J. Appl. Math. & Informatics Vol. **42**(2024), No. 6, pp. 1389 - 1399
https://doi.org/10.14317/jami.2024.1389

RESULTS OF CONFORMABLE FRACTIONAL WEIGHTED NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper investigates the existence, uniqueness, and stability properties of solutions for a nonlinear neutral fractional differential system (NFDS) with infinite delay (ID). The system incorporates the concept of conformable fractional derivatives ($\mathcal{CFD}$). Our analysis employs the Banach fixed point theorem to establish the existence and uniqueness of solutions. To illustrate the theoretical framework, we conclude by presenting an illustrative example.

AMS Mathematics Subject Classification : 26A33, 34A08.
Key words and phrases : Fractional-order, $\mathcal{CFD}$, existence and uniqueness, fixed-point theorem.

1. Introduction

Fractional calculus is an extension of classical derivatives that studies the non-integer order of differentiation and integration. This field provides a comprehensive framework for modeling phenomena characterized by memory and hereditary properties. In this introduction, we explore the historical evolution, fundamental principles, and expanding applications of fractional calculus across various scientific and engineering disciplines [18, 19, 12].

The roots of fractional calculus trace back to the 17th century when mathematicians such as Leibniz and L'Hopital first pondered the concept of fractional derivatives. Over the centuries, prominent figures like Euler, Riemann, and Grunwald significantly contributed to the theoretical foundations of this discipline. Despite its early beginnings, fractional calculus remained relatively obscure for many years [29, 23, 4]. However, recent decades have witnessed a resurgence of interest, driven by the efficacy of fractional derivatives and integrals in modeling complex dynamics observed in real-world systems [13, 10, 3].

Received July 5, 2024. Revised October 1, 2024. Accepted October 3, 2024. *Corresponding author.
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Fractional calculus introduces fractional-order derivatives and integrals, generalizing traditional integer-order operators. These operators provide a detailed representation of processes with "memory", where past states influence future behavior in a non-local manner. This capability makes fractional calculus a powerful tool for modeling a variety of phenomena, including: Anomalous diffusion [27] (processes where the rate of diffusion is not governed by the classical Fickian law, often observed in complex materials and biological systems); viscoelasticity [28] (materials that exhibit both elastic and viscous behavior, with fractional derivatives capturing the memory effects crucial for accurate modeling); fractal systems (structures exhibiting self-similarity at different scales, where fractional calculus provides a natural framework due to its inherent non-local nature) and control theory [11] (designing control systems for complex dynamical processes, where fractional derivatives can model systems with memory or long-range dependence).

The $\mathcal{CFD}$ is a relatively new definition of a fractional derivative, introduced by Abdel-Rahman Khalil et al. in 2014 [22]. Unlike some classical fractional derivatives (e.g., Riemann-Liouville, Caputo), the conformable derivative offers several advantages [2, 16, 7, 9]. Conformable fractional differential equations (CFDEs) have emerged as a significant area of investigation within the broader field of fractional calculus.

In 2017, Bayour and Torres [5] explored the existence results for the following system:

$$\begin{cases} \mathcal{D}_0^{\rho}\omega(\curlyvee) = \digamma(\curlyvee, \omega(\curlyvee)), \ \curlyvee \in [0, Z], \ Z > 0, \ \rho \in (0, 1], \\ \omega(0) = \omega_0, \end{cases} \tag{1}$$

where $\mathcal{D}_0^{\rho}$ signify the $\mathcal{CFD}$ of order ρ. Subsequently, Zhong et al. [32] discussed the existence and stability outcomes for the system (1) with non-local conditions under appropriate conditions. Wang and Bai [30] applied upper and lower solution techniques alongside monotone iterative methods to establish solutions within the anti-periodic impulsive framework.

Bouaouid et al. [7] adopted a semigroup theory perspective to investigate various problems similar to (1), including scenarios with non-local conditions under different assumptions. More recently, Hannabou et al. [16] explored a novel class of non-local integro-differential equations (IDEs) involving the $\mathcal{CFD}$. Their approach leverages the theory of operator semigroups and fractional calculus to define a solution concept within this domain. The analysis utilizes powerful tools from fixed-point theory, employing appropriate theorems to guarantee the existence and uniqueness of solutions. Li et al. [26] focused on conformable neutral systems, establishing existence results using fixed-point theorems. Xiao et al. [31] investigated conformable stochastic functional differential equations (CSFDEs) of neutral type, proving existence and uniqueness, followed by stability analysis. Abbas and Benchohra [1] considered the system (1), addressing existence and uniqueness for systems with both finite and infinite delays using fixed-point theory, and further extended their results to neutral-type systems. Hilal et al. [20] examined fractional conformable neutral-type systems with a non-local condition, proving the existence and uniqueness of mild solutions via fixed-point theorems. Building on Xiao et al. [31], another study explored optimal control for conformable fractional neutral stochastic integro-differential systems [8]. Most recently, Krim et al. [24] examined implicit CFDEs, establishing existence results using a specific contraction mapping in b-metric spaces. Despite these advancements, a comprehensive investigation into the existence, uniqueness, and stability properties for neutral-type CFDEs with ID, particularly for systems resembling model (2), remains absent in the current literature. This gap highlights a potential area for further research.

Inspired by the aforementioned research [24, 1], this paper delves into the existence and uniqueness of a novel class of nonlinear implicit neutral CFDEs. The system under investigation is formulated as:

$$\begin{cases} \left({}^{\mathcal{CFD}}\mathfrak{D}_{0+}^{\rho} - \zeta\right)\left[\omega(\Upsilon) - \mathcal{H}(\Upsilon, \omega_\Upsilon, B_1\omega(\Upsilon))\right] = F\left(\Upsilon, \omega_\Upsilon, B_1\omega(\Upsilon)\right), \\ \Upsilon \in \mathfrak{I} = [0, Z],\ \zeta, Z > 0, \\ \omega(\Upsilon) = \xi(\Upsilon),\ \Upsilon \in (-\infty, 0], \end{cases} \tag{2}$$

where $F, \mathcal{H} : \mathfrak{I} \times \mathcal{W} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions with $\mathcal{H}(0, \omega_0, 0) = 0$, ${}^{\mathcal{CFD}}\mathfrak{D}_{0+}^{\rho}$ is the $\mathcal{CFD}$ of order $\rho \in (0, 1)$, and $\mathcal{W}$ represents a space with properties relevant to the system, which will be formally introduced as the phase space later.

The term B_1 is defined as:

$$B_1\omega(\Upsilon) = \int_0^{\Upsilon} G(\Upsilon, \sigma)\omega(\sigma)d\sigma,$$

where $G \in C\left(D, \mathbb{R}^+\right)$ denotes the set of all positive functions that are continuous on $D = \left\{(\Upsilon, \sigma) \in \mathbb{R}^2 : 0 \le \sigma \le \Upsilon < Z\right\}$. Additionally, it is given that $B_1^* = \sup_{\Upsilon \in [0,Z]} \int_0^{\Upsilon} G(\Upsilon, \sigma)d\sigma < \infty$.

For any $\Upsilon \in \mathfrak{I}$, we define $\omega_\Upsilon \in \mathcal{W}$ by

$$\omega_\Upsilon(\theta) = \omega(\Upsilon + \theta), \quad \text{for} \quad \theta \in (-\infty, 0].$$

Our work is meticulously organized to present the concepts and results in a logical sequence. Section 2 lays the groundwork by introducing the notation and revisiting foundational concepts from fractional calculus. Additionally, relevant auxiliary results are presented in this section to equip the reader for the subsequent analysis. Section 3 constitutes the core of the paper: existence and uniqueness results for problem (2) under Banach contraction principle. Finally, Section 4 presents an illustrative academic example to showcase the significance of our main findings.

2. Preliminaries

To lay the groundwork for subsequent analysis, this section establishes key notions, definitions, and preliminary results that will be employed throughout this work.

Let $\mathcal{C}(\mathfrak{I}, \mathbb{R})$ represent the set of all real continuous functions, and let $L^1(\mathfrak{I}, \mathbb{R})$ denote the space of all locally Lebesgue integrable real functions. We also consider $\mathcal{C}^{\rho}_{1-\beta}(\mathfrak{I}, \mathbb{R})$, the Banach space of all continuous functions $\omega : \mathfrak{I} \to \mathbb{R}$ such that $\lim_{\Upsilon \to 0} \omega(\Upsilon)$ exists with the norm $\|\omega\|_{\mathcal{C}^{\rho}_{1-\beta}} = \max\{|\omega(\Upsilon)| : \Upsilon \in \mathfrak{I}\}$.

Consider the space

$$\Theta = \{\omega : (-\infty, Z] \to \mathbb{R}, \omega|_{(-\infty,Z]} \in \mathcal{W}, \omega|_{\mathfrak{I}} \in \mathcal{C}^{\rho}_{1-\beta}(\mathfrak{I}, \mathbb{R})\},$$

where $\omega|_{\mathfrak{I}}$ is the restriction of ω to $[0, Z]$.

Definition 2.1. [22] The $\mathcal{CFD}$ of a function $f : [0, \infty) \to \mathbb{R}$ of order $0 < \rho \leq 1$ is defined by

$$\mathfrak{D}^{\rho} f(\Upsilon) = \lim_{\varepsilon \to 0} \frac{f\left(\Upsilon + \varepsilon \Upsilon^{1-\rho}\right) - f(\Upsilon)}{\varepsilon},$$

provided the limit exists.

In the case where $\Upsilon = 0$, we modify the definition as follows:

$${}^{\mathcal{CFD}}\mathfrak{D}^{\rho} f(0) = \lim_{\Upsilon \longrightarrow 0^+} {}^{\mathcal{CFD}}\mathfrak{D}^{\rho} f(\Upsilon).$$

Theorem 2.2. [22] *Let $\rho \in (0, 1]$ and f_1, f_2 be ρ-differentiable at a point $\Upsilon > 0$. As a result, we have*

(i) ${}^{\mathcal{CFD}}\mathfrak{D}^{\rho}(f_1 f_2) = f_1 {}^{\mathcal{CFD}}\mathfrak{D}^{\rho}(f_2) + f_2 {}^{\mathcal{CFD}}\mathfrak{D}^{\rho}(f_1)$.

(ii) ${}^{\mathcal{CFD}}\mathfrak{D}^{\rho} f(\Upsilon) = \Upsilon^{1-\rho} \mathfrak{D} f(\Upsilon)$, *where f is differentiable and* $\mathfrak{D} = \dfrac{d}{d\Upsilon}$.

Definition 2.3. [2] Let $\rho \in (0, 1]$. The $\mathcal{CF}$ integral starting from a point d of a function $f : [0, \infty) \to \mathbb{R}$ of order ρ is described as

$$I^{\rho}(f)(\Upsilon) = \int_0^{\Upsilon} \sigma^{\rho-1} f(\sigma) d\sigma.$$

Theorem 2.4. [22] *If $f(\cdot)$ is a continuous function in the domain of $I^{\rho}(\cdot)$, then for all $\Upsilon > 0$, we have*

$${}^{\mathcal{CFD}}\mathfrak{D}^{\rho}(I^{\rho} f(\Upsilon)) = f(\Upsilon).$$

Proposition 2.5. [22] *If $f(\cdot)$ is a differentiable function, then for all $\Upsilon > 0$, we have*

$$I^{\rho}\big({}^{\mathcal{CFD}}\mathfrak{D}^{\rho} f(\cdot)\big) = f(\Upsilon) - f(0).$$

Now, we construct the phase space axioms. Let $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ be a seminormed linear space consisting of functions mapping $(-\infty, 0]$ into $\mathbb{R}$ and satisfy the subsequent axioms, which are derived from the original formulations by Hale and Kato [15].

(B1) If $\omega : (-\infty, 0] \to \mathbb{R}$, and $\omega_0 \in \mathcal{W}$, then there exist constants $\eta_1, \eta_2, \eta_3 > 0$ such that for each $\Upsilon \in \mathfrak{I}$, the following hold:
  (a) $\omega_\Upsilon \in \mathcal{W}$,
  (b) $\|\omega_\Upsilon\|_{\mathcal{W}} \leq \eta_1 \|\omega_0\|_{\mathcal{W}} + \eta_2 \sup_{\psi \in [0,\Upsilon]} |\omega(\psi)|$,
  (c) $\|\omega(\Upsilon)\| \leq \eta_3 \|\omega_\Upsilon\|_{\mathcal{W}}$.

(B2) For the function $\omega(\cdot)$ as described in (B1), ω_Υ is continuous on $\mathfrak{I}$ and maps into the space $\mathcal{W}$.

(B3) The space $\mathcal{W}$ possesses the property of completeness.

Definition 2.6. A function $\omega \in \Theta$ constitutes a solution to the system (2) if and only if it adheres to the subsequent integral equation:

$$\begin{aligned}\omega(\Upsilon) &= \mathcal{H}\left(\Upsilon, \omega_\Upsilon, B_1\omega(\Upsilon)\right) + e^{\zeta \frac{\Upsilon^\rho}{\rho}} \xi(0) \\ &\quad + e^{\zeta \frac{\Upsilon^\rho}{\rho}} \int_0^\Upsilon \sigma^{\rho-1} e^{-\zeta \frac{\sigma^\rho}{\rho}} F\left(\sigma, \omega_\sigma, B_1\omega(\sigma)\right) d\sigma, \quad \Upsilon \in \mathfrak{I}. \qquad (3)\end{aligned}$$

3. Main results

This section utilizes the Banach contraction principle to establish the existence and uniqueness of a solution for the system defined by equation (2) on the interval $(-\infty, Z]$. We begin by presenting the following assumptions, which will be crucial for deriving the main results:

(HF) The function $F : \mathfrak{I} \times \mathcal{W} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists\ \mathcal{M}_F, \widetilde{\mathcal{M}}_F > 0$ such that

$$|F(\Upsilon, \xi, v) - F(\Upsilon, \overline{\xi}, \overline{v})| \leq \mathcal{M}_F \|\xi - \overline{\xi}\|_{\mathcal{W}} + \widetilde{\mathcal{M}}_F |v - \overline{v}|$$

for each $\Upsilon \in \mathfrak{I}$, $\xi, \overline{\xi} \in \mathcal{W}, v, \overline{v} \in \mathbb{R}$.

(HH) The function $\mathcal{H} : \mathfrak{I} \times \mathcal{W} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists\ \mathcal{M}_{\mathcal{H}}, \widetilde{\mathcal{M}}_{\mathcal{H}} > 0$ such that

$$|\mathcal{H}(\Upsilon, \xi, v) - \mathcal{H}(\Upsilon, \overline{\xi}, \overline{v})| \leq \mathcal{M}_{\mathcal{H}} \|\xi - \overline{\xi}\|_{\mathcal{W}} + \widetilde{\mathcal{M}}_{\mathcal{H}} |v - \overline{v}|$$

for each $\Upsilon \in \mathfrak{I}$, $\xi, \overline{\xi} \in \mathcal{W}, v, \overline{v} \in \mathbb{R}$.

Theorem 3.1. *Suppose F and $\mathcal{H}$ are satisfy the conditions (HF) and (HH) respectively. If*

$$\begin{aligned}\Lambda_1 = \Bigg\{ &\left(\mathcal{M}_{\mathcal{H}} + \frac{\mathcal{M}_F}{\zeta}\left(e^{\zeta \frac{Z^\rho}{\rho}} - 1\right)\right)\eta_2 \\ &+ \left(\widetilde{\mathcal{M}}_{\mathcal{H}} + \frac{\widetilde{\mathcal{M}}_F}{\zeta}\left(e^{\zeta \frac{Z^\rho}{\rho}} - 1\right)\right) B_1^* \Bigg\} < 1, \qquad (4)\end{aligned}$$

then the system (2) *has a unique solution on* $(-\infty, Z]$.

Proof. Define the operator $\Upsilon : \Theta \to \Theta$ by

$$(\Upsilon\omega)(\Upsilon) = \begin{cases} \mathcal{H}\left(\Upsilon, \omega_\Upsilon, B_1\omega(\Upsilon)\right) + e^{\zeta\frac{\Upsilon^\rho}{\rho}}\xi(0) \\ \quad + e^{\zeta\frac{\Upsilon^\rho}{\rho}}\displaystyle\int_0^\Upsilon \sigma^{\rho-1}e^{-\zeta\frac{\sigma^\rho}{\rho}}F\left(\sigma, \omega_\sigma, B_1\omega(\sigma)\right)d\sigma, \quad \Upsilon \in \Im, \\ \xi(\Upsilon), \quad (-\infty, 0]. \end{cases}$$

We consider the function $\omega_1 : (-\infty, Z] \to \mathbb{R}$ defined by

$$\omega_1(\Upsilon) = \begin{cases} 0, & \text{if} \quad \Upsilon \in \Im \\ \xi(\Upsilon), & \text{if} \quad \Upsilon \in (-\infty, 0]. \end{cases}$$

Then $\omega_{10} = \xi$. For each $\omega_2 \in \mathcal{C}(\Im, \mathbb{R})$, with $\omega_2(0) = 0$, we denote by $\overline{\omega}_2$ the function described by

$$\overline{\omega}_2(\Upsilon) = \begin{cases} \omega_2(\Upsilon), & \text{if} \quad \Upsilon \in \Im \\ 0, & \text{if} \quad \Upsilon \in (-\infty, 0]. \end{cases}$$

If $\omega(\cdot)$ fulfills (3), we can express $\omega(\Upsilon)$ as $\omega(\Upsilon) = \omega_1(\Upsilon) + \overline{\omega}_2(\Upsilon)$ for $\Upsilon \in \Im$, indicating that $\omega_\Upsilon = \omega_{1\Upsilon} + \overline{\omega}_{2\Upsilon}$ for every $\Upsilon \in \Im$. Furthermore, the function $\omega_2(\cdot)$ meets

$$\begin{aligned} \omega_2(\Upsilon) &= \mathcal{H}(\Upsilon, \omega_{1\Upsilon} + \overline{\omega}_{2\Upsilon}, B_1(\omega_1(\Upsilon) + \overline{\omega}_2(\Upsilon))) + e^{\zeta\frac{\Upsilon^\rho}{\rho}}\xi(0) \\ &\quad + e^{\zeta\frac{\Upsilon^\rho}{\rho}}\int_0^\Upsilon \sigma^{\rho-1}e^{-\zeta\frac{\sigma^\rho}{\rho}}F\left(\sigma, \omega_{1\sigma} + \overline{\omega}_{2\sigma}, B_1(\omega_1(\sigma) + \overline{\omega}_2(\sigma))\right)d\sigma, \ \Upsilon \in \Im. \end{aligned} \tag{5}$$

Setting

$$\widetilde{\Theta} = \{\omega_2 \in \Theta : \omega_{20} = 0\}$$

and let $\|\cdot\|_{\widetilde{\Theta}}$ be the norm in $\widetilde{\Theta}$ defined by

$$\|\omega_2\|_{\widetilde{\Theta}} = \|\omega_{20}\|_{\mathcal{W}} + \sup_{\Upsilon\in\Im}|\omega_2(\Upsilon)| = \sup_{\Upsilon\in\Im}|\omega_2(\Upsilon)|, \quad \omega_2 \in \widetilde{\Theta},$$

then $(\widetilde{\Theta}, \|\cdot\|_{\widetilde{\Theta}})$ is a Banach space. Define the operator $\Upsilon_1 : \widetilde{\Theta} \to \widetilde{\Theta}$ by

$$\begin{aligned} (\Upsilon_1\omega_2)(\Upsilon) &= \mathcal{H}(\Upsilon, \omega_{1\Upsilon} + \overline{\omega}_{2\Upsilon}, B_1(\omega_1(\Upsilon) + \overline{\omega}_2(\Upsilon))) + e^{\zeta\frac{\Upsilon^\rho}{\rho}}\xi(0) \\ &\quad + e^{\zeta\frac{\Upsilon^\rho}{\rho}}\int_0^\Upsilon \sigma^{\rho-1}e^{-\zeta\frac{\sigma^\rho}{\rho}}F\left(\sigma, \omega_{1\sigma} + \overline{\omega}_{2\sigma}, B_1(\omega_1(\sigma) + \overline{\omega}_2(\sigma))\right)d\sigma, \ \Upsilon \in \Im. \end{aligned}$$

Thus, having a fixed point for the operator Υ is equivalent to having a fixed point for Υ_1. We now focus on proving the existence of a fixed point for Υ_1.

Note 3.2. *From the phase space axioms (B1) and conditions (HF)-(HH), we have the following estimations: For any $\omega_2, \omega_2^* \in \widetilde{\Theta}$ and for all $\Upsilon \in \Im$, we have*

$$\begin{aligned} &|\mathcal{H}(\Upsilon, \omega_{1\Upsilon} + \overline{\omega}_{2\Upsilon}, B_1(\omega_1(\Upsilon) + \overline{\omega}_2(\Upsilon))) - \mathcal{H}(\Upsilon, \omega_{1\Upsilon} + \overline{\omega}^*_{2\Upsilon}, B_1(\omega_1(\Upsilon) + \overline{\omega}^*_2(\Upsilon)))| \\ &\leq \mathcal{M}_{\mathcal{H}}\|\overline{\omega}_{2\Upsilon} - \overline{\omega}^*_{2\Upsilon}\|_{\mathcal{W}} + \widetilde{\mathcal{M}}_{\mathcal{H}}B_1|\overline{\omega}_2(\Upsilon) - \overline{\omega}^*_2(\Upsilon)|. \end{aligned} \tag{6}$$

Since

$$\begin{aligned}\|\overline{\omega}_{2\curlyvee}-\overline{\omega}^*_{2\curlyvee}\|_{\mathcal{W}} &\leq \eta_1\|\omega_{20}-\omega^*_{20}\|_{\mathcal{W}}+\eta_2\sup_{0\leq\psi\leq\curlyvee}|\omega_2(\psi)-\omega^*_2(\psi)|\\ &\leq \eta_1(0)+\eta_2\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}\\ &\leq \eta_2\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}.\end{aligned}$$

Thus (6) *becomes*

$$\begin{aligned}&|\mathcal{H}(\curlyvee,\omega_{1\curlyvee}+\overline{\omega}_{2\curlyvee},B_1(\omega_1(\curlyvee)+\overline{\omega}_2(\curlyvee)))-\mathcal{H}(\curlyvee,\omega_{1\curlyvee}+\overline{\omega}^*_{2\curlyvee},B_1(\omega_1(\curlyvee)+\overline{\omega}^*_2(\curlyvee)))|\\ &\leq(\mathcal{M}_{\mathcal{H}}\eta_2+\widetilde{\mathcal{M}}_{\mathcal{H}}B^*_1)\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}.\end{aligned}\tag{7}$$

In the similar manner

$$\begin{aligned}&|F(\curlyvee,\omega_{1\curlyvee}+\overline{\omega}_{2\curlyvee},B_1(\omega_1(\curlyvee)+\overline{\omega}_2(\curlyvee)))-F(\curlyvee,\omega_{1\curlyvee}+\overline{\omega}^*_{2\curlyvee},B_1(\omega_1(\curlyvee)+\overline{\omega}^*_2(\curlyvee)))|\\ &\leq(\mathcal{M}_F\eta_2+\widetilde{\mathcal{M}}_F B^*_1)\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}.\end{aligned}\tag{8}$$

Let $\omega_2,\omega^*_2\in\widetilde{\Theta}$. Then, for every $\curlyvee\in\mathfrak{I}$, the following holds:

$$\begin{aligned}&|(\Upsilon_1\omega_2)(\curlyvee)-(\Upsilon_1\omega^*_2)(\curlyvee)|\\ &\leq(\mathcal{M}_{\mathcal{H}}\eta_2+\widetilde{\mathcal{M}}_{\mathcal{H}}B^*_1)\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}\\ &\quad+(\mathcal{M}_F\eta_2+\widetilde{\mathcal{M}}_F B^*_1)\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}e^{\zeta\frac{\curlyvee^\rho}{\rho}}\int_0^{\curlyvee}\sigma^{\rho-1}e^{-\zeta\frac{\sigma^\rho}{\rho}}d\sigma\\ &\leq(\mathcal{M}_{\mathcal{H}}\eta_2+\widetilde{\mathcal{M}}_{\mathcal{H}}B^*_1)\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}\\ &\quad+(\mathcal{M}_F\eta_2+\widetilde{\mathcal{M}}_F B^*_1)\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}\cdot\frac{1}{\zeta}\left(e^{\zeta\frac{Z^\rho}{\rho}}-1\right)\\ &\leq\Bigg\{\left(\mathcal{M}_{\mathcal{H}}+\frac{\mathcal{M}_F}{\zeta}\left(e^{\zeta\frac{Z^\rho}{\rho}}-1\right)\right)\eta_2\\ &\quad+\left(\widetilde{\mathcal{M}}_{\mathcal{H}}+\frac{\widetilde{\mathcal{M}}_F}{\zeta}\left(e^{\zeta\frac{Z^\rho}{\rho}}-1\right)\right)B^*_1\Bigg\}\|\omega_2-\omega^*_2\|_{\widetilde{\Theta}}.\end{aligned}$$

By virtue of condition (4), the operator Υ satisfies the conditions of the Banach contraction principle [14]. This guarantees the existence of a unique fixed point which is the unique solution of [5] on $\mathfrak{I}$. Define $\omega(\curlyvee)=\omega_1(\curlyvee)+\overline{\omega}_2(\curlyvee)$. Then, $\omega(\curlyvee)$ represents the unique solution to the structure (2) on $(-\infty,Z]$. □

4. Example

For the sake of clarity, we showcase the aforementioned results through a practical example in this section.

Let $\beta\in\mathbb{R}_+$. We define a Banach space $\mathcal{W}_\beta$ of functions on the interval $(-\infty,0]$ as follows:

$$\mathcal{W}_\beta=\{\omega\in\mathcal{C}(-\infty,0],\mathbb{R}):\lim_{\curlyvee\to-\infty}e^{\beta\curlyvee}\omega(\curlyvee)\text{ exists in }\mathbb{R}\}.$$

The norm on $\mathcal{W}_\beta$ is defined as:

$$\|\omega\|_\beta = \sup_{-\infty<\Upsilon\leq 0}\left\{e^{\beta\Upsilon}|\omega(\Upsilon)|\right\}.$$

This space satisfies all the phase space axioms. Consider the subsequent conformable fractional neutral infinite delay system

$$\begin{cases} \left({}^{\mathcal{CFD}}\mathfrak{D}_{0^+}^{\rho} - \zeta\right)\left[\omega(\Upsilon) - \left\{\dfrac{e^{-\Upsilon-\Upsilon\beta}\|\omega_\Upsilon\|}{9\left(e^{\Upsilon}+e^{-\Upsilon}\right)}\right.\right. \\ \left.\left. +\dfrac{e^{-\Upsilon}}{16\left(e^{\Upsilon}+e^{-\Upsilon}\right)}\sin\left(\left|\displaystyle\int_0^{\Upsilon}(\Upsilon-\sigma)\omega(\sigma)d\sigma\right|\right)\right\}\right] = \dfrac{e^{-\Upsilon\beta}|\omega_\Upsilon|}{\left(25+e^{-\Upsilon}\right)\left(1+\|\omega_\Upsilon\|\right)} \\ +\dfrac{e^{-\Upsilon}}{16\left(e^{\Upsilon}+e^{-\Upsilon}\right)}\sin\left(\left|\displaystyle\int_0^{\Upsilon}(\Upsilon-\sigma)\omega(\sigma)d\sigma\right|\right), \quad \Upsilon\in\mathfrak{J}=(0,1], \\ \omega(\Upsilon)=\xi(\Upsilon),\ \Upsilon\in(-\infty,0]. \end{cases} \tag{9}$$

We define the functions as follows:

$$\mathcal{F}(\Upsilon,u,v) = \frac{e^{-\Upsilon\beta}u}{\left(25+e^{-\Upsilon}\right)(1+u)} + \frac{e^{-\Upsilon}}{16\left(e^{\Upsilon}+e^{-\Upsilon}\right)}\sin(|B_1v|), \quad \Upsilon\in\mathfrak{J}$$

and

$$\mathcal{H}(\Upsilon,u,v) = \frac{e^{-\Upsilon-\Upsilon\beta}u}{9\left(e^{\Upsilon}+e^{-\Upsilon}\right)} + \frac{e^{-\Upsilon}}{16\left(e^{\Upsilon}+e^{-\Upsilon}\right)}\sin(|B_1v|), \quad \Upsilon\in\mathfrak{J},$$

where $B_1v=\int_0^{\Upsilon}(\Upsilon-\sigma)vd\sigma$. Here $B_1^*=\sup_{\Upsilon\in[0,1]}\int_0^{\Upsilon}(\Upsilon-\sigma)d\sigma=\frac{1}{2}<\infty$.

For any u,v,u^*,v^* from $\mathcal{W}_\beta$ and $\Upsilon\in\mathfrak{J}$, we obtain

$$\begin{aligned} |\mathcal{F}(\Upsilon,u,v)-\mathcal{F}(\Upsilon,u^*,v^*)| &\leq \frac{e^{-\Upsilon\beta}}{25+e^{-\Upsilon}}\left|\frac{u}{1+u}-\frac{u^*}{1+u^*}\right| \\ &\quad+\frac{e^{-\Upsilon}}{16\left(e^{\Upsilon}+e^{-\Upsilon}\right)}|B_1v-B_1v^*| \\ &= \frac{e^{-\Upsilon-\Upsilon\beta}}{25e^{\Upsilon}+1}\left|\frac{u-u^*}{(1+u)(1+u^*)}\right| + \frac{B_1^*}{16\left(e^{2\Upsilon}+1\right)}|v-v^*| \\ &\leq \frac{e^{-\Upsilon}}{25}e^{-\Upsilon\beta}|u-u^*| + \frac{B_1^*}{16}|v-v^*| \\ &\leq \frac{1}{25}\|u-u^*\|_\beta + \frac{1}{16}\cdot B_1^*|v-v^*| \\ |\mathcal{H}(\Upsilon,u,v)-\mathcal{H}(\Upsilon,u^*,v^*)| &\leq \frac{e^{-\Upsilon-\Upsilon\beta}}{9\left(e^{\Upsilon}+e^{-\Upsilon}\right)}|u-u^*| + \frac{e^{-\Upsilon}}{16\left(e^{\Upsilon}+e^{-\Upsilon}\right)}|B_1v-B_1v^*| \\ &\leq \frac{1}{9\left(e^{2\Upsilon}+1\right)}\|u-u^*\|_\beta + \frac{B_1^*}{16\left(e^{2\Upsilon}+1\right)}|v-v^*| \\ &\leq \frac{1}{9}\|u-u^*\|_\beta + \frac{1}{16}\cdot B_1^*|v-v^*|. \end{aligned}$$

Therefore the conditions (HF) and (HH) are satisfied with $\mathcal{M}_F = \dfrac{1}{25}, \widetilde{\mathcal{M}}_F = \widetilde{\mathcal{M}}_{\mathcal{H}} = \dfrac{1}{16}$ and $\mathcal{M}_{\mathcal{H}} = \dfrac{1}{9}$. It can be observed that (4) is satisfied by considering $\eta_2 = \zeta = Z = 1$ and $\rho = \dfrac{1}{2}$. Moreover

$$\Lambda_1 = \left\{ \left(\mathcal{M}_{\mathcal{H}} + \frac{\mathcal{M}_F}{\zeta}\left(e^{\zeta \frac{Z^\rho}{\rho}} - 1 \right) \right) \eta_2 + \left(\widetilde{\mathcal{M}}_{\mathcal{H}} + \frac{\widetilde{\mathcal{M}}_F}{\zeta}\left(e^{\zeta \frac{Z^\rho}{\rho}} - 1 \right) \right) B_1^* \right\} \approx 0.598 < 1.$$

Thus by Theorem 3.1, the system (9) has a unique solution on $(-\infty, 1]$.

5. Conclusion

In this study, we applied the Banach contraction principle to establish the necessary conditions for the existence and uniqueness of solutions for a nonlinear weighted NFDS. A compelling example was provided to validate the results, which are pioneering for fractional differential equations (FDEs) incorporating the conformable fractional derivative ($\mathcal{CFD}$). By leveraging a suitable fixed-point theorem to prove existence, controllability, and stability in models with non-instantaneous impulses, there is significant potential to advance the efficacy of ongoing and future research in this area.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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