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A COMPREHENSIVE GENERALIZATION OF CLASSICAL FIBONACCI SEQUENCES, BINET FORMULA AND IDENTITIES

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Abstract. This article presents a fundamental generalization of the classical Fibonacci sequence. We introduce a general 2^{nd} order recurrence relation, $V_n = pV_{n-1} + qV_{n-2}$, $q \neq 0, n \geq 2$, with initial terms $V_0(= a), V_1(= b)$, a, b, p, and q are any non-zero real numbers. We derive an explicit generalized form of the generating function and comprehensive Binet's formula, which comprehend this concept to various sequences that follow similar recurrence relations. Furthermore, we analyze more generalized and specialized cases, uncovering new and existing identities for well-known sequences such as Fibonacci, Lucas, Pell, Pell-Lucas, Goksal Bilgici, and others. Our analysis implicitly reveals identities like Cassini's, Catalan's, d'Ocagne's, and Gelin-Cesàro in the generalized form. Additionally, tabular and graphical representations are provided to illustrate the relationships between the terms of these sequences.

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1. Introduction

In the realm of mathematics, the most intriguing number sequence is the classical Fibonacci sequence. For both amateur mathematicians and professionals seeking innovative insights, it continues to offer numerous opportunities due to its wonderful and incredible properties [11]. Several generalizations of the classical Fibonacci sequence are available in the literature, achieved either by altering the initial conditions [2, 7, 9, 10, 11, 15, 21] or by modifying the recurrence relation [4, 12, 13, 14, 16, 17, 18, 20, 22, 23]. Among them, sequences generated by recurrence relations stand out as the most prominent paradigms of

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recursive sequences. Classical Fibonacci, Lucas, Pell, Pell-Lucas, modified Pell sequences, and their generalizations possess several fascinating properties and find applications in numerous areas of pure and applied mathematical sciences, such as graph theory [6, 10], algebra [19, 5], quasi crystals [19, 3] and computer algorithms [18, 1, 19]. The well-known Fibonacci sequence is defined by the recurrence relation and initial conditions are as follows:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ F_0 = 0, F_1 = 1.$$

In this paper, both the initial terms and the recurrence relation are considered in their generalized forms, as described in the following definition.

Definition 1.1. We define a generalization of the Classical Fibonacci sequence $\{V_n\}_{n=0}^{\infty}$ by the following recurrence relations:

$$V_n = pV_{n-1} + qV_{n-2} \tag{1}$$

with the initial conditions,$V_0 = a, V_1 = b$, where a, b, p, and q are any non-zero real numbers. This formula (1) holds true for every integer $n \geq 2$.

Ten terms of Generalized Fibonacci sequences

$$\begin{bmatrix}
a \\
b \\
q\cdot a + p\cdot b \\
p\cdot q\cdot a + \left(p^2 + q\right)\cdot b \\
\left(p^2\cdot q + q^2\right)\cdot a + \left(p^3 + 2\cdot p\cdot q\right)\cdot b \\
\left(p^3\cdot q + 2\cdot p\cdot q^2\right)\cdot a + \left(p^4 + 3\cdot p^2\cdot q + q^2\right)\cdot b \\
\left(p^4\cdot q + 3\cdot p^2\cdot q^2 + q^3\right)\cdot a + \left(p^5 + 4\cdot p^3\cdot q + 3\cdot p\cdot q^2\right)\cdot b \\
\left(p^5\cdot q + 4\cdot p^3\cdot q^2 + 3\cdot p\cdot q^3\right)\cdot a + \left(p^6 + 5\cdot p^4\cdot q + 6\cdot p^2\cdot q^2 + q^3\right)\cdot b \\
\left(p^6\cdot q + 5\cdot p^4\cdot q^2 + 6\cdot p^2\cdot q^3 + q^4\right)\cdot a + \left(p^7 + 6\cdot p^5\cdot q + 10 p^3\cdot q^2 + 4\cdot p\cdot q^3\right)\cdot b \\
\left(p^7\cdot q + 6\cdot p^5\cdot q^2 + 10 p^3\cdot q^3 + 4\cdot p\cdot q^4\right)\cdot a + \left(p^8 + 7\cdot p^6\cdot q + 15 p^4\cdot q^2 + 10 p^2\cdot q^3 + q^4\right)\cdot b \\
\left(p^8\cdot q + 7\cdot p^6\cdot q^2 + 15 p^4\cdot q^3 + 10 p^2\cdot q^4 + q^5\right)\cdot a + \left(p^9 + 8\cdot p^7\cdot q + 21\cdot p^5\cdot q^2 + 20 p^3\cdot q^3 + 5\cdot p\cdot q^4\right)\cdot b \\
\left(p^9\cdot q + 8\cdot p^7\cdot q^2 + 21\cdot p^5\cdot q^3 + 20 p^3\cdot q^4 + 5\cdot p\cdot q^5\right)\cdot a + \left(p^{10} + 9\cdot p^8\cdot q + 28 p^6\cdot q^2 + 35 p^4\cdot q^3 + 15\cdot p^2\cdot q^4 + q^5\right)\cdot b
\end{bmatrix} =$$

	0
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89

FIGURE 1. First ten terms of the sequence (1) and the corresponding terms of the classical Fibonacci sequence

Utilizing (1), the first ten terms of the sequence are displayed in Figure1. On substituting $a = 0$, $b = 1$, $p = 1$, and $q = 1$ into these terms of Figure1 yields the first 10 terms of the classical Fibonacci sequence. Likewise, terms of the others existing known sequences can be obtained by substituting the analogous values of a, b, p, and q. In the literature [8, 11], numerous generalizations of Fibonacci sequences exist. Among them, some of the recognized generalizations with recurrence relations and initial conditions are as follows:

The **Lucas sequence**

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2,\ L_0 = 2, L_1 = 1.$$

The **Pell sequence**

$$P_n = 2P_{n-1} + P_{n-2},\ n \geq 2,,P_0 = 2, P_1 = 1.$$

The **Modified Pell sequence**

$$q_n = 2q_{n-1} + q_{n-2},\ n \geq 2,,q_0 = 1, q_1 = 1.$$

The **Pell-Lucas sequence**

$$Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2,\ Q_0 = 2, Q_1 = 2.$$

The **Goksal Bilgici sequences**

$$f_n = 2sq_{n-1} + (t^2 - s)q_{n-2},\ n \geq 2,q_0 = 0, q_1 = 1.$$

and

$$l_n = 2sl_{n-1} + (t^2 - s)l_{n-2},\ n \geq 2,\ l_0 = 2, l_1 = 2s.$$

The **Jacobsthal sequences**

$$J_n = J_{n-1} + 2J_{n-2},\ n \geq 2,\ J_0 = 0, J_1 = 1.$$

The **Jacobsthal-Lucas sequences**

$$L_n = L_{n-1} + 2J_{n-2},\ n \geq 2,\ L_0 = 2, L_1 = 1.$$

Evidently, for $(p, q) = (1, 1)$ and $(a, b) = (1, 1)$, $(p, q) = (1, 1)$ and $(a, b) = (2, 1)$, $(p, q) = (2, 1)$ and $(a, b) = (2, 1)$, $(p, q) = (2, 1)$ and $(a, b) = (2, 2)$, $(p, q) = (1, 2)$ and $(a, b) = (0, 1)$, $(p, q) = (1, 2)$ and $(a, b) = (2, 1)$ and $(p, q) = \left(2s, t^2 - s\right)$ and $(a, b) = (0, 1)$, $(p, q) = \left(2s, t^2 - s\right)$ and $(a, b) = (2, 2s)$, where s and t are any non − zero real numbers, the sequence $\{V_n\}$ defined in (1) reduces the Classical Fibonacci, Lucas, Pell, Modified Pell, Pell-Lucas Jacobsthal, Jacobsthal-Lucas and Goksal Bilgici sequences respectively.

2. Main results

The explicit **Generalized Generating function** of the sequence defined in (1).

Theorem 2.1 (Generalized Generating Functions)**.** *The generalized generating function of the sequence defined in (1) is*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - pV_0)x}{(1 - px - qx^2)} = \frac{a + (b - pa)x}{(1 - px - qx^2)}.$$

Proof. Let

$$V(x) = \sum_{n=0}^{\infty} V_n x^n, \tag{2}$$

$$pxV(x) = px \sum_{n=0}^{\infty} V_n x^n \tag{3}$$

$$qx^2 V(x) = qx^2 \sum_{n=0}^{\infty} V_n x^n \tag{4}$$

Then (2)-(3)-(4) gives

$$\left(1-px-qx^2\right)V(x)=V_0+\left(V_1-pV_0\right)x$$

Hence

$$\sum_{n=0}^{\infty}V_nx^n=\frac{V_0+(V_1-pV_0)x}{(1-px-qx^2)}=\frac{a+(b-pa)x}{(1-px-qx^2)}. \tag{5}$$

□

2.1. Special Cases of the generating function). *Case 1: Fibonacci sequences* Substituting $p=1, q=1$ and $V_0=\ F_0=0,\ V_1=F_1=1$ in (5), subsequently, the generating function in (5) simplifies to:

$$F_n=F_{n-1}+F_{n-2}\left(V_n=pV_{n-1}+pV_{n-2}\right),\ n\geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty}F_nx^n=\frac{F_0+\left(F_1-pF_0\right)x}{(1-px-qx^2)}=\frac{1}{(1-x-x^2)}. \tag{6}$$

Therefore, (6) represents the generating function for the well-known classical Fibonacci sequences.

Case 2: Lucas sequence

Substituting $p=1, q=1$ and $V_0=\ l_0=2,\ V_1=l_1=1$ in (5), then the generating function in (5), simplifies to:

$$l_n=2l_{n-1}+l_{n-2},\ n\geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty}l_nx^n=\frac{2-x}{(1-x-x^2)}. \tag{7}$$

Thus (7) is in agreement with the generating function for Lucas sequence.

Case 3: Pell sequence

Substituting $p=2, q=1$ and $V_0=\ P_0=0,\ V_1=P_1=1$ in (5), then the generating function in (5), simplifies to:

$$P_n=2P_{n-1}+P_{n-2},\ \ n\geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty}P_nx^n=\frac{1}{(1-2x-x^2)}. \tag{8}$$

Thus (8) is in agreement with the generating function for Pell sequence.

Case 4:Modified Pell sequence

Substituting $p=2, q=1$ and $V_0=\ f_0=1,\ V_1=f_1=1$ in (5), then the generating function in (5), simplifies to:

$$f_n=2f_{n-1}+f_{n-2},\ \ n\geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty}f_nx^n=\frac{1-x}{(1-2x-x^2)}. \tag{9}$$

Thus (9) is in agreement with the generating function for Modified Pell sequence.

Case 5: Pell-Lucas sequence Substituting $p=2, q=1$ and $V_0=\ Q_0=2,\ V_1=Q_1=2$ in (5),then the generating function in (5), simplifies to:

$$Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty} f_n x^n = \frac{2-2x}{(1-2x-x^2)}. \tag{10}$$

Thus (10) is in agreement with the generating function for Pell-Lucas sequence.
Case 6: Goksal Bilgici sequences
Substituting $p = 2a, q = b - a^2$ and $V_0 = f_0 = 0,\ V_1 = f_1 = 1$ and $p = 2a, q = b - a^2$ and $V_0 = l_0 = 2,\ V_1 = l_1 = 2a$ in (5), then the corresponding generating functions for the Goksal Bilgici sequences are:

$$f_n = 2af_{n-1} + (b-a^2)f_{n-2},\ n \geq 2$$
$$\text{and}$$
$$l_n = 2al_{n-1} + (b-a^2)l_{n-2},\ n \geq 2 \text{ are}$$

$$\sum_{n=0}^{\infty} f_n x^n = \frac{x}{(1-2ax-(b-a^2)x^2)}. \tag{11}$$

and

$$\sum_{n=0}^{\infty} l_n x^n = \frac{2-2ax}{(1-2ax-(b-a^2)x^2)}. \tag{12}$$

Thus (11) and (12) are in agreement with the generating function for the well-known Goksal Bilgici sequences.
Case 7: Jacobsthal Sequences Substituting $p = 1, q = 2$ and $V_0 = J_0 = 0,\ V_1 = J_1 = 1$ in (5), then the corresponding generating functions for the Jacobsthal sequence is

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x}{(1-x-2x^2)}. \tag{13}$$

Case 8: Jacobsthal-Lucas Sequences Substituting $p = 1, q = 2$ and $V_0 = J_0 = 2,\ V_1 = J_1 = 1$ in (5), then the corresponding generating functions for the Jacobsthal-Lucas sequence is

$$\sum_{n=0}^{\infty} J_n x^n = \frac{2-x}{(1-x-2x^2)}. \tag{14}$$

□

3. Binet's Formula for the Generalized Fibonacci Sequence

The following theorem gives the generalized form of Binet formula for the sequence defined in (1).

Theorem 3.1 (Generalized Binet's formula)**.** *The generalized Binet's formula for the sequence defined in (1) is*

$$V_n = V_0 \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) + (V_1 - pV_0) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right),$$

where $\alpha, \beta = \frac{p \pm \sqrt{p^2+4q}}{2}$ *are the roots of the equation* $x^2 - px - q = 0$.

Proof. Consider partial fraction decomposition of the right-hand side of the generating function (5) of the sequence defined in (1)

$$\frac{V_0 + (V_1 - pV_0)\, x}{(1 - px - qx^2)} \equiv \frac{(\alpha V_0 + V_1 - pV_0)}{(\alpha - \beta)\,(1 - \alpha x)} - \frac{(\beta V_0 + V_1 - pV_0)}{(\alpha - \beta)\,(1 - \beta x)}$$

. On simplification we have

$$= \frac{V_0}{(\alpha - \beta)} \left[\frac{\alpha}{(1 - \alpha x)} - \frac{\beta}{(1 - \beta x)} \right] + \frac{(V_1 - pV_0)}{(\alpha - \beta)} \left[\frac{1}{(1 - \alpha x)} - \frac{1}{(1 - \beta x)} \right]$$

we have

$$\sum_{n=0}^{\infty} V_n x^n = \left[V_0 \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) + (V_1 - pV_0) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \right] x^n. \tag{15}$$

Thus the generalized form of the Binet formula for the generalized Fibonacci sequence $\{V_n\}_{n=0}^{\infty}$ is

$$V_n = V_0 \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) + (V_1 - pV_0) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{16}$$

where α and,β are the roots of the equation $x^2 - px - q = 0$. defined in (1). □

3.1. Special Cases of Binet formula. *Case 1: Fibonacci sequences*

Substitute $p = 1, q = 1$ and $V_0 = \ F_0 = 0,\ V_1 = F_1 = 1$ in (16), we have

$$V_n = 0 \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) + (1 - 0) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$

$$F_n = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right),\ \alpha, \beta = \frac{1 \pm \sqrt{5}}{2}. \tag{17}$$

Thus (17), is the Binet's formula for the classical Fibonacci sequence.
Case 2: Lucas sequence
Substitute $p = 1, q = 1$ and $V_0 = \ l_0 = 2,\ V_1 = l_1 = 1$ in (16), then the Binet's formula for the Lucas sequence (5),is

$$l_n = 2 \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) + (1 - 2) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$

which is simplifies to

$$l_n = \alpha^n + \beta^n,\ . \tag{18}$$

where $\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$. Thus (18) is the Binet's formula for the for Lucas sequence.

Case 3: Pell sequence
Substitute $p = 2, q = 1$ and $V_0 = P_0 = 0,\ V_1 = P_1 = 1$ for Pell sequence $P_n = 2P_{n-1} + P_{n-2},\ n \geq 2$ in (16), we have

$$P_n = 0\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + (1)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

then the Binet's formula for the Pell sequence(5),

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ \alpha, \beta = 1 \pm \sqrt{2}. \tag{19}$$

Thus (19) is in agreement with the Binet's formula for Pell sequence.

Case 4:Modified Pell sequence
Substitute $p = 2, q = 1$ and $V_0 = f_0 = 1,\ V_1 = f_1 = 1$ in (16), then the Binet's formula for the modified Pell sequence $f_n = 2f_{n-1} + f_{n-2},\ n \geq 2$ then we have in (16), we have

$$q_n = 1\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + (1)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

, which is simplifies to

$$q_n = \alpha^n + \beta^n,\ \alpha, \beta = 1 \pm \sqrt{2} \tag{20}$$

Thus (20) is in agreement with the Binet's formula for Modified Pell sequence.
Case 5: Pell-Lucas sequence
Substitute $p = 2, q = 1$ and $V_0 = Q_0 = 2,\ V_1 = Q_1 = 2$ in (16), then the Binet's formula in for the Pell-Lucas sequence $Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2$ then we have

$$Q_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + (2-4)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

,

$$\Rightarrow Q_n = 2\left[\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right] = 2q_n,\ \alpha, \beta = 1 \pm \sqrt{2}. \tag{21}$$

Thus (21) is in agreement with the Binet's formula for Pell-Lucas sequence.
Case 6: Goksal Bilgici sequences
Substitute $p = 2a, q = b - a^2$ and $V_0 = f_0 = 0,\ V_1 = f_1 = 1$
and $p = 2a, q = b - a^2$ and $V_0 = l_0 = 2,\ V_1 = l_1 = 2a$ in (16), then the corresponding Binet's formula for the Goksal Bilgici sequences are

$$f_n = 0.\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + (1 - 2a.0)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

and

$$l_n = 2.\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + (2a - 4a)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

where

$$\because \alpha \pm \beta = a \pm \sqrt{b}$$

.

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{22}$$

and

$$l_n = \alpha^n + \beta^n \tag{23}$$

Thus (22) and (23) are in agreement with the Binet's formul for the Goksal Bilgici sequences. *Case 7: Jacobstal sequence*
Substitute $p = 1, q = 2$ and $V_0 = \ J_0 = 0,\ V_1 = J_1 = 1$
in (16), then the corresponding Binet's formula for the Jacobstal sequence is

$$L_n = 2.\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - \frac{\alpha^n - \beta^n}{\alpha - \beta} L_n = \alpha^n + \beta^n \ \ (\ \because 2\alpha\text{-}1\text{=}3,\ 2\beta\text{-}1\text{=-}3,\ \alpha\text{-}\beta\text{=}3). \tag{24}$$

Here α, β are roots of the equation $x^2 - x - 2 = 0$, this formula is the same as is in [8].

Case 8: Jacobstal-Lucas sequence
Substitute $p = 1, q = 2$ and $V_0 = \ J_0 = 2,\ V_1 = J_1 = 1$
in (16), then the corresponding Binet's formula for the Jacobstal-Lucas sequence is

$$L_n = \alpha^n + \beta^n \ \ (\ \because 2\alpha\text{-}1\text{=}3,\ 2\beta\text{-}1\text{=-}3,\ \alpha\text{-}\beta\text{=}3) \tag{25}$$

Here α, β are roots of the equation $x^2 - x - 2 = 0$, this formula is the same as is in [8].

TABLE 1. **Generating and Binet's formulas of some of the well known sequences as special cases of this generalized $V_n = pV_{n-1} + qV_{n-2}$ sequence,$V_0(= a), V_1(= b)$, a, b, p, and q**

Sequence, Initial conditions Constant Coefficients	n^{th} term	Generating function	Binet's formula
Generalized Sequence $V_0(= a), V_1(= b)$, a, b, p, q are arbitrary	$V_n = pV_{n-1} + qV_{n-2}$	$\sum\limits_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-pV_0)x}{(1-px-qx^2)} = \frac{a+(b-pa)x}{(1-px-qx^2)}.$	$V_n = V_0\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) + (V_1 - pV_0)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$
Classical Fibonacci $F_0(= 0), F_1(= 1)$ $p = 1, q = 1$	$F_n = F_{n-1} + F_{n-2}$	$\sum\limits_{n=0}^{\infty} F_n x^n = \frac{1}{(1-x-x^2)}.$	$F_n = \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right),\ \alpha, \beta = \frac{1\pm\sqrt{5}}{2}$
Lucas $l_0(= 2), l_1(= 1)$ $p = 2, q = 1$	$l_n = 2l_{n-1} + l_{n-2}$	$\sum\limits_{n=0}^{\infty} l_n x^n = \frac{2-x}{(1-x-x^2)}$	$l_n = \alpha^n + \beta^n,$
Pell sequence $P_0(= 2), P_1(= 1)$ $p = 2, q = 1$	$P_n = 2P_{n-1} + P_{n-2}$	$\sum\limits_{n=0}^{\infty} P_n x^n = \frac{1}{(1-2x-x^2)}$	$P_n = \frac{\alpha^n-\beta^n}{\alpha-\beta},\ \alpha, \beta = 1 \pm \sqrt{2}$
Modified Pell sequence $V_0(= 1), V_1(= 1)$ $p = 2, q = 1$	$V_n = 2V_{n-1} + V_{n-2}$	$\sum\limits_{n=0}^{\infty} V_n x^n = \frac{1-x}{(1-2x-x^2)}$	$V_n = \alpha^n + \beta^n,\ \alpha, \beta = 1 \pm \sqrt{2}$
Pell-Lucas sequence $Q_0(= 2), Q_1(= 2)$ $p = 2, q = 1$	$Q_n = 2Q_{n-1} + Q_{n-2}$	$\sum\limits_{n=0}^{\infty} Q_n x^n = \frac{2-2x}{(1-2x-x^2)}$	$Q_n = 2q_n,\ \alpha, \beta = 1 \pm \sqrt{2}$
Goksal Bilgici sequence-1st $f_0(= 0), f_1(= 1)$ $p = 2s, q = (t^2 - s)$	$f_n = 2sf_{n-1} + (t^2 - s)f_{n-2}$	$\sum\limits_{n=0}^{\infty} f_n x^n = \frac{x}{(1-2ax-(b-a^2)x^2)}$	$f_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$
Goksal Bilgici sequence-2nd $l_0(= 2), l_1(= 2s)$ $p = 2s, q = (t^2 - s)$	$l_n = 2sl_{n-1} + (t^2 - s)l_{n-2}$	$\sum\limits_{n=0}^{\infty} l_n x^n = \frac{2-2ax}{(1-2ax-(b-a^2)x^2)}$	$l_n = \alpha^n + \beta^n$

Jacobsthal sequences $J_0(=0), J_1(=1)$ $p=1, q=2$	$J_n = J_{n-1} + 2J_{n-2}$	$\sum_{n=0}^{\infty} J_n x^n = \frac{x}{(1-x-2x^2)}$	$L_n = \alpha^n + \beta^n$ ($\because$ 2α-1=3, 2β-1=-3, α-β=3) .
Jacobsthal-Lucas sequences $L_0(=2), L_1(=1)$ $p=1, q=2$	$L_n = L_{n-1} + 2L_{n-2}$	$\sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{(1-x-2x^2)}$	$L_n = \alpha^n + \beta^n$, ($\because$ 2α-1=3, 2β-1=-3, α-β=3)

4. Generalized Identities

Theorem 4.1. *For every integer n, in the sequence defined in (1) is*

$$V_{-n} = \frac{1}{(-q)^n}\left[V_n - (2V_1 - pV_0)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right]$$

where V_n, α, β are the roots of the equation $x^2 - px - q = 0$. are defined in (16).

Proof. Using the generalized Binet formula (16) and replacing n with $-n$ in it we have

$$V_{-n} = V_0\left(\frac{\alpha^{-n+1} - \beta^{-n+1}}{\alpha - \beta}\right) + (V_1 - pV_0)\left(\frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta}\right), \tag{26}$$

Now

$$\frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = -\frac{1}{(-q)^n}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) \because \alpha\beta = -q \tag{27}$$

$$\left(\frac{\alpha^{-n+1} - \beta^{-n+1}}{\alpha - \beta}\right) = -\frac{1}{(-q)^n}\left(p\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) - \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)\right) \tag{28}$$

using

$$\alpha^2 = \alpha p + q, \beta^2 = \beta p + q$$

. Using (27) and (28) in (26), we have

$$V_{-n} = \frac{1}{(-q)^n}\left[V_n - (2V_1 - pV_0)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right] \tag{29}$$

□

Corollary 4.2 (Fibonacci)**.** *If $p = 1, q = 1$ and $V_0 = F_0 = 0,\ V_1 = F_1 = 1$ in (29), then it reduces to $F_{-n} = (-1)^{n+1}F_n$.*

Proof. On substituting $p = 1, q = 1$ and $V_0 = F_0 = 0,\ V_1 = F_1 = 1$ in (29) we have

$$V_{-n} = \frac{1}{(-q)^n}\left[V_n - (2V_1 - pV_0)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right]$$

$$F_{-n} = \frac{1}{(-1)^n}\left[F_n - (2 - 0)F_n\right]$$

$$F_{-n} = \frac{1}{(-1)^n}\left[-F_n\right] = (-1)^{n+1}F_n. \tag{30}$$

Thus (30) is the corresponding identity for the classical Fibonacci sequence. □

Corollary 4.3 (Lucas)**.** *If $p = 1, q = 1$ and $V_0 = \ l_0 = 2,\ V_1 = l_1 = 1$ in (29), then (29) reduces to $l_{-n} = (-1)^{-n} l_n$.*

Proof. On substituting $p = 1, q = 1$ and $V_0 = \ l_0 = 2,\ V_1 = l_1 = 1$ in (29),in we have

$$l_{-n} = \frac{1}{(-1)^n}\left[l_n - (2-2)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right]$$

$$l_{-n} = (-1)^{-n} l_n \tag{31}$$

Thus (31) is the corresponding identity for the Lucas sequence. □

Corollary 4.4 (Pell)**.** *If $p = 1, q = 1$ and $V_0 = \ P_0 = 2,\ V_1 = P_1 = 1$ in (29), then (29) reduces to $P_{-n} = (-1)^{n+1} P_n$.*

Proof. On substituting $p = 1, q = 1$ and $V_0 = \ P_0 = 2,\ V_1 = P_1 = 1$ in (29), in we have

$$V_{-n} = \frac{1}{(-q)^n}\left[V_n - (2V_1 - pV_0)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right]$$

$$P_{-n} = \frac{1}{(-1)^n}\left[P_n - 2P_n\right] = (-1)^{n+1} P_n. \tag{32}$$

Thus (32) is the corresponding identity for the Pell sequence. □

Corollary 4.5 (Modified Pell)**.** *If $p = 2, q = 1$ and $V_0 = \ q_0 = 1,\ V_1 = q_1 = 1$ in (29), then (29) reduces to $q_{-n} = (-1)^n q_n$.*

Proof. On substituting $p = 2, q = 1$ and $V_0 = \ q_0 = q,\ V_1 = q_1 = 1$ in (29),in we have

$$q_{-n} = \frac{1}{(-1)^n}\left[q_n - (2-2)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right]$$

$$q_{-n} = (-1)^{-n} q_n \tag{33}$$

Thus (33) is the corresponding identity for the Modified Pell sequence. □

Corollary 4.6 (Pell-Lucas)**.** *If $p = 2, q = 1$ and $V_0 = \ Q_0 = 2,\ V_1 = Q_1 = 2$ in (29), then (29) reduces to $Q_{-n} = (-1)^n Q_n$.*

Proof. On substituting $p = 2, q = 1$ and $V_0 = \ Q_0 = 2,\ V_1 = Q_1 = 2$ in (29), we have

$$Q_{-n} = \frac{1}{(-1)^n}\left[Q_n - (4-4)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right]$$

$$Q_{-n} = (-1)^{-n} Q_n \tag{34}$$

Thus (34) is the corresponding identity for the Pell-Lucas sequence. □

Corollary 4.7 (Goksal Bilgici)**.** *If $p = 2a, q = b - a^2$ and $V_0 = \ f_0 = 0,\ V_1 = f_1 = 1$
and $p = 2a, q = b - a^2$ and $V_0 = \ l_0 = 2,\ V_1 = l_1 = 2a$ in (29), then (29) reduces to $f_{-n} = -\frac{1}{(b-a^2)^n} f_n$ and $l_{-n} = -\frac{1}{(b-a^2)^n} l_n$.*

Proof. Substituting $p = 2a, q = b - a^2$ and $V_0 = \ f_0 = 0,\ V_1 = f_1 = 1$ and $p = 2a, q = b - a^2$ and $V_0 = \ l_0 = 2,\ V_1 = l_1 = 2a$ in (29, we have

$$f_{-n} = \frac{1}{(b-a^2)^n}\left[f_n - 2\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right]$$

and

$$l_{-n} = \frac{1}{(a^2-b)^n}\left[l_n - (4a - 4a)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right]$$

$$f_{-n} = \frac{-1}{(b-a^2)^n} f_n, \tag{35}$$

and

$$l_{-n} = \frac{-1}{(b-a^2)^n} l_n, \tag{36}$$

Thus (35) and (36) are the corresponding identity for the Goksal Bilgici sequences. □

Theorem 4.8 (Cassini identity)**.** *For every integer n, for the sequence defined in (1). Then*

$$V_{n+1}V_{n-1} - V_n^2 = (-q)^{n-1}\left[qV_0^2 + (\alpha+\beta)V_0V_1 - {V_1}^2\right]$$

. Here $\alpha + \beta = p$.

Proof. Using the definition in (1) and the generalized Fibonacci sequence (16) we have

$$V_{n+1}V_{n-1} - V_n^2 = (-q)^{n-1}\left[qV_0^2 + (\alpha+\beta)V_0V_1 - {V_1}^2\right] \tag{37}$$

The above result can be written as

$$V_{n+1}V_{n-1} - V_n^2 = (-q)^{n-1}\left[qa^2 + pab - b^2\right], \alpha + \beta = p.$$

□

Corollary 4.9 (Fibonacci)**.** *If $p = 1, q = 1$ and $V_0 = \ F_0 = 0,\ V_1 = F_1 = 1$ in (**??**), then it reduces to*

$$F_{n+1}F_{n-1} - {F_n}^2 = (-1)^n. \tag{38}$$

Corollary 4.10 (Lucas)**.** *If $p = 1, q = 1$ and $V_0 = \ l_0 = 2,\ V_1 = l_1 = 1$ in (37), then it reduces to*

$$l_{n+1}l_{n-1} - l_n^2 = 5(-1)^{n-1} \tag{39}$$

Corollary 4.11 (Pell)**.** *If $p = 1, q = 1$ and $V_0 = \ P_0 = 2,\ V_1 = P_1 = 1$ in (37), then (37) reduces to*

$$P_{n+1}P_{n-1} - P_n^2 = 5(-1)^{n-1} \tag{40}$$

Corollary 4.12 (Modified Pell)**.** *If $p = 2, q = 1$ and $V_0 = \ q_0 = 1,\ V_1 = q_1 = 1$ in (37), then (37) reduces to*

$$q_{n+1}q_{n-1} - q_n^2 = 2(-1)^{n-1} \tag{41}$$

Corollary 4.13 (Pell-Lucas)**.** *If* $p = 2, q = 1$ *and* $V_0 = \ Q_0 = 2,\ V_1 = Q_1 = 2$ *in (37), then (37) reduces to*

$$Q_{n+1}Q_{n-1} - Q_n^2 = 8(-1)^{n-1} \tag{42}$$

Corollary 4.14 (Goksal Bilgici)**.** *If* $p = 2a, q = b - a^2$ *and* $V_0 = \ f_0 = 0,\ V_1 = f_1 = 1$
and $p = 2a, q = b-a^2$ *and* $V_0 = \ l_0 = 2,\ V_1 = l_1 = 2a$ *in (37), then (37) reduces to*

$$f_{n+1}f_{n-1} - f_n^2 \ = -\left(a^2 - b\right)^{n-1}. \tag{43}$$

and

$$l_{n+1}l_{n-1} - l_n^2 = 4b\left(a^2 - b\right)^{n-1}. \tag{44}$$

Theorem 4.15 (Catalan identity)**.** *For every integer* n *and* r*, generalized sequence defined in(1)we have*

$$V_{n+r}V_{n-r} - {V_n}^2 = (-q)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)^2\left(qa^2 + pba - b^2\right)$$

.

Proof. Using the definition in (1) and the generalized Fibonacci sequence (16) we have

$$V_{n+r}V_{n-r} - {V_n}^2 = -(\alpha\beta)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)^2\left[(\alpha\beta)\,V_0^2 - (\alpha+\beta)\,V_0V_1 + {V_1}^2\right].$$

which can be written as

$$V_{n+r}V_{n-r} - {V_n}^2 = (-q)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)^2\left(qa^2 + pba - b^2\right). \tag{45}$$

□

Theorem 4.16. *For every integer n,r and s, generalized sequence defined in (1) and (16) we have*

$$V_{n+r}V_{n+s} - V_nV_{n+r+s} = (-q)^n\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)\left(\frac{\alpha^s - \beta^s}{\alpha - \beta}\right)\left(qa^2 + pba - b^2\right)$$

.

Proof. Using the definition in (1) for the generalized Fibonacci sequence and (16) we have

$$V_{n+r}V_{n+s} - V_nV_{n+r+s} = (-q)^n\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)\left(\frac{\alpha^s - \beta^s}{\alpha - \beta}\right)\left(qa^2 + pba - b^2\right) \tag{46}$$

□

Corollary 4.17. *On substituting the appropriate values for p, q in the recurrence relation defined in (1) and the initial conditions* V_0 *and* V_1 *for Fibonacci,Lucas,Pell, modified Pell, Pell-Lucas and Goksal Bilgici sequences in (46), then the corresponding identity for these sequences is obtained.*

Theorem 4.18 (d'Ocagne's Identity). *For every integer m and n generalized sequence defined in (1) and (16) we have*

$$V_m V_{n+1} - V_{m+1} V_n = -(-q)^n \left(\frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta}\right)\left(qa^2 + pba - b^2\right)$$

.

Proof. content... Using the definition in (1) for the generalized Fibonacci sequence and (16) we have

$$V_m V_{n+1} - V_{m+1} V_n = -(-q)^n \left(\frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta}\right)\left(qa^2 + pba - b^2\right). \tag{47}$$

□

Corollary 4.19. *On substituting the appropriate values for p, q in the recurrence relation defined in (1) and the initial conditions V_0 and V_1 for Fibonacci,Lucas,Pell, modified Pell, Pell-Lucas and Goksal Bilgici sequences in (46), then the corresponding identity for these sequences is obtained.*

Theorem 4.20 (Gelin Cesaro identity). *For every integer n,r, we have*

$$\begin{aligned}
&V_{n-2}V_{n-1}V_{n+1}V_{n+2} - {V_n}^2 \\
&= {V_0}^4 \nabla \Big(\Delta_{n-1}\Delta_n\Delta_{n+2}\Delta_{n+3} - (\Delta_{n+1})^4\Big) \\
&+ {V_0}^3 (V_1 - pV_0)\nabla\Big((\Delta_{n+2})^2\Delta_{n-1}\Delta_n + (\Delta_{n-1})^2\Delta_{n+2}\Delta_{n+3} - 4(\Delta_{n+1})^3\Delta_n \\
&\quad + \Delta_n\Delta_{n+3}(\Delta_{n-1}\Delta_{n+1} + \Delta_{n-2}\Delta_{n+2})\Big) \\
&+ {V_0}^2 (V_1 - pV_0)^2\nabla\Big(\Delta_{n-2}\Delta_{n+3}(\Delta_{n-1}\Delta_{n+2} + \Delta_n\Delta_{n+1}) + \Delta_{n-1}\Delta_n\Delta_{n+1}\Delta_{n+2} \\
&\quad + (\Delta_{n-1})^2(\Delta_{n+2})^2 + (\Delta_{n-1})^2\Delta_{n+1}\Delta_{n+3} + (\Delta_{n+2})^2\Delta_{n-2}\Delta_n - 6(\Delta_{n+1})^2(\Delta_n)^2\Big) \\
&+ V_0 (V_1 - pV_0)^3\nabla\Big(\Delta_{n-2}\Delta_{n+1}(\Delta_{n-1}\Delta_{n+1}\Delta_{n+3} \\
&\quad + \Delta_{n-2}\Delta_{n+1}) + \Delta_{n-1}\Delta_{n-2}(\Delta_{n+2})^2 + (\Delta_{n-1})^2\Delta_{n+1}\Delta_{n+2}\Big) \\
&+ (V_1 - pV_0)^4\nabla\Big(\Delta_{n-2}\Delta_{n-1}\Delta_{n+1}\Delta_{n+2} - (\Delta_n)^4\Big)
\end{aligned} \tag{48}$$

where

$$\Delta_n = (\alpha^n - \beta^n) = \left(\sqrt{p^2 + 4q}\right)\sum_{j=0}^{\left[\frac{n}{2}\right]} C\binom{n-j-1}{j} p^{n-j-1} q^j, \nabla = {}^1/_{(\alpha-\beta)^4}$$

Theorem 4.21 (d'Ocagne's Identity). *For every integer m and n generalized sequence defined in (1) and (16) we have*

$$V_m V_{n+1} - V_{m+1} V_n = -(-q)^n \left(\frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta}\right)\left(qa^2 + pba - b^2\right)$$

.

Proof. Using the definition in (1) for the generalized Fibonacci sequence and (16) we have

$$V_m V_{n+1} - V_{m+1} V_n = -(-q)^n \left(\frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta} \right) \left(qa^2 + pba - b^2 \right). \qquad (49)$$

□

Theorem 4.22. *For every integers m and n in the generalized sequence defined in (1) and numbers defined in (16) we have*

$$V_{n+m} \pm V_{n-m} = \left[(R_{n+m} + R_{n-m}) \right] (V_1 - pV_0) \pm \left[(R_{n+m+1} + R_{n-m+1}) \right] V_0$$

,

$$\begin{aligned} V_{n+m} V_{n-m} &= (R_{n-m} R_{n+m}) (V_1 - pV_0)^2 + (R_{n-m+1} R_{n+m+1}) {V_0}^2 \\ &+ (-q)^{m+n+1} l_{-2m+1} + (-q)^{m-n+} l_{2m+1} + \frac{1}{(p^2 + 4q)} (V_1 - pV_0) V_0 \end{aligned}$$

where $R_i = \left(\frac{\alpha^i - \beta^i}{\alpha - \beta} \right)$.

Theorem 4.23. *For every integer n in the generalized sequence defined in (1) and numbers in (16) we have*

$$V_{n+1} - \alpha V_n = \left[V_0 \beta + (V_1 - pV_0) \right] \beta^n$$
$$V_{n+1} - \beta V_n = \left[V_0 \alpha + (V_1 - pV_0) \right] \alpha^n$$
$$(V_{n+1} - \alpha V_n)(V_{n+1} - \beta V_n) = \left(a^2 q^2 - pab + b^2 \right) (-q)^n$$

Theorem 4.24. *For every integer n in the generalized sequence defined in (1) and numbers defined in (16) we have*

$$\lim_{n \to \infty} \frac{V_{n+1}}{V_n} = \begin{cases} \alpha \text{ if } p > 0, \; q \geq 0, \; p, q \in N \\ \beta \text{ if } p < 0, \; q, \; p^2 + 4q > 0, p, q \in R \end{cases}$$

5. Discussion and Conclusion

In this article, an advanced generalization of the Fibonacci sequence, which we call the generalized Fibonacci sequence is considered which is Unlike other generalizations as its parameters for the recurrence relation and for the initial terms, a, b, p, and q can be any real numbers. First ten terms are of this sequence are exhibited in Figure-1 1 in general form and with values and with some Table-2 (tablualr form). How these sequences progresses versus increasing n is exhibited graphically in Figure-2. Terms of the any knwon sequence can be checked and verified on imposing the restriction on parameters p,q, a and b. Generalized generating function (3) and Binet formula (5) are obtained, then utilizing these, Generating function and Binet formulas of known existing sequences are also verified. Special cases of generating function and Binet formulas of this generalization are also displayed in the tabular form in Table-1. Some identities in their generalized forms are also obtained and can be specialized to the existing identities by simply substituting the values of a, b, p, and q. Finally

Table ratio of $(n+1)$th to nth terms as n approaches to infinity is illustrated in tabular form in Table-3 and graphically in Figure-3.

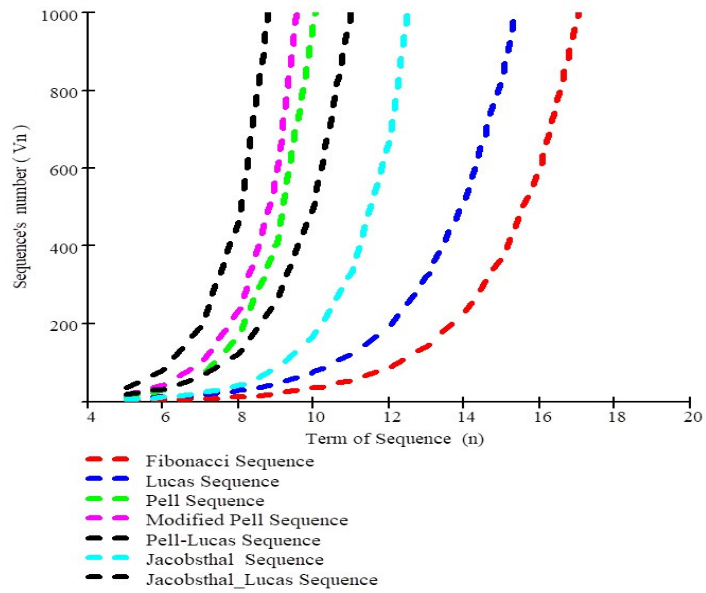

FIGURE 2. Progress of the Sequence grows using the generalized term of (1).

TABLE 2. **Generalization of Classical fibnoacci Sequence**

Initial Conditions	$V_0=0, V_1=1, p=1, q=1$	$V_0=2, V_1=1, p=1, q=1$	$V_0=0, V_1=1, p=2, q=1$	$V_0=1, V_1=1, p=2, q=1$	$V_0=2, V_1=2, p=2, q=1$	$V_0=0, V_1=1, p=1, q=2$
n=	Fibonacci	Lucas	Pell	Modified-Pell	Pell-Lucas	Jacobstal
1	0	2	0	1	2	0
2	1	1	1	1	2	1
3	1	3	2	3	6	1
4	2	4	5	7	14	3
5	3	7	12	17	34	5
6	5	11	29	41	82	11
7	8	18	70	99	198	21
8	13	29	169	239	478	43
9	21	47	408	577	1154	85
10	34	76	985	1393	2788	171
—	—	—	—	—	—	—
100	3.542×10^{20}	7.921×10^{20}	6.669×10^{37}	9.474×10^{37}	1.895×10^{38}	2.11310×10^{29}

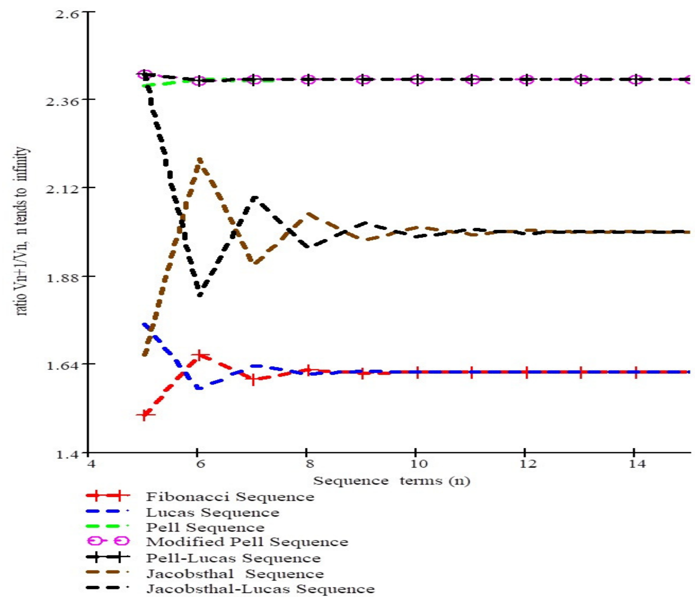

FIGURE 3. Ratio $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ for the well known sequences using the generalized term of (1).

TABLE 3. $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ **of Generalization of Classical fibnoacci Sequence**

Initial Conditions / Recurrence Relation	$V_0 = 0, V_1 = 1, p = 1, q = 1$	$V_0 = 2, V_1 = 1, p = 1, q = 1$	$V_0 = 0, V_1 = 1, p = 2, q = 11$	$V_0 = 1, V_1 = 1, p = 2, q = 1$	$V_0 = 2, V_1 = 2, p = 2, q = 1$	$V_0 = 0, V_1 = 1, p = 1, q = 2$	$V_0 = 2, V_1 = 1, p = 1, q = 2$
Sequence	Fibonacci	Lucas	Pell	Modified-Pell	Pell-Lucas	Jacobstal	Jacobstal Lucas
Vn+1/Vn	1.618	1.618	2.414	2.414	2.414	2.00	2.00

Similar expression for the Goksal Bilgici [**?**] is

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} or \lim_{n\to\infty} \frac{l_{n+1}}{l_n} \to \begin{cases} a + \sqrt{a^2 + b^2 - a} \text{ if } a > 0, a \in N, b \in R \\ a - \sqrt{a^2 + b^2 - a} \text{ if } a < 0, a \in N, b \in R. \end{cases}$$

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References

1. J. Atkins and R. Geist, *Fibonacci numbers and computer algorithms*, College Math. J. **18** (1987), 328-337.
2. G. Bilgici, *New generalizations of Fibonacci and Lucas sequences*, Amer. Math. Sc. **8** (2014), 1429–1437.
3. P. Chebotarevy, *Spanning forests and the golden ratio*, Discrete Appl. Math. **156** (2008), 813–821.
4. M. Edson and O. Yayenie, *A New Generalization of Fibonacci Sequences and Extended Binet's*, Integers **9** (2009), 639-654.
5. A.J. Feingold, *A hyperbolic gcm lie algebra and the fibonacci numbers*, Proceedings of the american mathematical society **3** (1980).
6. M.L. Fredman and R.E. Tarjan, *Fibonacci heaps and their uses in improved network optimization algorithms*, J. ACM. **34** (1987), 596-615.
7. A.F. Horadam, *A generalized Fibonacci sequence*, Amer. Math. Monthly **68** (1961), 455-459.
8. A.F. Horadam, *Jacobsthal Representation Numbers*, Fibonacci Quart. **34** (1996), 40-54.
9. D.V. Jaiswal, *On a generalized Fibonacci sequence*, Labdev J. Sci. Tech. Part A **7** (1969), 67-71.
10. S.T. Klein, *Combinatorial representation of generalized Fibonacci numbers*, Fibonacci Quart. **29** (1991), 124-131.
11. T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.
12. A.T. Krassimir, A.C. Liliya and S.D. Dimitar, *A new perspective to the generalization of the Fibonacci sequence*, Fibonacci Quart. **23** (1985), 21-28.
13. G.Y. Lee, S.G. Lee and H.G. Shin, *On the k-generalized Fibonacci matrix Q_k*, Linear Algebra Appl. **251** (1997), 73-88.
14. G.Y. Lee, S.G. Lee, J.S. Kim and H.K. Shin, *The Binet formula and representations of k-generalized Fibonacci numbers*, Fibonacci Quart. **39** (2001), 158-164.
15. J.C. Pond, *Generalized Fibonacci Summations*, Fibonacci Quart. **6** (1968), 97-108.
16. S.P. Pethe and C.N. Phadte, *A generalization of the Fibonacci sequence*, Applications of Fibonacci numbers **5** (1992), 465-472.
17. G. Sburlati, *Generalized Fibonacci sequences and linear congruence*, Fibonacci Quart. **40** (2002), 446-452.
18. I. Stojmenovic, *Recursive Algorithms in Computer Science Courses: Fibonacci Numbers and Binomial Coefficients*, IEEE Trans. Educ. **43** (2000), 273-276.
19. J. de Souza, E.M.F. Curado and M.A. Rego-Monteiro, *Generalized Heisenberg Algebras and Fibonacci Series*, College Math. J. phys, A. Math. Gen. **39** (1987), 10415.
20. J.E. Walton and A.F. Horadam, *Some further identities for the generalized Fibonacci sequence H_n*, Fibonacci Quart. **12** (1974), 272-280.
21. M. Waddill and L. Sacks, *Another generalized Fibonacci sequence*, Fibonacci Quart. **5** (1967), 209–222.
22. O. Yayenie, *A note on generalized Fibonacci sequences*, Appl. Math. Comput. **217** (2011), 5603–5611.
23. L.J. Zai and L.J. Sheng, *Some properties of the generalization of the Fibonacci sequence*, Fibonacci Quart. **25** (1987), 111-117.

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