}^{BS}g)$ the cotangent bundle of M equipped* *with the Berger-type deformed Sasaki metric. Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map. The tension field of the map*

$$\begin{array}{rcl}\psi : (T^*M, {}^{BS}g) & \longrightarrow & (N, h) \\ (x, p) & \longmapsto & \phi(x)\end{array}$$

is given by

$$\tau(\psi) = \tau(\phi) \circ \pi. \tag{22}$$

Proof. Let $\{F_a\}_{a=\overline{1,4m}}$ be a local orthonormal frame on T^*M defined by (21) as above. Since ψ is written in the form $\psi = \phi \circ \pi$, we have

$$\begin{aligned}
\tau(\psi) =& \tau(\phi \circ \pi) \\
=& d\phi(\tau(\pi)) + Tr_{{}^{BS}g} \nabla d\phi(d\pi, d\pi) \\
=& \sum_{i=1}^{2m} \Big(\nabla^N_{d\phi(d\pi({}^He_i))} d\phi(d\pi({}^He_i)) - d\phi(\nabla_{d\pi({}^He_i)} d\pi({}^He_i)) \Big) \\
&+ \sum_{j=1}^{2m-1} \Big(\nabla^N_{d\phi(d\pi({}^V\omega^j))} d\phi(d\pi({}^V\omega^j)) - d\phi(\nabla_{d\pi({}^V\omega^j)} d\pi({}^V\omega^j)) \Big) \\
&+ \nabla^N_{d\phi(d\pi(\frac{1}{\sqrt{\lambda}}{}^V\omega^m))} d\phi(d\pi(\frac{1}{\sqrt{\lambda}}{}^V\omega^m)) - d\phi(\nabla_{d\pi(\frac{1}{\sqrt{\lambda}}{}^V\omega^m)} d\pi(\frac{1}{\sqrt{\lambda}}{}^V\omega^m)) \\
=& \sum_{i=1}^{m} \Big((\nabla^N_{d\phi(e_i\circ\pi)} d\phi(e_i \circ \pi)) - d\phi(\nabla_{e_i\circ\pi} e_i \circ \pi) \Big) \\
=& \sum_{i=1}^{m} \Big(\nabla^N_{d\phi(e_i)} d\phi(e_i) \circ \pi - d\phi((\nabla_{e_i} e_i) \circ \pi) \Big) \\
=& \sum_{i=1}^{m} \Big(\nabla^N_{d\phi(e_i)} d\phi(e_i) - d\phi(\nabla_{e_i} e_i) \Big) \circ \pi \\
=& \tau(\phi) \circ \pi.
\end{aligned}$$

□

From Theorem 4.20, we obtain

Theorem 4.21. *Let (M^{2m}, J, g) be a standard Kähler manifold, (N^n, h) be a Riemannian manifold and let $(T^*M, {}^{BS}g)$ the cotangent bundle of M equipped with the Berger-type deformed Sasaki metric. Let $\phi : (M, g) \longrightarrow (N, h)$ a smooth map. The map*

$$\begin{array}{rcl}\psi : (T^*M, {}^{BS}g) & \longrightarrow & (N, h) \\ (x, p) & \longmapsto & \phi(x)\end{array}$$

is harmonic if and only if ϕ is harmonic.

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