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THE SPLIT (D,C) NUMBER OF A GRAPH AND ITS IMPLICATIONS

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Abstract. In this paper, we introduce the integrated color variable, called the domination chromatic number, or the (D, C) - number, for connected graphs. We explore the concept of dominating chromatic sets in the graph G, known as (D, C) - sets for G, and split dominating chromatic sets, called split (D, C) - sets, for various connected graphs. A set $S \subseteq V$ of vertices in the graph G is called a (D, C) - set for the graph G if it is both a dominating set and a chromatic set of the graph G. The smallest size of such a set is called the (D, C) - number for G, represented as : $\gamma_{\chi}(G) = \min\{|S| : S \text{ is a } (D, C)\text{- set of } G\}$. A set $S \subseteq V$ of vertices in the graph G is called a split (D, C) - set for G if it is both a (D, C) - set and the induced subgraph $\langle V \setminus S\rangle$ is disconnected. The smallest size of a split (D, C) - set is called the split (D, C) - number, represented as: $\gamma_{\chi_S}(G) = \min\{|S| : S \text{ is a split } (D, C)\text{- set of } G\}$. This paper also discusses the characterization of this parameter and optimized dominating sets. We identify the split (D, C) - number for some standard graphs and examine the realization problem for K - coloring a graph G. For any two positive integers λ and p where $2 \leq \lambda \leq p$, there exists a connected graph G with order p such that $\gamma_{\chi_S}(G) = \lambda$.

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1. Introduction

A graph $G = (V, E)$ is defined as an ordered pair of two sets: V, the vertex set, and E, the edge set of the graph G. The number of elements in the vertex set V is called the order of the graph G, while the number of elements in the edge set E is referred to as the size of the graph G. The subject Graph Theory

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is the study of the properties and structures of graphs, which are mathematical representations of sets of objects (called vertices or nodes) connected by links (or edges). It provides a framework for analyzing relationships and interactions in various fields, such as computer science, biology, social networks, and logistics.

If a graph G has at most one edge connecting each pair of vertices and no vertex is connected to itself, then G is called a simple graph. A graph G is classified as acyclic if it contains no cycles. A tree is defined as a connected acyclic graph. In a tree, a support vertex is a vertex adjacent to a leaf, where a leaf is a vertex with degree 1. A vertex v is referred to as a cut vertex of G if the removal of v, denoted by $G \setminus \{v\}$, disconnects the graph. The set $\mathrm{Cut}(G)$ denotes the collection of all cut vertices in G. For further details on terminology and graph-theoretical notation, refer to [2, 3, 8].

Throughout this paper, we consider only undirected, connected, finite, and simple graphs (i.e., without loops or multiple edges). The open neighborhood of a vertex v, denoted $N(v)$, is the set of all vertices adjacent to v. The closed neighborhood of a vertex v is denoted $N[v]$, is defined as $N[v] = N(v) \cup \{v\}$. The concept of a dominating set in graphs, introduced by O. Ore [16], has a notable historical connection to the game-chess. A foundational exploration of domination concepts in graph theory was led by T. W. Haynes et al. [6, 9].

A dominating set in a graph G is a subset $D \subseteq V$ such that every vertex in V is either in D or is adjacent to at least one vertex in D. In other words, every vertex outside of D (i.e., $V \setminus D$) is dominated by at least one vertex in D. In terms of neighborhood, If a set $D \subseteq V$ of vertices is said to be a dominating set, then the closed neighborhood $N[D] = V$. The domination number $\gamma(G)$ is the minimum size of dominating sets of G.

In the graph theory, the concept of domination in graphs leads to several important parameters (dominated) in graph theory. These parameters capture different ways of measuring the control or influence of certain vertices over others in the graph. Some of the key domination parameters in graphs are domination number, the total domination number, independent domination number and connected domination number of a graph, refer to [10, 11, 15, 18, 20].

The domination parameters have a wide range of applications. This article explores the split and non-split characteristics of domination sets within a graph. Specifically, a dominating set D of a graph G is called a split dominating set for G if the induced subgraph $\langle V \setminus D \rangle$ is disconnected, and a non-split dominating set for G if it is connected. The split domination number $\gamma_s(G)$ is defined as the minimum cardinality among all split dominating sets. For further details, see [13, 14, 17, 19].

As noted in [5], a vertex coloring of a graph G is a function $f : V \to \mathbb{N}$ such that $f(u) = f(v) \Rightarrow \{u, v\} \notin E(G)$ for all $u, v \in V$. A graph G is said to be k - colorable if it has a proper k - vertex coloring. The smallest positive integer k for which G is k- colorable is called the chromatic number $\chi(G)$ of the graph G. A graph G is k-chromatic if it is connected and $\chi(G) = k$. The notion of chromatic sets in graphs is then defined by selecting vertices from each color class of G, as explored in [4]. For further reading on graph coloring, see [12].

Let G be a k-chromatic graph. A subset $C \subseteq V$ of the vertices of G is called a chromatic set of G if C contains all vertices of G with distinct colors. The minimum cardinality of all chromatic sets of G is known as the chromatic number $\chi(G)$ of G. Thus,

$$\chi(G) = \min\{|C| : C \text{ is a chromatic set of } G\}.$$

In this paper, integrating the domination and chromatic parameters of a graph into a single parameter (namely, dominating chromatic number). It offers a powerful tool for analyzing and optimizing graph networks. These parameters provide structural insights and optimization possibilities that are valuable in both theoretical research and real-world applications. The motivation behind integrating domination and chromatic sets into what we call (D, C) - sets stems from both theoretical and practical considerations. The combination of these two fundamental parameters - domination and coloring - addresses key challenges in graph optimization, resource management, and network stability. Below are detailed insights into the motivations behind this integration, refer to [1, 7].

2. The (D, C) - number & Split (D, C) - number of a graph

Definition 2.1. *A set $S \subseteq V$ of vertices is said to be a (D, C) - set for G if it is both dominating and chromatic set of G. The minimum cardinalities among the (D, C) sets is called (D, C) - number, denoted by $\gamma_\chi(G)$*

$$\text{i.e.,} \quad \gamma_\chi(G) = \min\{|S| : S, \text{ a } (D, C) - \text{set of } G\}.$$

Obviously, the parameter $\gamma(G)$ and $\chi(G)$ may be distinct or not. In general

$$1 \leq \gamma_\chi(G) \leq p$$

where p is the order of G. This means that for any graph G, the minimum (D, C) - set may vary greatly depending on the structure of the graph.

Definition 2.2. *A set $S \subseteq V$ of vertices in a connected graph G is said to be a split (D, C) set of G if*

(i) S is a (D, C) set

(ii) induced subgraph $\langle V \setminus S \rangle$ is disconnected.

Definition 2.3. *The split (D, C) set of minimum order is called minimum split (D, C) number and its cardinality is called split (D, C) - number of G, denoted by $\gamma_{\chi S}(G)$.*

$$\text{i.e.,} \quad \gamma_{\chi S}(G) = \min\{|S| : S \text{ is a split } (D, C) \textit{ set}\}$$

Definition 2.4. *A split (D, C) set $S \subseteq V$ of minimum cardinality is called γ_{χ_S} - set of the graph G. That is, $\gamma_{\chi_S}(G)$ is size of the γ_{χ_S} - set of G.*

Establish this concept by an example,

Example 1. *Consider a 2-coloring graph G i.e. $\chi(G) = 2$ which is shown in the Figure 2.1.*

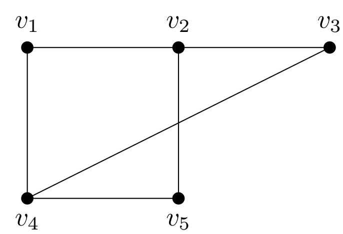

FIGURE 2.1. A graph G with $\gamma_{\chi_S}(G) = 3$

The 2-element chromatic sets of the graph G are $S_1 = \{v_1, v_2\}$ $S_2 = \{v_1, v_4\}$ $S_3 = \{v_2, v_3\}$ $S_4 = \{v_2, v_5\}$ $S_5 = \{v_3, v_4\}$ $S_6 = \{v_4, v_5\}$. These sets are also dominating. So that (D, C) - number $\gamma_\chi(G) = 2$.

But the induced subgraph $\langle V \setminus S_i \rangle$ is not disconnected for $i = 1$ to 6, we treat the split (D, C) - number $\gamma_{\chi_S}(G)$ as zero. i.e., $\gamma_{\chi_S}(G) = 0$.

If G is upgraded to 3-colorable pattern, then the possible three element chromatic sets are listed:

$$T_1 = \{v_1, v_2, v_4\} \ \& \ T_2 = \{v_2, v_4, v_5\}$$

Clearly, T_1 & T_2 are also dominating sets for the given graph G. The induced subgraph $\langle V \setminus T_i \rangle$ is disconnected, which is shown in the Figure 2.2. Therefore, the split (D, C) - number $\gamma_{\chi_S}(G) = 3$.

FIGURE 2.2. Disconnected Induced subgraph $\langle V \setminus T_i \rangle$ for G

Example 2. *Consider a tree T_4 on four vertices, which is given in the Figure 2.3.*

The two element chromatic sets of the tree T_4 are $\{v_1, v_2\}, \{v_2, v_3\}$ and $\{v_2, v_4\}$ respectively. These sets are also dominating in nature. If we choose $S = \{v_1, v_2\}$ or $\{v_2, v_3\}$ or $\{v_2, v_4\}$, then $\langle V \setminus S \rangle$ is disconnected. Therefore, S is γ_{χ_S} - set of

G and $\gamma_{\chi_S}(T_4) = |S| = 2$. If we add one or two vertices to any of the tree leafs of the given tree T_4, then we obtain a new tree on five or six vertices denoted by T_5 or T_6. Therefore,

$$\gamma_{\chi_S}(T_5) = \gamma_{\chi_S}(T_6) = 2.$$

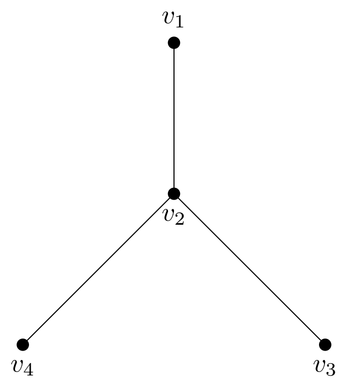

FIGURE 2.3. A Tree with $\gamma_{\chi_S}(T_4) = 2$

Example 3. *Consider a 2-coloring tree G, which is given in the Figure 2.4.*

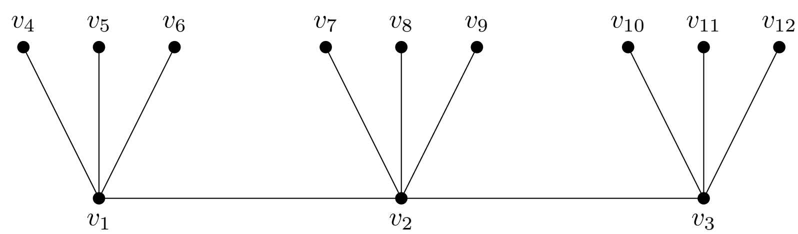

FIGURE 2.4. A 2-coloring Tree with $\gamma_{\chi_S}(G) = 3$

The set $S = \{v_1, v_2.v_3\}$ is the only dominating set of G but it is not chromatic set because G is 2-colorable. Also $\langle V \setminus S \rangle$ is disconnected. Therefore, $\gamma(G) = 3$ and $\gamma_{\chi_S}(G) = 0$. If we allow tree G to three color pattern, then we can find the split (D, C)-number for the tree G, $\gamma_{\chi_S}(G) = 3$.

In the next section, we present some basic observations about the discussed parameter. Characterizing this parameter in connected graphs is highly valuable.

3. Basic Observations

Proposition 3.1. *For a given k-coloring G, not all dominating sets are (D, C) sets. But converse always true.*

When we consider a complete graph $G = K_p$, each singleton set $\{v_i\}$ acts as a dominating set for the graph G for $i = 1, 2, \ldots, p$. Therefore, the size of the dominating basis for the graph G is $\gamma(G) = 1$, the domination number for G. However, these singleton sets do not qualify as valid (D, C) sets in G.

Since all vertices are adjacent, the set $\{v_1, v_2, \ldots, v_p\}$ is the only valid (D, C) set for G. The relationship between (D, C) sets and split (D, C) sets also deserves attention. Therefore,

Proposition 3.2. *For a given k-coloring of G, not all (D, C) sets are split (D, C) sets. But converse always true.*

Theorem 3.3. *For any connected graph G,*

(i) $\gamma_\chi(G) \geq \max\{\gamma(G), \chi(G)\}$
(ii) $\gamma_{\chi_S}(G) \geq \max\{\gamma_\chi(G), \chi(G)\}$

Proof. (i) Due to the Proposition 3.1, $\gamma_\chi(G) \geq \gamma(G)$.

Since the (D, C) set of G is always a chromatic set, $\gamma_\chi(G) \geq \chi(G)$. It follows that the (D, C) number bounded below by $\max\{\gamma(G), \chi(G)\}$. Therefore the result follows.

(ii) Due to proposition 3.2, $\gamma_{\chi_S}(G) \geq \gamma(G)$ and the split (D, C) set is always a chromatic set and $\langle V \setminus S\rangle$ is disconnected $\Rightarrow \gamma_{\chi_S}(G) \geq \chi(G)$. It follows that the split (D, C) number bounded below by $\max\{\gamma_\chi(G), \chi(G)\}$. Therefore the result follows. □

Next result gives the upper bound of any connected graph G with $cut(G) = \phi$ where $cut(G)$ is the set of all cut vertices of G.

4. Characterization Results

Theorem 4.1. *Let G be a connected graph with n vertices. Then $\gamma_{\chi_S}(G) = 2$ if and only if there exists a vertex $u \in \text{cut}(G)$ such that $d(u)$ is atmost $n - 1$.*

Proof. Assume $\gamma_{\chi_S}(G) = 2$, there exists a γ_{χ_S}- set $S = \{u, v\}$ such that $\langle V \setminus S\rangle$ is disconnected. Therefore, we cannot find another vertex $v \in \text{cut}(G)$. So $d(u)$ must be atmost $n - 1$. Conversly assume that a vertex $u \in \text{cut}(G)$ and $d(u) = n - 1$. Then we can find $u \in V$ such that {u,v} is a split (D, C) set due to the disconnectedness of $\langle V \setminus \{u, v\}\rangle$. Therefore, $\gamma_{\chi_S}(G) = 2$. □

Corollary 4.2. *For a connected graph G of order $n > 4$ with $Cut(G) \neq \phi$, $\gamma_{\chi_S}(G) \geq 2$.*

Theorem 4.3. *For any connected graph G with $\operatorname{cut}(G) \neq \phi$,*

$$\gamma_{\chi S}(G) \leq \gamma(G) + \chi(G) - \lambda$$

where λ is the cardinality of the optimised dominating set.

Proof. Consider a k-coloring graph G, a minimum dominating set $S \subseteq V$ turns into an optimized dominating set only when S contains maximum number of vertices of chromatic sets. For the time being S_λ denotes optimized version of minimum dominating sets. Since $\operatorname{cut}(G) \neq \phi$, there exist at least one vertex u such that $G \setminus \{u\}$ is disconnected. It follows that, split (D, C)-number coincide with (D, C)-number. Therefore, it is enough to prove that

$$\gamma_\chi(G) \leq \gamma(G) + \chi(G) - \lambda$$

We claim that $A = S_\lambda \cup S$ is a (D, C) set where S is the chromatic set such that $S \not\subseteq S_\lambda$. Since $S_\lambda \subseteq A$, A is a dominating set. Therefore, it is a chromatic set also. That is, $S_\lambda \cup S$ is a (D, C) set. Thus

$$\begin{array}{rrcl} & A & = & S_\lambda \cup S \\ \Rightarrow & |A| & = & |S_\lambda \cup S| \\ & & = & |S_\lambda| + |S| - |S_\lambda \cap S| \\ & & \leq & \gamma(G) + (\chi(G) - \lambda) - 0 \\ & \gamma_\chi(G) & \leq & \gamma(G) + \chi(G) - \lambda \end{array}$$

Hence the result follows. □

Theorem 4.4. *For any k coloring of G, split (D, C)-Number*

$$\gamma_{\chi S}(G) = \begin{cases} \gamma_\chi(G) & \text{if } cut(G) \neq \phi \\ 0 & \text{if } cut(G) = \phi \end{cases}$$

Proof. Let $C = \{v_1, v_2, \ldots v_m\}$ be a split (D, C) set of G. Then $\gamma_{\chi S}(G) \leq m$. Since G is k-colorable, then $m \geq k$. Since C is a split (D, C) set of G, closed neighborhood $N[C] = V$, C is chromatic & $\langle V \setminus C \rangle$ is disconnected.

Case (i) $\operatorname{Cut}(G) \neq \phi$

We have $\gamma_{\chi S}(G) \leq |C|$. It follows that, $\gamma_{\chi S}(G) \leq \gamma_\chi(G)$. But $\gamma_\chi(G) \leq \gamma_{\chi S}(G)$ is always true, due to Theorem 3.3. Hence $\gamma_{\chi S}(G) = \gamma_\chi(G)$.

Case (ii) $\operatorname{Cut}(G) = \phi$

Since $\operatorname{Cut}(G) = \phi$, the set $\langle V \setminus C \rangle$ is not disconnected. Therefore, $\gamma_{\chi S}(G) = 0$. Hence result follows. □

Theorem 4.5. *A tree T has support adjacent to more than one pendent vertex or T has a non-support if and only if every (D, C) - set is also a split (D, C) - set, that is,*

$$\gamma_\chi(G) = \gamma_{\chi S}(G).$$

Proof. Let S be a (D,C) - set of T with $\gamma_\chi(G) = |S|$.

Case 1: T has a support vertex u such that u is adjacent to more than one pendant vertex. Therefore, the support vertex u must belongs to the (D,C) - set and $\langle V \setminus S\rangle$ is disconnected. So, $\gamma_\chi(G) = \gamma_{\chi S}(G)$.

Case 2: T has a non-support vertex w, then (D,C) - set S contains either w or at least one support or one non-support adjacent to w. Therefore $\langle V \setminus S\rangle$ is disconnected. hence, $\gamma_\chi(G) = \gamma_{\chi S}(G)$. □

5. Split (D,C)- number of Standard Graphs

Theorem 5.1. *If G is a path graph of order n, then*

$$\gamma_{\chi S}(G) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & n \geq 4 \\ 0, & \text{otherwise} \end{cases}$$

Proof. Let $G = P_n$ be a path graph with vertices $\{v_1, v_2, \ldots, v_n\}$. Consider the following cases on different orders n.

Case 1: (when $n \leq 3$)

If $n = 1$, we cannot find split (D,C)- number from G. So, $\gamma_{\chi S}(G) = 0$. If $n = 2$, all the vertices of G are pendant, it means that set of all pendant vertices is itself a chromatic set. But its complement does not induce a disconnected graph. i.e., $\gamma_{\chi S}(G) = 0$. Finally $n = 3$, all the vertices are not pendant. So the chromatic set contain at least one pendant vertex and the remaining are internal vertices. So the (D,C) - number become 2. But its complement does not induce a disconnected graph. Hence,

$$\gamma_{\chi S}(G) = 0.$$

Case 2:(when $n \geq 4$)

In this case maximum degree of P_n, $\Delta(P_n) = 2$. It shows that any given vertex of P_n may only dominate at most two vertices. i.e., $\lceil \frac{n}{3} \rceil$ becomes the lower bound of the dominating sets. If we starting with second vertex at any one of the pendant vertex of P_n and choose the third vertex there after. This shows that the upper bound reaches to $\lceil \frac{n}{3} \rceil$ where $n \not\cong 0 (\mod 3)$. Therefore, we can easily upgrade the chromatic set to dominating sets. Hence the split (D,C) - number of P_n coincide with $\lceil \frac{n}{3} \rceil$

$$\text{i.e., } \gamma_{\chi S}(G) = \left\lceil \frac{n}{3} \right\rceil \text{ for } n \geq 4.$$

□

In the case of cycle graph $G = C_n$ for $n = 3, 4$ we cannot find split (D,C) number of G, so we treat it as $\gamma_{\chi S}(G) = 0$. In this case $n = 5, 6$ the split (D,C)

number coincide with its chromatic number.

$$i.e., \gamma_{\chi S}(G) = \chi(G) = \begin{cases} 3 \text{ if } n = 5 \\ 2 \text{ if } n = 6 \end{cases}$$

When $n \geq 7$, we can upgrade the chromatic set of G to its dominating set. Therefore split (D, C) - number coincide with chromatic number of G.

$$i.e., \gamma_{\chi S}(G) = \gamma(G) = \left\lceil \frac{n}{3} \right\rceil, for\ n \geq 7.$$

Therefore,

Theorem 5.2. *Let $G = C_n$ be a cycle graph of order n, then*

$$\gamma_{\chi S}(G) = \begin{cases} 0 & if \quad n = 3, 4 \\ 3 & if \quad n = 5 \\ 2 & if \quad n = 6 \\ \left\lceil \frac{n}{3} \right\rceil & if \quad n \geq 7 \end{cases}$$

Theorem 5.3. *For a complete graph $G = K_n$, $\gamma_{\chi S}(K_n)$=0.*

Proof. Since G is complete, V is the only chromatic set in G. So V is the (D, C) - set and $\gamma_\chi(G) = n$. So split (D, C) - set does not exist. That is, $\gamma_{\chi S}(G) = 0$. □

Theorem 5.4. *For a complete bipartite graph $G = K_{m,n}$, $\gamma_{\chi S}(G) = 0$.*

Proof. Since G is complete bipartite, we can have two vertex partitions V_1 and V_2. Let $S = \{u, v\}$ such that $u \in V_1$ and $v \in V_2$, then S is a (D, C) - set and $\langle V \setminus S \rangle$ is connected. Therefore, $\gamma_{\chi S}(G) = 0$. □

The split (D, C) - number of some special graphs are given in the Table 1.

Table 1. Special Graphs with split (D,C)-number

Graphs G	$\chi(G)$	$\gamma(G)$	$\gamma_\chi(G)$	$\gamma_{\chi S}(G)$
Comb graph $C_{n,n}(n \geq 3)$	2	n	n	n
Star graph $S_n(n \geq 4)$	2	1	2	2
Friendship graph F_n	3	1	3	3
Helm graph $H_n(n \geq 3)$	$n+1$	n	$n+1$	$n+1$

Next, show how the parameter is realized using a cycle graph with q vertices.

6. Realization result

Theorem 6.1. *For any two positive integers λ and p such that the condition $2 \leq \lambda \leq p$, there exists a connected graph G such that $\gamma_{\chi S}(G) = \lambda$ and order of the graph, $|V(G)| = p$.*

Proof. Consider a cycle graph C_q with vertex set $V(C_q) = \{x_1, x_2, \ldots x_q\}$. Join one copy of complete bipartite graph $K_{1,p-q}$ to any one of the x_i in $V(C_q)$ where $i = 1, 2, 3, \ldots, q$, we get a new graph G (See the Figure 6.1) with order

$$\begin{aligned} |V(G)| &= |V(C_q)| + |V(K_{1,p-q})| \\ &= q + p - q = p. \end{aligned}$$

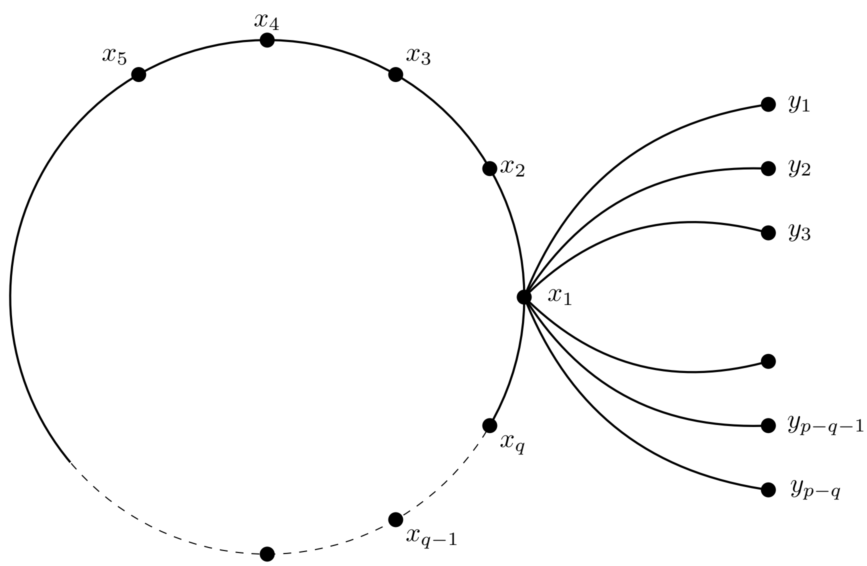

FIGURE 6.1. A graph G with $\gamma_{\chi_S}(G) = \lambda$ and $|V(G)| = p$.

If q is even, the chromatic number of C_q is always 2 and for odd positive integer q, it will become 3. In both cases, domination number and chromatic number is at most q. i.e., there exist a (D, C) set S such that $|S| = \lambda \leq q$ and $\lambda \geq 2$. It follows that $\langle V \setminus S \rangle$ is also disconnected. So, the split (D, C) - number $\gamma_{\chi_S}(G) = \lambda$. Hence the result follows. □

7. Conclusion

The split domination chromatic number is a new term in Graph Theory. It has many uses in various areas like network design, facility location, social network analysis, and wireless mesh networks. This concept is also connected to other well-known graph parameters, such as the domination number and the chromatic number.

The domination-chromatic number, formed through (D, C) - sets in graphs, combines both dominating and chromatic properties. By merging these two key ideas, it improves the understanding of graph structures and gives insights into their behavior. Some benefits include better structural efficiency and more optimized use of graph resources. Readers can connect the concept to different practical areas, contribute more to the field of graph theory, and advance science and technology.

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Data availability : Not applicable

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