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ON THE OPTIMAL CONTROL POLICY OF A FRACTIONAL ORDER BIOLOGICAL MODEL†

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Abstract. A fractional-order biological model with Crowley-Martin functional response is considered and investigated. Prey species grow according to the logistic growth function and it is subject to harvesting. The existence, uniqueness, and boundedness of its solution are shown. The considered model has three fixed points. Further, the local behavior of all possible equilibrium(fixed) points is studied and analyzed for the considered system and its discretization. The results show that the fixed points are locally stable under some conditions. Furthermore, a non-constant harvesting to find the optimal harvest policy is employed. Also, it is found that constant harvesting can not give the optimal profit. Numerical outcomes are illustrated to confirm and show the analytic outcomes.

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1. Introduction

Fractional order derivatives have drawn considerable significance in discussing the dynamical behaviors of biological models due to their efficiency in precisely describing many nonlinear real-world phenomena. As a result, fractional-order derivatives have increasing attention from researchers and scientists. Therefore, they describe their mathematical systems by a set of fractional-order derivatives, see [1, 2, 3, 16, 19, 23, 25]. Also,fractional-order derivatives are used in electrical circuits theory, control theory etc[7, 22]. Several definitions are considered in the literature for the fractional-order derivatives, the well-known ones Grunwald-Letnikov, Riemann-Liouville, Fabrizio, Caputo, Riesz fractional-order

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derivatives, weyl, Marechand and many other definitions, for more details see [4, 21, 24]. The definition of Caputo has a significant property which is the fractional derivative of any constant function is zero. So it is used by many authors in their systems. The authors of [22] gave a physical and geometrical interpretation of fractional differentiation and integration. For more results and details about the non-integer derivatives dynamic system, see [17, 18, 23] and references therein.
Optimal control theory is an essential tool that can be used to solve various problems in the real world, including biological situations, it is a very useful that can be used to make perfect and better decisions. In particular, it is applied to manage many renewable resources such as fish populations, plant populations, etc. Many articles in the literature discuss the optimal harvesting to get the optimal gains of renewable resources- see [8, 5, 10, 11, 12, 30, 31, 32, 33].
In this work, we consider and investigate a fractional order two-dimensional prey-predator system with Corwely-Martin functional response with harvesting, and then we investigate its discretization system. First, we consider a constant harvesting rate, then we extend the considered model to non-constant harvesting to find the optimal harvest strategy by employing the discrete optimal control theory.
This paper contains six sections. In section two, the formulation of the mathematical model is done. The boundedness and existence with non-negatively of its solution are shown and proved. Section three discusses the discretization of the mathematical model and its behavior. In section four, the Optimal Harvesting approach is employed. Numerical simulations of the theoretical findings are confirmed and presented in section five. Conclusions are given in section six.

2. The fractional-order system and its equilibria

In [6] P.H. Crowley and E.K. Martin discussed two dimensions prey-predator model as follows:

$$\begin{aligned} \frac{dX(T)}{dT} &= X(T)(a - X(T)) - \frac{bX(T)Y(T)}{(1+\alpha X(T))(1+\beta Y(T))}, \\ \frac{dY(T)}{dT} &= \frac{dX(T)Y(T)}{(1+\alpha X(T))(1+\beta Y(T))} - cY(T). \end{aligned} \tag{1}$$

Where $X(T)$ and $Y(T)$ represent the number of prey and predator species at time T, respectively. The parameters $a, b, c,$ and d take a positive constant. The α is the handling time, and β stands for the magnitude of interference among predators.
We study and introduce the fractional order mathematical model with constant harvesting on the prey species. Then, we extend the considered model to the non-constant harvesting process to get the maximum gain. The model (1) becomes

as follows:

$$D^{\theta}x(t) = x(t)(a - x(t)) - \frac{bx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - hx(t),$$
$$D^{\theta}y(t) = \frac{dx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - cy(t). \tag{2}$$

Here h is the harvesting rate. First, it is considered to be constant and then it will be a non-constant rate. θ stands for the fractional order such that $\theta \in (0, 1]$. $D^{\theta}x(t)$ is the Caputo differentiation.

Definition 2.1 ([13, 23])**.** The Caputo definition is given by the following:
$D^{\theta}_{t}f(x) = I^{l-\theta}f(x), \quad \theta > 0$
where $l = [\theta]$ and $I^{\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-\mu)^{\gamma-l}u(t)dt, \quad \gamma > 0.$

I^{γ} represents the γ order Riemann-Liuouvilleo integral operator,while D^{θ} represents the $\theta-$ order Caputo operator and $\Gamma(.)$ stands for the gamma function.

The existence and uniqueness of solution for a general fractional-order derivative is given in [9].

Lemma 2.2. *Let* $f : [t_0, \infty) \times \Omega \to \mathbb{R}^n$, *and* $0 < \theta \leq 1$. *If* $f(t, x)$ *has the locally Lipchitz condition to the variable* x, *then the initial value problem* $D^{\theta}_{t}x(t) = f(t, x),\ t > t_0 \quad x_0$ *has a unique solution on* $[t_0, \infty) \times \Omega$.

Some helpful results and theorems are needed throughout this work. The following lemmas are shown in [14, 20, 29].

Lemma 2.3. *Assume that* $f(t) \in C[a, b] \quad , \quad 0 < \theta \leq 1$, *and* $D^{\theta}f(t) \in C(a, b]$, *then:*

(1) $f(t) = f(a) + \frac{1}{\Gamma\theta)}(D^{\theta}f(\xi)(t-a)^{\theta}, \quad$ *and,* $a \leq \xi \leq s \quad \forall\ s\ \in\ (a, b]$.

(2) *If* $D^{\theta}f(t) \geq 0 \quad \forall\ t\ \in\ (a, b),$ *then* $f(t)$ *is a non-decreasing function* $\forall\ t\ \in\ [a, b]$ *and if* $D^{\theta}f(t) \leq 0, \quad \forall t\ \in (a, b)$, *then* $f(t)$ *is a non-increasing function* $\forall\ t\ \in\ [a, b]$.

Lemma 2.4. (1) *Suppose the following Cauchy problem*

$$D^{\theta}x(t) = ax(t) + f(t),$$
$$x(t_0) = b \qquad b \in \mathbb{R}.$$

Where $a \in \mathbb{R}$ *and* $0 < \theta \leq 1$ *then the solution is as follows:*

$$x(t) = bE_{\theta}[a(t-t_0)^{\theta}] + \int_{t_0}^{t}(t-s)^{\theta-1}E_{\theta,\theta}[a(t-s)^{\theta}]f(s)ds$$

, and the solution to the following problem

$$D^{\theta}x(t) = ax(t),$$
$$x(t_0) = b \qquad b \in \mathbb{R}.$$

is given by $x(t) = bE_\theta[a(t-t_0)^\theta]$.

(2) *Consider that* $u(t)$ *is a continuous function on* $[t_0, \infty]$ *such that*

$$D^\theta u(t) \le -au(t) + \mu,$$

where $t_0 \ge 0$ *represents the initial time and* $(a, \mu) \in \mathbb{R}^2, a \neq 0$. *Then, its solution is as follows:*

$$u(t) \le (u_{t_0} - \frac{\mu}{a})E_\theta[-a(t-t_0)^\theta] + \frac{\mu}{a}.$$

Theorem 2.5. *Let* $\Omega_+ = \{(x,y) \in R^2 \,\|x \ge 0, y \ge 0\}$ *be the set of all non-negative real numbers in* R^2 *, then each solution of the model (2) with* $x_0 \ge 0$ *and* $y_0 \ge 0$ *is uniformly bounded and non-negative.*

Proof. Consider that $x(0) > 0$ for $t = 0$, $x(t) \ge 0 \quad \forall \quad t \ge 0$ is not true. This implies

$$\begin{aligned} x(t) > 0 &\qquad 0 \le t < t_1, \\ x(t) = 0 &\qquad t = t_1, \\ x(t) < 0 &\qquad t > t_1, \end{aligned}$$

for a constant $t_1 > 0$.

The first equation of the model(2) implies that $D^\theta_{t_1} = 0$ at $t = t_1$. Apply part 1 in the Lemma 1, we get $x(t_1^+) = 0$, contradicts the fact $x(t_1^+) < 0, i.e., x(t) < 0, t > t_1$. Hence, we get $x(t) \ge 0, \quad \forall \ t \ge 0$. The same argument can show that $y(t) \ge 0 \ \forall \ t \ge 0$.

For boundedness, let $V(t) = \frac{x(t)}{b} + \frac{y(t)}{d}$ so that

$$\begin{aligned} D^\theta V(t) &= \frac{D^\theta x(t)}{b} + \frac{D^\theta y(t)}{d} \\ &= \frac{1}{b}[x(t)(a - x(t)) - \frac{bx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - hx(t)] \\ &+ \frac{1}{d}[\frac{dx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - cy(t)]. \end{aligned}$$

Hence,

$$\begin{aligned} D^\theta V(t) + \xi V(t) &= \frac{ax}{b} - \frac{x^2}{b} - \frac{hx}{b} - \frac{cy}{d} + \frac{\xi x}{b} + \frac{\xi y}{d} \\ &= -\frac{x^2}{b} + (a+\xi-h)\frac{x}{b} + (\xi - c)\frac{y}{d} \\ &- \frac{1}{b}[x^2 - (a+\xi-h)x + \frac{(a+\xi-h)^2}{4} - \frac{(a+\xi-h)^2}{4}] + (\xi - c)\frac{y}{d} \\ &= -\frac{1}{b}(x - \frac{a+\xi-h}{2})^2 + \frac{(a+\xi-h)^2}{4} + (\xi - c)\frac{y}{d} \\ &\le \frac{(a+\xi-h)^2}{4b} + (\xi - c)\frac{y}{d}. \end{aligned}$$

If we choose ξ such that $\xi < c$, then $D^{\theta}V(t) + \xi V(t) \leq \frac{(a+\xi-h)^2}{4b}$.
Therefore, $V(t) \leq (V(0) - \frac{L}{\xi})E_{\alpha}(-\xi t^{\alpha}) + \frac{1}{\xi}(1 - E_{\alpha}(-\xi t^{\alpha})$ and $V(t) \to \xi$ as $t \to$ inf and $0 \leq V(t) \leq \frac{1}{\xi}$. □

Theorem 2.6. *Let $\Omega = \{(x,y) \in R^2 \; |V(t) \leq \frac{1}{\xi} + \epsilon \quad \forall \quad \epsilon > 0\}$, then for each $x_1 = (x_0, y_0) \in \{(x,y) \in R^2 \; |max\{|x|, |y| \leq \gamma\}$ for a sufficiently large γ the system (2) with initial condition x_1 has a unique solution $s = (x,y) \in \Omega$, that is defined $\forall \; t \geq 0$.*

Proof. Assume that $H(s) = (H_1(s), H_2(s))$ be a mapping where

$$H_1 = x(t)(a - x(t)) - \frac{bx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - hx(t),$$
$$H_2 = \frac{dx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - cy(t).$$

Then for any $s = (x,y), \bar{s} = (\bar{x}, \bar{y}) \in \Omega$, we have

$$\begin{aligned}
&\|H(s) - H(\bar{s})\| \\
&= |H_1(s) - H_1(\bar{s})| + |H_2(s) - H_2(\bar{s})| \\
&= \left| x(t)(a - x(t)) - \frac{bx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - hx(t) - (\bar{x}(a - \bar{x}) - \frac{b\bar{x}\bar{y}}{(1+\alpha\bar{x})(1+\beta\bar{y})} - h\bar{x}) \right| \\
&+ \left| \frac{dx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - cy - (\frac{d\bar{x}\bar{y}}{(1+\alpha\bar{x})(1+\beta\bar{y})} - c\bar{y}) \right| \\
&= \left| a(x - \bar{x} - (x^2 - \bar{x}^2) - \frac{bx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} + \frac{b\bar{x}\bar{y}}{(1+\alpha\bar{x})(1+\beta\bar{y})} - h(x - \bar{x}) \right| \\
&+ \left| -c(y - \bar{y}) + \frac{dx(t)y(t)}{(1+\alpha x(t))(1+\beta y(t))} - \frac{d\bar{x}\bar{y}}{(1+\alpha\bar{x})(1+\beta\bar{y})} \right| \\
&= \left| a(x - \bar{x}) - (x - \bar{x})(x + \bar{x}) - h(x - \bar{x}) - [\frac{ZZ}{z_1}] \right| \\
&+ \left| -c(y - \bar{y}) + \frac{dy(x - \bar{x}) - d\bar{x}(y - \bar{y}) + d\beta y\bar{y}(x - \bar{x}) + d\alpha x\bar{x}(y - \bar{y})}{z_1} \right| \\
&\leq a\,|(x - \bar{x})| + |(x + \bar{x})|\,|(x - \bar{x})| + h\,|(x - \bar{x})| + b\,|\bar{x}|\,|y - \bar{y}| + b\,|y|\,|(x - \bar{x})| \\
&+ b\beta\,|y|\,|\bar{y}|\,|(x - \bar{x})| + b\alpha\,|\bar{x}|\,|\bar{y}|\,|(x - \bar{x})| + c\,|(y - \bar{y})| \\
&+ d\,|y|\,|(x - \bar{x})| + d\,|\bar{x}|\,|(y - \bar{y}| + d\beta\,|y|\,|\bar{y}|\,|(x - \bar{x})| + d\alpha\,|x|\,|\bar{x}|\,|(y - \bar{y})| \\
&\leq [a + |(x + \bar{x})| + h + b\,|y| + b\beta\,|y|\,|\bar{y}| + b\alpha\,|\bar{x}|\,|\bar{y}| + d\,|y| + d\beta\,|y|\,|\bar{y}|]\,|(x - \bar{x})| \\
&+ [b\,|\bar{x}| + c + d\,|\bar{x}| + d\alpha\,|\bar{x}|\,|x|]\,|(y - \bar{y})| \\
&\leq L(|(x - \bar{x})| + |(y - \bar{y})|).
\end{aligned}$$

Where $ZZ = b\bar{x}(y - \bar{y}) + by(x - \bar{x}) + b\beta y\bar{y}(x - \bar{x}) - b\alpha\bar{x}\bar{y}(x - \bar{x})$, $z_1 = [(1+\alpha x(t))(1+\beta y(t))][(1+\alpha\bar{x})(1+\beta\bar{y})]$, $L = Max\{M_1, M_2\}$, $M_1 = (a + h + |(x + \bar{x})| + b\,|y| + b\beta\,|y|\,|\bar{y}| + b\alpha\,|\bar{x}|\,|\bar{y}| + d\,|\bar{y}| + d\beta\,|y|\,|\bar{y}|)$ and $M_2 = b\,|\bar{x}| + c + d\,|\bar{x}| + d\alpha\,|\bar{x}|\,|x|$. Therefore, $H(s)$ has the Lipchitz condition. Hence, from Lemma 2, we get the result. □

Remark 2.1. We can find the equilibrium(fixed) points of model (2) by setting $D^{\theta}x(t) = 0$ and $D^{\theta}y(t) = 0$, then the equilibrium points are as follows :

(1) The point $E_0 = (0,0)$ stands for the trivial equilibrium point that always exists.
(2) If $a > h$, then the point $E_1 = (a-h, 0)$ exists.
(3) The positive (interior) equilibrium point $E_2 = (x^*, y^*)$ exists only if $a \in (x^*+h, x^*+h+\frac{b}{k\alpha})$, where x^* represents the positive root of the equation $-\beta d\alpha x^3 + (a\beta d\alpha - \beta d - \beta d * h\alpha)x^2 + (a\beta d - bd - \beta dh)x + cb + cb\alpha = 0$ and $y^* = \frac{(x^* k - ak + hk)}{(ak\beta - x^* k\beta - b - \beta hk)}$ with $k = (1 + \alpha x^*)$.

Remark 2.2. The local asymptotically stable of an equilibrium point E^* of the model (2) is established if $|arg(\lambda_i)| > \frac{\theta\pi}{2}$ for $i = 1, 2$, where λ_i represent the eigenvalues of the Jacobian matrix $J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ of the considered system (2) evaluated at E^* [16, 17].

The Jacobian matrix of the considered system (2) at point (X, Y) is given by

$$J(X,Y) = \begin{bmatrix} a - 2aX - \frac{bY}{k^2 k_1} - h & -\frac{bX}{kk_1^2} \\ \frac{dY}{k^2 k_1} & \frac{dX}{kk_1^2} - c \end{bmatrix}$$

and $f(\lambda) = \lambda^2 + P\lambda + Q$ is the characteristic polynomial of $J(X,Y)$ with $P = c - \frac{dX}{kk_1^2} - a + 2aX + \frac{bY}{k^2 k_1} + h$ and $Q = \frac{dX}{kk_1^2} - \frac{2ad(X)^2}{kk_1^2} - \frac{dhX}{kk_1^2} + 2acX - ac + \frac{bcY}{k^2 k_1} + ch$. Where $k = 1 + \alpha X$ and $k_1 = 1 + \beta Y$.

Theorem 2.7. (1) *The trivial equilibrium(fixed) point E_0 is locally stable if and only if $a < h$.*

(2) *If $a < \frac{1}{2}$ and $\frac{d(a-h)}{k} < c$, then the equilibrium point E_1 is stable point.*

Proof. (1) It is easy to check that the eigenvalues of the Jacobian matrix J_{E_0} at the point E_0 are $\lambda_1 = a - h$ and $\lambda_2 = -c$. So that the results are obtained.

(2) The Jacobian matrix J_{E_0} at the E_1 is :

$$J_{E_1} = \begin{bmatrix} a - 2a(a-h) - h & -\frac{b(a-h)}{k} \\ 0 & \frac{d(a-h)}{k} - c \end{bmatrix}.$$

Then the eigenvalues of the matrix J_{E_1} at the point E_1 are $\lambda_1 = a - 2a(a-h) - h$ and $\lambda_2 = \frac{d(a-h)}{k} - c$. Therefore, the results can easily be obtained.

□

The following proposition is proved in [1] which is needed to study the behavior of the model (2) at the positive equilibrium point E_2.

Lemma 2.8. [1] *Consider that* $p(x) = x^2 + a_1 x + a_2$ *is a polynomial of degree 2 if one of the following holds*

(1) *Routh-Hurwitz conditions, namely* $a_1 > 0$ *and* $a_2 > 0$.

(2) $a_1 < 0, \quad 4a_2 > a_1^2$ *and* $\left| tan^{-1}(\frac{\sqrt{4a_2 - a_1^2}}{a_1} \right| > \frac{\theta\pi}{2}$,

then the roots $r_i \quad i = 1, 2$ *of* p *satisfy* $|arg(r_i)| > \frac{\theta\pi}{2}$.

Proof. See[1]. □

Theorem 2.9. *The unique positive equilibrium point* E_2 *of the model (2) is locally stable if one of the following statements holds.*
a) $P > 0$, *and* $Q > 0$
b)$P < 0$, $4Q > P^2$ *and* $tan^{-1}(\frac{\sqrt{4Q - P^2}}{P}) > \frac{\theta\pi}{2}$, *where* P *and* Q *are defined as before.*

Proof. The Jacobian matrix of the system (2) at the point E_2 is :

$$J_{E_2} = \begin{bmatrix} a - 2ax^* - \frac{by^*}{k^2 k_1} - h & -\frac{bx^*}{kk_1^2} \\ \frac{dy^*}{k^2 k_1} & \frac{dx^*}{kk_1^2} - c \end{bmatrix}$$

Hence, the characteristic polynomial of J_{E_2} is $f(\lambda) = \lambda^2 + P\lambda + Q = 0$ with $P = c - \frac{dx^*}{kk_1^2} - a + 2ax^* + \frac{by^*}{k^2 k_1} + h$ and $Q = \frac{dx^*}{kk_1^2} - \frac{2ad(x^*)^2}{kk_1^2} - \frac{dhx^*}{kk_1^2} + 2acx^* - ac + \frac{bcy^*}{k^2 k_1} + ch$. Where $k = 1 + \alpha x^*$ and $k_1 = 1 + \beta y^*$.
Now, applying Proposition 2.8 by setting $a_1 = P$ and $a_2 = Q$, the results are obtained. □

3. The discretization system and its behavior

In this part, we will study and discuss the discretization of the considered system (2). We apply the discretization method to the considered model (2), then the discretized system is

$$\begin{aligned} x_{n+1} &= x_n + \frac{S^\theta}{\Gamma(\theta+1)}[x_n(a - x_n) - \frac{bx_n y_n}{(1+\alpha x_n)(1+\beta y_n)} - hx_n], \\ y_{n+1} &= y_n + \frac{S^\theta}{\Gamma(\theta+1)}[\frac{dx_n y_n}{(1+\alpha x_n)(1+\beta y_n)} - cy_n]. \end{aligned} \tag{3}$$

The definitions of the a, b, c, d, h, θ are the same as the previous interpretation. S is a positive number. The Jacobian matrix of the discretized model(3) at any point (x, y) is then

$$J = \begin{bmatrix} 1 + ma - 2max - \frac{mby}{k^2 k_1} - mh & -\frac{mbx}{kk_1^2} \\ \frac{mdy}{k^2 k_1} & 1 + \frac{mdx}{kk_1^2} - mc \end{bmatrix},$$

where $m = \frac{S^\theta}{\Gamma(\theta+1)}$, $k = 1 + \alpha x$ and $k = 1 + \beta y$.

Remark 3.1. Consider the following discrete system

$$\overrightarrow{x}_{t+1} = f(\overrightarrow{x}_t) \quad t = 1, 2, 3, ... \tag{4}$$

The point E^* is called a fixed(equilibrium) point of (4) if $E^* = f(E^*)$ and it is called stable point if $|\lambda_i| < 1$ for $i = 1, 2, ..., n$, where λ_i stand for the eigenvalues of the Jacobian matrix J evaluated at E^*. E^* is called a non-hyperbolic point if $|\lambda_i| = 1$ for some i.

Theorem 3.1. (1) *The trivial fixed point E_0 of the discretized model (3) is locally stable if $a \in (h - \frac{2}{m}, h)$ and $c < \frac{2}{m}$. The point E_0 is a non-hyperbolic point if $a = h$ or $a = h - \frac{2}{m}$ or $c = \frac{2}{m}$.*

(2) *The fixed point E_1 of the discretized system (3) is locally stable if $m < \frac{2}{(\frac{2a}{k^2}-1)(a-h)}$ with $\frac{k^2}{a} < 2$ and $c \in (\frac{d(a-h)}{k}, \frac{d(a-h)}{k} + \frac{2}{m})$. It is non-hyperbolic point if $m = \frac{2}{(\frac{2a}{k^2}-1)(a-h)}$ with $\frac{k^2}{a} < 2$ or $c = \frac{d(a-h)}{k}$ or $c = \frac{d(a-h)}{k} + \frac{2}{m}$.*

Proof. (1) The eigenvalues of J, the Jacobian matrix of the discretized model(3) at E_0, are $\lambda_1 = 1 + ma - mh$ and $\lambda_1 = 1 - mc$. Therefore, the results can be easily obtained.

(2) The Jacobian matrix J at E_1 of the discretized model (3) is given by

$$J_{E_1} = \begin{bmatrix} 1 + ma - \frac{2ma(a-h)}{k^2} - mh & -\frac{bm(a-h)}{k} \\ 0 & 1 + \frac{md(a-h)}{k} - mc \end{bmatrix}.$$

Therefore, the eigenvalues are $\lambda_1 = 1 + ma - \frac{2ma(a-h)}{k^2} - mh$ and $\lambda_2 = 1 + \frac{md(a-h)}{k} - mc$. So that if $0 < m < \frac{2}{(\frac{2a}{k^2}-1)(a-h)}$ with $\frac{k^2}{a} < 2$, then $0 < m((\frac{2a}{k^2} - 1)(a-h)) < 2$ and $-1 < -1 + m((\frac{2a}{k^2} - 1)(a-h)) < 1$ this implies $\left|1 + ma - \frac{2ma(a-h)}{k^2} - mh\right| < 1$ and then $|\lambda_1| < 1$.
Now if $\frac{d(a-h)}{k} < c < \frac{d(a-h)}{k} + \frac{2}{m}$, then $-1 < 1 + \frac{md(a-h)}{k} - mc < 1$ this implies that $|\lambda_2| < 1$. Therefore, the point is locally stable. □

We use the Schur-Chohn criteria for studying the dynamic behavior of the positive fixed point. The next lemma appeared in [26].

Lemma 3.2. *Let $P_2(x) = x^2 + p_1x + q_1$ be a polynomial of degree 2 then its roots are inside the unit disk if and only if $P_2(1) > 0$, $P_2(-1) > 0$ and $q_1 < 1$.*

Proof. see[26]. □

Theorem 3.3. *The positive fixed point E_2 of the model (4)is local stable if the following conditions hold:*

(1) $axA + cB > dxC + ac,$

(2) $mxD + E > mB_1 + mxC_1,$

(3) $mE_1 + a + \frac{dx}{kk_1^2} < xH + H_1.$

where $A = \frac{d}{kk_1^2} + 2c \quad B = h + \frac{by}{k^2k_1}, \quad C = \frac{2ax}{kk_1^2} + \frac{by}{k^3k_1^3} + \frac{h}{kk_1^2}, \quad D = \frac{2d}{kk_1^2} + \frac{mad}{kk_1^2} + 2mac, \quad E = 4 + 2am + \frac{m^2bcy}{k^2k_1} + m^2hc \quad B_1 = 2c + 2h + mac + \frac{2mby}{k^2k_1}, \quad C_1 = 4a - \frac{2madx}{kk_1^2} + \frac{mdh}{kk_1^2} + \frac{mbdy}{k^3k_1^3}, \quad E_1 = \frac{adx}{kk_1^2} + 2acx + \frac{bcy}{k^2k_1} + hc, \quad H = 2a + \frac{mbdy}{k^3k_1^3} + \frac{mdh}{kk_1^2}, \quad H_1 = \frac{by}{k^2k_1} + h + c + amc.$

Proof. The Jacobian matrix at E_2 of the discretized model (3) is given by

$$J = \begin{bmatrix} 1 + ma - 2max - \frac{my}{k^2k_1} - mh) & -\frac{mbx}{kk_1^2} \\ \frac{mdy}{k^2k_1} & 1 + m(\frac{mdx}{kk_1^2} - mc) \end{bmatrix},$$

where $m = \frac{S^\theta}{\Gamma(\theta+1)}$, $k = 1 + \alpha x$ and $k = 1 + \beta y$. The characteristic polynomial of J_{E_2} is then $F(\lambda) = \lambda^2 + p_1\lambda + q_1 = 0$ with $p_1 = -2 - \frac{mdx}{kk_1^2} + mc - am + 2amx + \frac{mby}{k^2k_1} + mh \quad q_1 = 1 + ma - 2max - \frac{mby}{k^2k_1} - mh + \frac{mdx}{kk_1^2} + \frac{m^2adx}{kk1^2} - \frac{2m^2adx^2}{kk_1^2} - \frac{m^2bdxy}{k^3k_1^3} - \frac{m^2dxh}{kk_1^2} - mc - m^2ac + 2m^2acx + \frac{m^2bcy}{k^2k_1} + m^2hc.$
Now if the condition 1 holds $axA + cB > dxC + ac$, then we have $ax(\frac{d}{kk_1^2} + 2c) + c(h + \frac{by}{k^2k_1}) > dx(\frac{2ax}{kk_1^2} + \frac{by}{k^3k_1^3} + \frac{h}{kk_1^2}) + ac$ this gives $\frac{m^2axd}{kk_1^2} + 2m^2axc + m^2ch + \frac{cm^2by}{k^2k_1} > \frac{2m^2adx^2}{kk_1^2} + \frac{m^2dxby}{k^3k_1^3} + \frac{m^2dxh}{kk_1^2}) + m^2ac$. Hence $\frac{m^2axd}{kk_1^2} + 2m^2axc) + m^2ch + \frac{cm^2by}{k^2k_1} \frac{2m^2adx^2}{kk_1^2} + \frac{m^2dxby}{k^3k_1^3} + \frac{m^2dxh}{kk_1^2} + m^2ac > 0$, by a simple substitution we have $P_2(1) > 0$.
If the condition 2 holds that means $mxD + E > mB_1 + mxC_1$, then we have $\frac{2mxd}{kk_1^2} + \frac{m^2xad}{kk_1^2} + 2m^2xac + 4 + 2am + \frac{m^2bcy}{k^2k_1} + m^2hc > 2mc + 2mh + m^2ac + \frac{2m^2by}{k^2k_1} + 4mxa - \frac{2m^2adx^2}{kk_1^2} + \frac{m^2dhx}{kk_1^2} + \frac{m^2xbdy}{k^3k_1^3}$, by a simple substitution we have $P_2(-1) > 0$.
If the condition 3 $mE_1 + a + \frac{dx}{kk_1^2} < xH + H_1$, then $\frac{amdx}{kk_1^2} + 2amcx + \frac{bmcy}{k^2k_1} + hmc + a + \frac{dx}{kk_1^2} < 2ax + \frac{mbdxy}{k^3k_1^3} + \frac{mdhx}{kk_1^2} + \frac{by}{k^2k_1} + h + c + amc$ this gives $\frac{amdx}{kk_1^2} + 2amcx + \frac{bmcy}{k^2k_1} + hmc + a + \frac{dx}{kk_1^2} - 2ax - \frac{mbdxy}{k^3k_1^3} - \frac{mdhx}{kk_1^2} - \frac{by}{k^2k_1} - h - c - amc < 0$ from this we have $q_1 < 1$. Therefore, by lemma(3.2), the positive point of the discretized model is locally stable. □

4. Optimal harvesting process

This section is concerned with using the discrete version optimal control theory. It is impossible to remove more than the population so we use an exponential cost function. Here the control variable is h_n^* and the aim is to maximize the net profit that described by the following objective function $J(h_n^*) = Max J(h_n), h_n \in U$. Here U is the set of other control variables, where

$$J(h_n) = \sum_{n}^{T} c_1(1 - e^{h_n})x_n - c_2(1 - e^{h_n})^2,$$

$$\begin{aligned} x_{n+1} &= x_n + \frac{S^\theta}{\Gamma(\theta+1)}[x_n(a - x_n) - \frac{bx_ny_n}{(1+\alpha x_n)(1+\beta y_n)} - (1 - e^{h_n})x_n], \\ y_{n+1} &= y_n + \frac{S^\theta}{\Gamma(\theta+1)}[\frac{dx_ny_n}{(1+\alpha x_n)(1+\beta y_n)} - cy_n], \end{aligned} \tag{5}$$

where c_1 and c_2 are positive values, and the cost function is given by $c_2(1 - e^{h_n})^2$. We solve the previous optimal control harvesting by using the maximum principle of Pontryagin[15, 27, 28].

Theorem 4.1. *Consider we have an optimal control solution h_n^* with the optimal state solutions x_n^* and y_n^* which maximizes the objective function $J(h_n^*)$ then the following shadow price functions λ_n, and μ_n for $n = 1, 2, ...T-1$ exist which satisfy:*

$$\begin{aligned} \lambda_n &= c_1(1 - e^{h_n}) + \lambda_{n+1}[1 + ma - 2max - \frac{my}{k^2k_1} - (1 - e^{h_n})] + \mu_{n+1}(-\frac{mbx}{kk_1^2}). \\ \mu_n &= \lambda_{n+1}\frac{mdy}{k^2k-1} + \mu_{n+1}[1 + m(\frac{mdx}{kk_1^2} - mc).] \end{aligned} \tag{6}$$

with $m = \frac{S^\theta}{\Gamma(\theta+1)}$, $\lambda_T = 0$, and $\mu_T = 0$. So the form of the characterization optimal solution is as follows: $h_n^* = ln(\frac{(c_1 - \lambda_{n+1})x_n}{2c_2})$.

Proof. Define the Hamiltonian functional for $n = 1, 2, ...T - 1$ as follows:

$$\begin{aligned} \mathcal{H}_n = {} & c_1(1 - e^{h_n})x_n - c_2(1 - e^{h_n})^2 + \lambda_{n+1}[x_n + m(x_n(a - x_n) - \frac{bx_ny_n}{(1+\alpha x_n)(1+\beta y_n)} \\ & - (1 - e^{h_n})x_n)] + \mu_{n+1}[y_n + m(\frac{dx_ny_n}{(1+\alpha x_n)(1+\beta y_n)} - cy_n)]. \end{aligned} \tag{7}$$

Then for $n = 1, 2, ...T - 1$, the necessary conditions are

$$\begin{aligned} &\lambda_n \\ &= \frac{\partial \mathcal{H}}{\partial x_n} = c_1(1 - e^{h_n}) + \lambda_{n+1}[1 + ma - 2max - \frac{my}{k^2k_1} - (1 - e^{h_n})] + \mu_{n+1}(-\frac{mbx}{kk_1^2}) \\ &\mu_n = \frac{\partial \mathcal{H}}{\partial y_n} = \lambda_{n+1}\frac{mdy}{k^2k-1} + \mu_{n+1}[1 + m(\frac{mdx}{kk_1^2} - mc)] \end{aligned} \tag{8}$$

From the optimality condition $\frac{\partial \mathcal{H}}{\partial h_n^*} = 0$, one can obtain the form of the characterization of the optimal solution, which is given by $h_n^* = ln(\frac{(c_1 - \lambda_{n+1})x_n}{2c_2})$. □

Therefore, the previous optimal control problem is numerically solved to get the optimal control solution.

5. Numerical outcomes

In this section, the confirmation of our theoretical results is done. In Figure 1 we have shown the local stability of the equilibrium point E_0, E_1 and E_2, respectively. The choice of parameters for the point E_0 is as follows: $a = 0.4$, $b = 0.7, \alpha = 0.5$, $\beta = 0.5$, $h = 0.5$, $c = 0.2$, $d = 0.5$ and $\theta = 0.95$. The initial values are $x_0 = 6$ and $y_0 = 1.4$. For the point E_1 the parameters are $a = 0.49$, $b = 0.6$, $\alpha = 0.5$, $\beta = 0.5$, $h = 0.05$, $c = 0.15$, $d = 0.4$ and the initial vales are $x_0 = 3.5$ and $y_0 = 0.2$. For the unique interior point E_1 the parameters are $a = 4$, $b = 0.6$, $\alpha = 0.5$, $\beta = 0.5$, $h = 0.2$, $c = 0.3$, $d = 0.3$ and the initial vales are $x_0 = 3.7$ and $y_0 = 0.8$. Figure 2 indicates the local behavior of the points with the same previous parameters except for the value of $\theta = 0.83$.

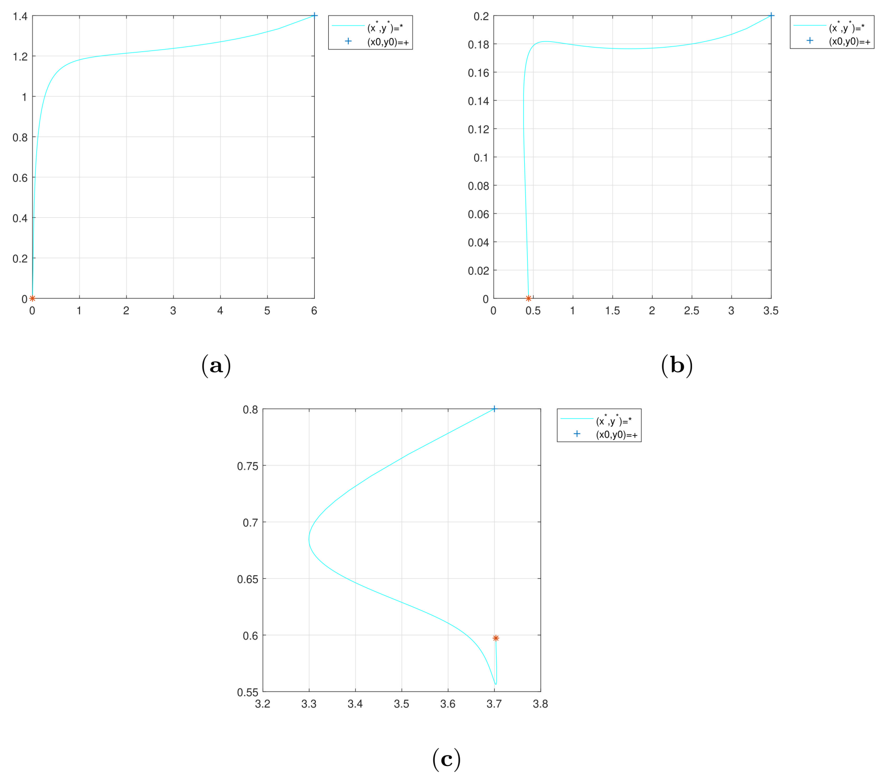
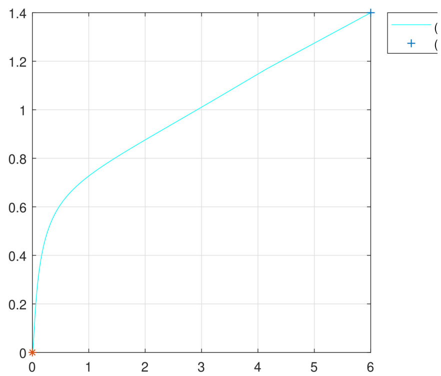
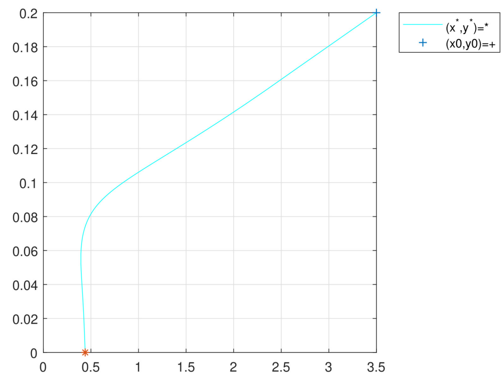


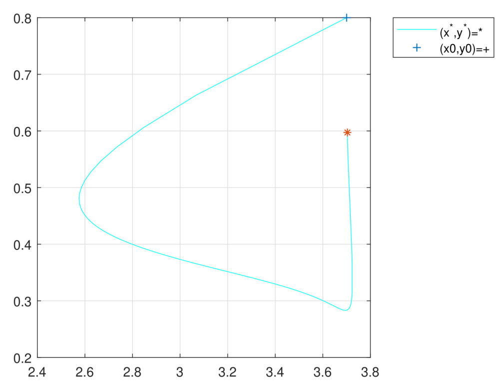
FIGURE 1. The local behavior of the equilibrium points of system (2). (a) For the point E_0. (b) For the point E_1. (c) For the positive point E_2. Here $\theta = 0.95$.



(a)



(b)



(c)

FIGURE 2. The same values of parameters are used in Figure 1, but $\theta = 0.83$. (a) For the point E_0. (b) For the point E_1. (c) For the unique positive point E_2.

In Figure 3, we have to show the behavior of the equilibrium points of the discretized model (3). We choose the parameters values of E_0 as follows: $a = 0.29,\ b = 0.7, \alpha = 0.5,\ \beta = 0.5,\ h = 0.3,\ c = 0.3,\ d = 0.2,\ S = 0.3$ and $x(1) = 2.3,\ y(1) = 0.2$ are the initial values with $\theta = 0.95$. For the point E_1, we set these parameters values $a = 0.9,\ b = 0.7,\ \alpha = 0.5,\ \beta = 0.5,\ h = 0.3,\ c = 0.5,\ d = 0.5,\ S = 0.5$ and the initial values are $x(1) = 0.9,\ y(1) = 0.5$ with $\theta = 0.95$. Next values of parameters are taken for the positive point $a = 2, b = 0.7,\ \alpha = 0.2,\ \beta = 0.2; h = .02,\ c = 0.03,\ d = 0.4,\ S = 0.5,\ \theta = 0.83$ and $x(1) = 1.7,\ y(1) = 5.59$ are the initial values . Figure 4 shows the local behavior of all equilibrium points of the discretized model (3) with the same previous values of parameters with another value of θ, namely $\theta = 0.83$.

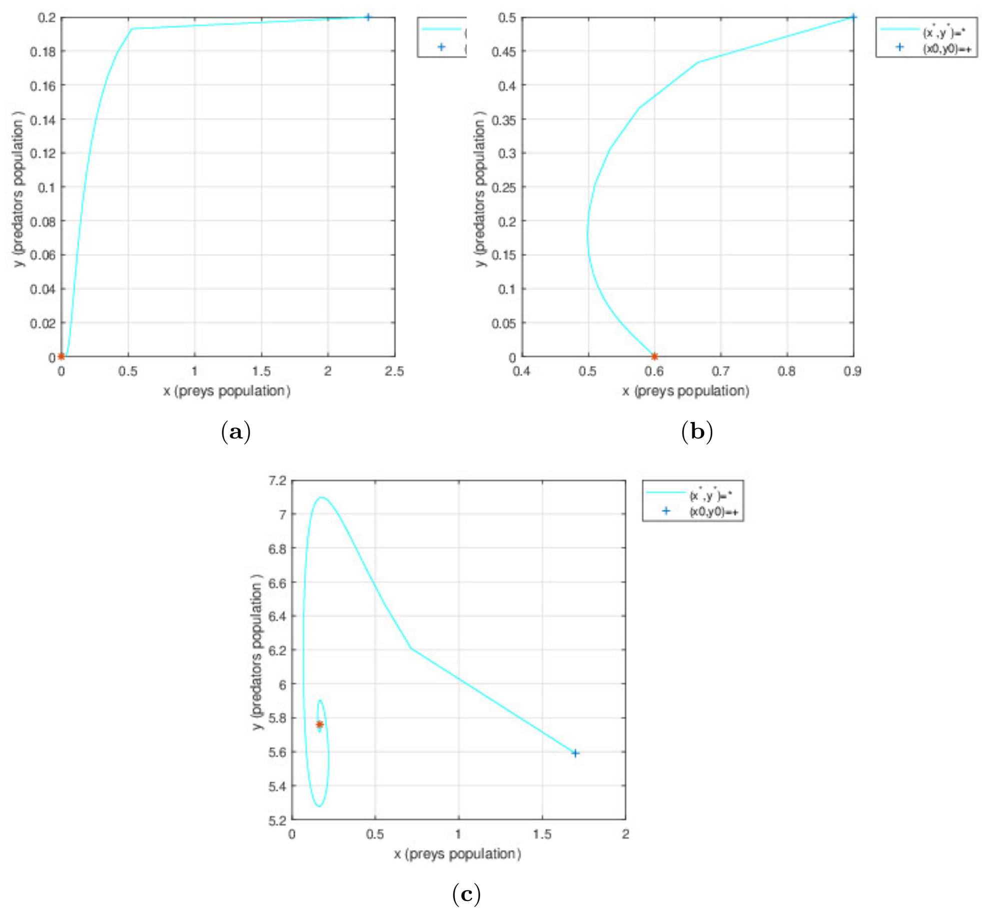
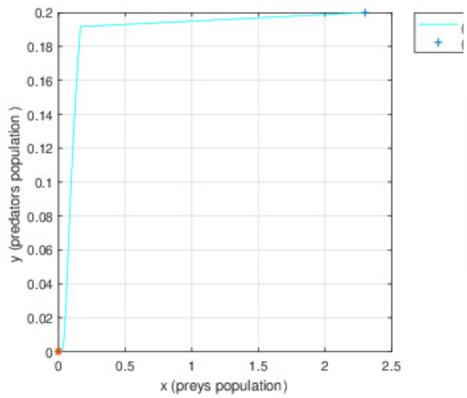
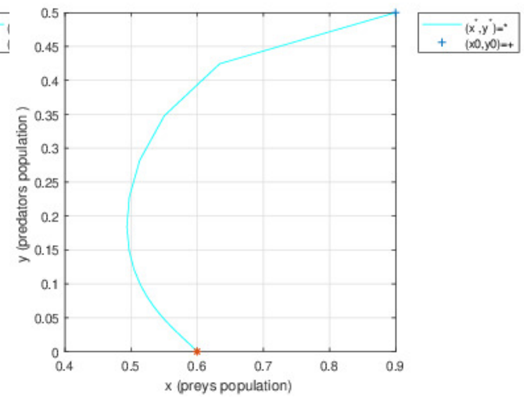


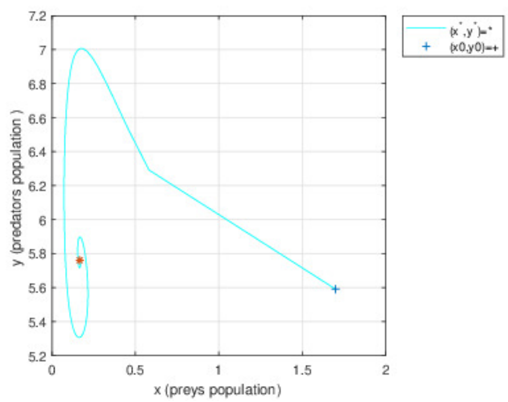
FIGURE 3. (a) for the point E_0. (b) For the point E_1. (c) For the unique positive point E_2. Here $\theta = 0.95$.



(**a**)



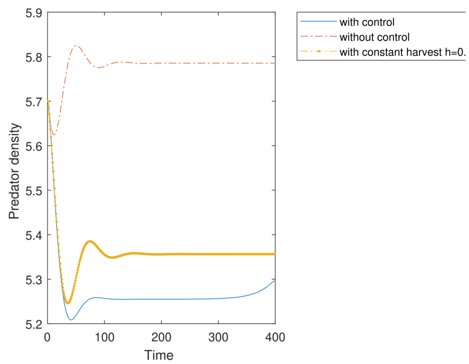
(**b**)



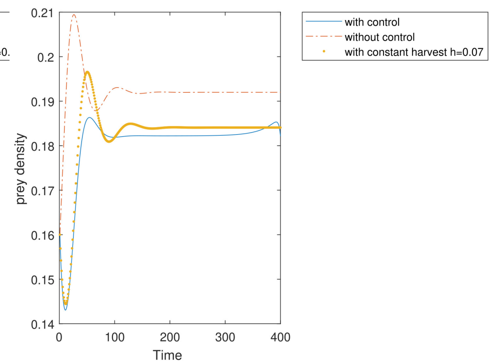
(**c**)

FIGURE 4. Here $\theta = 0.83$. (a) For the point E_0. (b) For the point E_1. (c) For the unique positive point E_2.

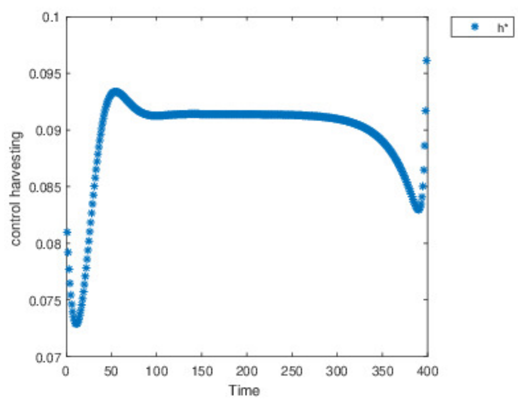
For the optimal control problem, the optimality system is numerically solved by an iterative method which is described in [14]. We begin with an initial value for the control variable, solve the state equation forward, and the adjoint system backward. We update the control variable until convergence of successive iterates is achieved. The next values of parameters are $a = 2$, $b = 0.7$, $\alpha = 0.2$, $\beta = 0.2$, $c = 0.03$, $d = 0.4$, $q = 1$, $\theta = 0.95$, $c1 = 0.01$, $c2 = 0.01$. The initial values are $x_0 = 0.16$ and $y_0 = 5.7$. We obtained the optimal value of the objective function $J(h^*) = 0.0324$, however, the values of J with different constant control strategies are $J(0.07) = 0.0314$, $J(0.08) = 0.0322$, $J(0.085) = 0.0323$, $J(0.0875) = 0.0322$, $J(0.09) = 0.0321$. Figure 5 shows the prey and predator populations with(without) control harvesting, the fixed harvesting strategy, and the optimal control variable as a function of time.



(**a**)



(**b**)



(**c**)

FIGURE 5. (a) For the predator population with(without) control harvesting(removing) and the fixed harvesting strategy (b) For the prey population. (c) For the optimal control(removing) as a function of time.

6. Conclusions

Fractional calculus has been successfully used and applied to describe or model many areas of science that cannot be set by other forms of equations. So, this article is concerned with a study of the two-dimensional biological system that is described by fractional-order derivative with CrowleyMartin functional response. In figures 1-2, we can see that the system (2.2) has three equilibrium points, namely, E_0, E_1 and$E2$, they also confirm the local stability of them for $\theta = 0.95$ and $\theta = 0.83$, respectively. Figures 3-4 confirm the behavior of the equilibrium points of the discretized model (3.1). them for $\theta = 0.95$ and $\theta = 0.83$, respectively. In Figure 5(a), one can see the impact of constant harvesting and non-constant harvesting on the density of the predator. In Figure 5(b), it can easily see the impact of constant harvesting and non-constant harvesting on the density of the prey. While Figure 5(c) shows the optimal control(removing) as a function of time. We also noted that the population™s extinction will occur with a heavily constant harvesting rate. It has been seen that the optimal profit does not occur at the constant harvesting rate. We followed the discrete Pontryagin maximum principle to get the optimality problem.

Conflicts of interest : The authors declare no conflict of interest.

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