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NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS WITH RESPECT TO THE q-SYMMETRIC POINTS DEFINED BY BERNOULLI POLYNOMIALS†

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Abstract. The objective of this paper is to introduce and investigate new subclass of bi-univalent functions with respect to the symmetric points in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ using Bernoulli polynomials. For functions belonging to this class, we obtain upper bounds for Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö inequalities $|a_3 - \mu a_2^2|$ for these new subclasses.

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1.Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$. If a function f is one-to-one in $\mathbb{D}$, then it is called univalent in $\mathbb{D}$.
Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$.
The inverse functions in the class $\mathcal{S}$ may not be defined on the entire unit disc $\mathbb{U}$ although the functions in the class $\mathcal{S}$ are invertible. However using Koebe-one quarter theorem [10] it is obvious that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disc of radius $\frac{1}{4}$. Hence every univalent function f has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}),$$

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and

$$f(f^{-1}(w)) = w\left(|w| < r_0(f) : r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$.

The expression $\sum$ as a non empty class of functions, as it contains at least the functions

$$f_1(z) = -\frac{z}{1-z'},\ f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z'}$$

with their corresponding inverses

$$f_1^{-1}(\omega) = \frac{\omega}{1+\omega'},\ f_2^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1}.$$

In addition, the Koebe function $f(z) = \frac{z}{(1-z)^2} \notin \sum$.

The study of analytical and bi-univalent functions is reintroduced in the publication of [29] and is then followed by work such as [5, 6, 13, 18, 19, 25]. The initial coefficient constraints have been determined by several authors who have presented new subclasses of bi-univalent functions [5, 6, 8, 26, 29, 30].

Let α and β be two analytic functions in $\mathbb{U}$. Then we say that α is subordinate to β, if a Schwarz function ω exists that is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1,\ (z \in \mathbb{U})$ such that

$$\alpha(z) = \beta(\omega(z)),\ (z \in \mathbb{U}).$$

This subordination is denoted by $\alpha \prec \beta$. Given that β is a univalent function in $\mathbb{U}$, then

$$\alpha(z) \prec \beta(z) \Longleftrightarrow \alpha(0) = \beta(0)\ and\ \alpha(\mathrm{U}) \subset \beta(\mathbb{U}).$$

By Loewner's technique, the Fekete-Szegö problem for the coefficients of $f \in \mathcal{S}$ in [18] is

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right)\ for\ 0 \leq \mu < 1.$$

The elementary inequality $|a_3 - a_2^2| \leq 1$ is obtained as $\mu \to 1$. The coefficient functional

$$F_\mu(f) = a_3 - a_2^2,$$

on the normalized analytic functions f in the open unit disk $\mathbb{U}$ also has a significant impact on geometric function theory. The Fekete-Szegö problem is known as the maximization problem for functional $|F_\mu(f)|$.

Researchers were concerned about several classes of univalent functions [11, 17, 22, 33] due to the Fekete-Szegö problem, proposed in 1933 [12] therefore, it stands to reason that similar inequalities were also discovered for bi-univalent functions and fairly recent publications can be cited to back up the claim that the subject still yields intriguing findings [1, 3, 35].

Because of their importance in probability theory, mathematical statistics, mathematical physics and engineering, orthogonal polynomials have been the subject of substantial research in recent years from a variety of angles. The classical orthogonal polynomials are the orthogonal polynomials that are most commonly used in applications (Hermite polynomials, Laguerre polynomials, Jacobi polynomials and Bernoulli). We point out [1, 2, 3, 4, 16, 31, 32] as more recent examples of the relationship between geometric function theory and classical orthogonal polynomials.

Fractional calculus, a classical branch of mathematical analysis whose foundations were laid by Liouville in an 1832 paper and is currently a very active research field [21], is one of many special functions that are studied. This branch of mathematics is known as the Bernoulli polynomials, named after Jacob Bernoulli $(1654-1705)$. A novel approximation method based on orthonormal Bernoulli's polynomials has been developed to solve fractional order differential equations of the Lane-Emden type [28], whereas in [7, 9, 20], Bernoulli polynomials are utilized to numerically resolve Freehold fractional integro differential equations with right sided Caputo derivatives.

The Bernoulli polynomials $\mathrm{B_n}(x)$ are often defined using the generating function:

$$F(x,t)=\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{\mathrm{B_n(x)}}{n!}t^n,\ \ |t|<2\pi, \tag{3}$$

where $\mathrm{B_n(x)}$ are polynomials in x, for each non negative integer n.

The Bernoulli [23] polynomials are easily computed by recursion since

$$\sum_{j=0}^{n-1}\binom{n}{j}\mathrm{B_j(x)}=nx^{n-1},\ \ n=2,3,\cdots. \tag{4}$$

The initial few polynomials of Bernoulli are

$$\mathrm{B_0(x)}=1,\ \mathrm{B_1(x)}=x-\frac{1}{2},\ \mathrm{B_2(x)}=x^2-x+\frac{1}{6},\ \mathrm{B_3(x)}=x^3-\frac{3}{2}x^2+\frac{1}{2}x,\cdots. \tag{5}$$

Jackson [14,15] at the beginning of the twentieth century studied consequences. The key concept is the q-derivative operator defined as follows:

$$D_qf(z)=\begin{cases}\frac{f(z)-f(qz)}{z(1-q)},\ z\neq 0\\ f'(0),\ z=0\end{cases}$$

which is said to be q-derivative (or difference) operator of a function f. By taking q-derivative of the function f in the form (1), we can see that

$$\mathrm{D}_qf(z)=1+\sum_{n=2}^{\infty}[n]_qa_nz^{n-1},\ \ z\neq 0,$$

where

$$[n]_q=\frac{1-q^n}{1-q}.$$

Note that as $q \longrightarrow 1^-$, $[n]_q \longrightarrow n$.
Using q-derivative operator and subordination now, we define new subclasses of bi-univalent functions, associated with Bernoulli polynomials. The class $\mathrm{S}_\mathrm{s}^\mathrm{q}$ of q-starlike functions with respect to symmetric points, was introduced by Sakaguchi [28] which consists of functions $f \in \mathrm{S}$ satisfying the condition

$$\Re\left\{\frac{z\mathrm{D}_q f(z)}{f(z)-f(-z)}\right\} > 0,\ (z \in \mathbb{U}).$$

The class C_s^q of q-convex functions with respect to symmetric points, was introduced by Wang [34] which consists of $f \in \mathcal{S}$ satisfying the condition

$$\Re\left\{\frac{\mathrm{D}_q[z\mathrm{D}_q f(z)]}{\mathrm{D}_q[f(z)-f(-z)]}\right\} > 0,\ (z \in \mathbb{U}).$$

In this paper, we introduce two subclasses of $\sum$: the class $\mathrm{S}_s^{\Sigma,q}(x)$ of functions bi-q-starlike with respect to the symmetric points and the relative class $\mathrm{C}_s^{\Sigma,q}(x)$ of functions bi-q-convex with respect to the symmetric points associated with Bernoulli polynomials. The definitions are as follows:

Definition 1.1. Let $f \in \mathrm{S}_s^{\Sigma,q}(x)$ be the function, the next subordinations holds:

$$\frac{2z\mathrm{D}_q f(z)}{f(z)-f(-z)} \prec F(x,z), \tag{6}$$

and

$$\frac{2\omega\mathrm{D}_q[g(\omega)]}{g(\omega)-g(-\omega)} \prec F(x,\omega), \tag{7}$$

where $z, \omega \in \mathbb{U}$, $F(x,z)$ is given by (3), and $g = f^{-1}$ is given by (2).

Definition 1.2. Let $f \in \mathrm{C}_s^{\Sigma,q}(x)$ be the function, the next subordinations holds:

$$\frac{2\mathrm{D}_q(z\mathrm{D}_q f(z))}{\mathrm{D}_q[f(-z)-f(z)]} \prec F(x,z), \tag{8}$$

and

$$\frac{2\mathrm{D}_q[\omega\mathrm{D}_q g(\omega)]}{\mathrm{D}_q[g(\omega)-g(-\omega)]} \prec F(x,\omega), \tag{9}$$

where $z, \omega \in \mathbb{U}$, $F(x,z)$ is given by [3] , and $g = f^{-1}$ is given by (2).

Lemma 1.3. *[24] Let* $c(z) = \sum_{n=1}^{\infty} c_n z^n,\ |c(z)| < 1,\ z \in \mathbb{U}$, *be an analytic function in* $\mathbb{U}$. *Then*

$$|c_1| \le 1,\ |c_n| \le 1 - |c_1|^2,\ n = 2, 3, \cdots.$$

2.Coefficients Estimates for the Class $\mathrm{S}_s^{\Sigma,q}(x)$

We obtain upper bounds of $|a_2|$ and $|a_3|$ for the functions belonging to class $\mathrm{S}_s^{\Sigma,q}(x)$.

Theorem 2.1. *If* $f \in \mathrm{S}_s^{\Sigma,q}(x)$, *then*

$$|a_2| \leq |\mathrm{B}_1(x)|\sqrt{6|\mathrm{B}_1(x)|}, \tag{10}$$

and

$$|a_3| \leq \frac{\mathrm{B}_1(x)}{[2]_q} + \frac{[\mathrm{B}_1(x)]^2}{2[2]_q}. \tag{11}$$

Proof. Let $f \in \mathrm{S}_s^{\sum,q}(x)$ and $g \in f^{-1}$. From definition in (6) and (7), we have

$$\frac{2z\mathrm{D}_q f(z)}{f(z) - f(-z)} = F(x, \psi(x)), \tag{12}$$

and

$$\frac{2\omega\mathrm{D}_q(g(\omega))}{g(\omega) - g(-\omega)} = F(x, \chi(\omega)), \tag{13}$$

where ψ and χ are analytic functions in $\mathbb{U}$ given by

$$\psi(z) = r_1 z + r_2 z^2 + \cdots, \tag{14}$$

$$\chi(\omega) = s_1\omega + s_2\omega^2 + \cdots, \tag{15}$$

and $\psi(0) = \chi(0) = 0$, and $|\psi(z)| < 1$, $|\chi(\omega)| < 1$, $z, \omega \in \mathbb{U}$.
As a result of Lemma 1.3,

$$|r_k| \leq 1 \ \textit{and}\ |s_k| \leq 1, \ k \in \mathbb{N}. \tag{16}$$

If we replace (14) and (15) in (12) and (13) respectively, we obtain

$$\frac{2z\mathrm{D}_q f(z)}{f(z) - f(-z)} = \mathrm{B}_0(x) + \mathrm{B}_1(x)\psi(z) + \frac{\mathrm{B}_2(x)}{2!}\psi^2(z) + \cdots, \tag{17}$$

and

$$\frac{2\omega\mathrm{D}_q(g(\omega))}{g(\omega) - g(-\omega)} = \mathrm{B}_0(x) + \mathrm{B}_1(x)\chi(\omega) + \frac{\mathrm{B}_2(x)}{2!}\chi^2(\omega) + \cdots. \tag{18}$$

In view (1) and (2), from (17) and (18), we obtain

$$1 + [2]_q a_2 z + [2]_q a_3 z^2 + \cdots = 1 + \mathrm{B}_1(x) r_1 z + \left[\mathrm{B}_1(x) r_2 + \frac{\mathrm{B}_2(x)}{2!} r_1^2\right] z^2 + \cdots$$

and

$$1 - [2]_q a_2\omega + [2]_q(2a_2^2 - a_3)\omega^2 + \cdots = 1 + \mathrm{B}_1(x) s_1\omega + \left[\mathrm{B}_1(x) s_2 + \frac{\mathrm{B}_2(x)}{2!} s_1^2\right]\omega^2 + \cdots,$$

which yields the following relations:

$$[2]_q a_2 = \mathrm{B}_1(x) r_1, \tag{19}$$

$$[2]_q a_3 = \mathrm{B}_1(x) r_2 + \frac{\mathrm{B}_2(x)}{2!} r_1^2, \tag{20}$$

and

$$-[2]_q a_2 = \mathrm{B}_1(x) s_1, \tag{21}$$

$$[2]_q(2a_2^2 - a_3) = \mathrm{B}_1(x) s_2 + \frac{\mathrm{B}_2(x)}{2!} s_1^2. \tag{22}$$

From (19) and (21), it follows that

$$r_1 = -s_1 \tag{23}$$

and

$$2[2]_q^2 a_2^2 = [\mathrm{B}_1(x)]^2(r_1^2 + s_1^2) \tag{24}$$

$$a_2^2 = \frac{[\mathrm{B}_1(x)]^2(r_1^2 + s_1^2)}{2[2]_q^2}. \tag{25}$$

Adding (20) and (22), using (24), we obtain

$$a_2^2 = \frac{[\mathrm{B}_1(x)]^3(r_2 + s_2)}{2([2]_q)([\mathrm{B}_1(x)]^2 - \mathrm{B}_2(x))}. \tag{26}$$

Using relation (5), from (16) for r_2 and s_2, we get (10).
Using (23) and (24), by subtracting (22) from (20), we get

$$\begin{aligned} a_3 &= \frac{[\mathrm{B}_1(x)](r_2 - s_2) + \frac{\mathrm{B}_2(x)}{2!}(r_1^2 - s_1^2)}{([2]_q)^2} + a_2^2 \\ &= \frac{[\mathrm{B}_1(x)](r_2 - s_2) + \frac{\mathrm{B}_2(x)}{2!}(r_1^2 - s_1^2)}{[2]_q^2} + \frac{[\mathrm{B}_1(x)]^2(r_1^2 + s_1^2)}{2([2]_q)^2}. \end{aligned} \tag{27}$$

Once again applying (23) and using (5), for the coefficients, r_1, s_1, r_2, s_2, we deduce (11). □

3.The Fekete-Szegö Problem for the Function Class $\mathrm{S}_s^{\Sigma,q}(x)$

We obtain the Fekete-Szegö inequality for the class $\mathrm{S}_s^{\Sigma,q}(x)$ due to the result of Zaprawa [35]

Theorem 3.1. *If f given by (1) is in the class $\mathrm{S}_s^{\sum,q}(x)$, where $\mu \in \mathbb{R}$, then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\mathrm{B}_1(x)}{[2]_q}, & |h(\mu)| \leq \frac{1}{4}, \\ 2\mathrm{B}_1(x)|h(\mu)|, & |h(\mu)| \geq \frac{1}{4}, \end{cases}$$

where

$$h(\mu) = [3]_q(1 - \mu)[\mathrm{B}_1(x)]^2.$$

Proof. If $f \in \mathrm{S}_s^{\Sigma,q}(x)$ is given by (1), from (25) and (26), we have

$$\begin{aligned} &a_3 - \mu a_2^2 \\ &= \frac{\mathrm{B}_1(x)(r_2 - s_2)}{2([2]_q)} + (1 - \mu)a_2^2 \\ &= \frac{\mathrm{B}_1(x)(r_2 - s_2)}{2([2]_q)} + \frac{(1 - \mu)[\mathrm{B}_1(x)]^3(r_2 + s_2)}{2([2]_q)([\mathrm{B}_1(x)]^2 - \mathrm{B}_2(x))} \\ &= \mathrm{B}_1(x)\left[\frac{r_2}{2([2]_q)} - \frac{s_2}{2([2]_q)} + \frac{(1 - \mu)[\mathrm{B}_1(x)]^2 r_2}{2([2]_q)([\mathrm{B}_1(x)]^2 - \mathrm{B}_2(x))} + \frac{(1 - \mu)[\mathrm{B}_1(x)]^2 s_2}{4([\mathrm{B}_1(x)]^2 - \mathrm{B}_2(x))}\right] \end{aligned}$$

$$= \mathrm{B}_1(x)\left[\left(h(\mu)+\frac{1}{4}\right)r_2+\left(h(\mu)-\frac{1}{4}\right)s_2\right],$$

where

$$h(\mu)=\frac{(1-\mu)[\mathrm{B}_1(x)]^2}{2([2]_q)([\mathrm{B}_1(x)]^2-\mathrm{B}_2(x))}.$$

Now, by using (5)

$$a_3-\mu a_2^2=\left(x-\frac{1}{2}\right)\left[\left(h(\mu)+\frac{1}{4}\right)r_2+\left(h(\mu)+\frac{1}{4}\right)s_2\right],$$

where

$$h(\mu)=3(1-\mu)\left(x-\frac{1}{2}\right)^2.$$

Therefore, given (5) and (16), we conclude that the necessary inequality holds. □

4.Coefficients Estimates for the Class $\mathrm{C}_s^{\sum,q}(x)$

We will obtain upper bounds of $|a_2|$ and $|a_3|$ for the functions belonging to a class $\mathrm{C}_s^{\sum,q}(x)$.

Theorem 4.1. *If* $f\in\mathrm{C}_s^{\sum,q}(x)$, *then*

$$|a_2|\le\frac{|\mathrm{B}_1(x)|\sqrt{|\mathrm{B}_1(x)|}}{\sqrt{\left|2[3]_q[\mathrm{B}_1(x)|]^2-2([2]_q^2)\mathrm{B}_2(x)\right|}},\tag{28}$$

and

$$|a_3|\le\frac{\mathrm{B}_1(x)}{2[3]_q}+\frac{[\mathrm{B}_1(x)]^2}{2([2]_q^3)}.\tag{29}$$

Proof. Let $f\in\mathrm{C}_s^{\sum,q}(x)$ and $g\in f^{-1}$. From definition in (8) and (9), we get

$$\frac{2\mathrm{D}_q[z\mathrm{D}_qf(z)]}{\mathrm{D}_q[f(z)-f(-z)]}=F(x,\psi(x)),\tag{30}$$

and

$$\frac{2\mathrm{D}_q[\omega\mathrm{D}_qg(\omega)]}{\mathrm{D}_q[g(\omega)-g(-\omega)]}=F(x,\chi(\omega)),\tag{31}$$

where ψ and χ are analytic functions in $\mathbb{U}$ given by

$$\psi(z)=r_1z+r_2z^2+\cdots,\tag{32}$$

$$\chi(\omega)=s_1\omega+s_2\omega^2+\cdots,\tag{33}$$

and $\psi(0)=\chi(0)=0$, and $|\psi(z)|<1$, $|\chi(\omega)|<1$, $z,\omega\in\mathbb{U}$.
As a result of Lemma 1.3,

$$|r_k|\le1 \textit{ and } |s_k|\le1,\ k\in\mathbb{N}.\tag{34}$$

If we replace (32) and (33) in (30) and (31) respectively, we obtain

$$\frac{2\mathrm{D}_q[z\mathrm{D}_q f(z)]}{\mathrm{D}_q[f(z)-f(-z)]} = \mathrm{B}_0(x) + \mathrm{B}_1(x)\psi(z) + \frac{\mathrm{B}_2(x)}{2!}\psi^2(z) + \cdots, \tag{35}$$

and

$$\frac{2\mathrm{D}_q[\omega\mathrm{D}_q g(\omega)]}{\mathrm{D}_q[g(\omega)-g(-\omega)]} = \mathrm{B}_0(x) + \mathrm{B}_1(x)\chi(\omega) + \frac{\mathrm{B}_2(x)}{2!}\chi^2(\omega) + \cdots. \tag{36}$$

In view (1) and (2), from (35) and (36), we obtain

$$1 + 2[2]_q a_2 z + 2[3]_q a_3 z^2 + \cdots = 1 + \mathrm{B}_1(x) r_1 z + \left[\mathrm{B}_1(x) r_2 + \frac{\mathrm{B}_2(x)}{2!} r_1^2\right] z^2 + \cdots$$

and

$$1 - 2[2]_q a_2 \omega + 2[3]_q (2a_2^2 - a_3)\omega^2 + \cdots = 1 + \mathrm{B}_1(x) s_1 \omega + \left[\mathrm{B}_1(x) s_2 + \frac{\mathrm{B}_2(x)}{2!} s_1^2\right] \omega^2 + \cdots,$$

which yields the following relations:

$$2[2]_q a_2 = \mathrm{B}_1(x) r_1, \tag{37}$$

$$2[3]_q a_3 = \mathrm{B}_1(x) r_2 + \frac{\mathrm{B}_2(x)}{2!} r_1^2, \tag{38}$$

and

$$-2[2]_q a_2 = \mathrm{B}_1(x) s_1, \tag{39}$$

$$2[3]_q (2a_2^2 - a_3) = \mathrm{B}_1(x) s_2 + \frac{\mathrm{B}_2(x)}{2!} s_1^2. \tag{40}$$

From (37) and (39), it follows that

$$r_1 = -s_1 \tag{41}$$

and

$$2[2]_q^4 a_2^2 = [\mathrm{B}_1(x)]^2 (r_1^2 + s_1^2) \tag{42}$$

$$a_2^2 = \frac{[\mathrm{B}_1(x)]^2 (r_1^2 + s_1^2)}{2[2]_q^4}. \tag{43}$$

Adding (38) and (40), using (43), we obtain

$$a_2^2 = \frac{[\mathrm{B}_1(x)]^3 (r_2 + s_2)}{2[2]_q ([3]_q [\mathrm{B}_1(x)]^2 - 2[2]_q \mathrm{B}_2(x))}. \tag{44}$$

Using relation (5), from (34) for r_2 and s_2, we get (28).
Using (41) and (42), by subtracting (40) from (38), we get

$$a_3 = \frac{[\mathrm{B}_1(x)](r_2 - s_2) + \frac{\mathrm{B}_2(x)}{2!}(r_1^2 - s_1^2)}{3[2]_q^2} + a_2^2$$

$$= \frac{[\mathrm{B}_1(x)](r_2 - s_2) + \frac{\mathrm{B}_2(x)}{2!}(r_1^2 - s_1^2)}{3[2]_q^2} + \frac{[\mathrm{B}_1(x)]^2(r_1^2 + s_1^2)}{2[2]_q^4}. \tag{45}$$

Once again applying (41) and using (5), for the coefficients, r_1, s_1, r_2, s_2, we deduce (29). □

5.The Fekete-Szegö Problem for the Function Class $\mathrm{C}_s^{\Sigma,q}(x)$

We obtain the Fekete-Szegö inequality for the class $\mathrm{C}_s^{\Sigma,q}(x)$ due to the result of Zaprawa[35]

Theorem 5.1. *If f given by (1) is in the class $\mathrm{C}_s^{\sum,q}(x)$, where $\mu \in \mathbb{R}$, then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\mathrm{B}_1(x)}{3[2]_q}, & |h(\mu)| \leq \frac{1}{12}, \\ 2\mathrm{B}_1(x)|h(\mu)|, & |h(\mu)| \geq \frac{1}{12}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)[\mathrm{B}_1(x)]^2}{2[2]_q([3]_q[\mathrm{B}_1(x)]^2 - 2[2]_q\mathrm{B}_2(x))}.$$

Proof. If $f \in \mathrm{C}_s^{\Sigma,q}(x)$ is given by (1), from (44) and (45), we have

$$\begin{aligned} & a_3 - \mu a_2^2 \\ &= \frac{\mathrm{B}_1(x)(r_2 - s_2)}{3[2]_q^2} + (1-\mu)a_2^2 \\ &= \frac{\mathrm{B}_1(x)(r_2 - s_2)}{3[2]_q^2} + \frac{(1-\mu)[\mathrm{B}_1(x)]^3(r_2 + s_2)}{2[2]_q([3]_q[\mathrm{B}_1(x)]^2 - 2[2]_q\mathrm{B}_2(x))} \\ &= \mathrm{B}_1(x)\left[\frac{r_2 - s_2}{3[2]_q^2} + \frac{(1-\mu)[\mathrm{B}_1(x)]^2 r_2}{2[2]_q(3[\mathrm{B}_1(x)]^2 - 4\mathrm{B}_2(x))} + \frac{(1-\mu)[\mathrm{B}_1(x)]^2 s_2}{2[2]_q(3[\mathrm{B}_1(x)]^2 - 4\mathrm{B}_2(x))}\right] \\ &= \mathrm{B}_1(x)\left[\left(h(\mu) + \frac{1}{12}\right) r_2 + \left(h(\mu) - \frac{1}{12}\right) s_2\right], \end{aligned}$$

where

$$h(\mu) = \frac{(1-\mu)[\mathrm{B}_1(x)]^2}{2[2]_q(3[\mathrm{B}_1(x)]^2 - 4\mathrm{B}_2(x))}.$$

Now, by using (5)

$$a_3 - \mu a_2^2 = \left(x - \frac{1}{2}\right)\left[\left(h(\mu) + \frac{1}{12}\right) r_2 + \left(h(\mu) - \frac{1}{12}\right) s_2\right],$$

where

$$h(\mu) = \frac{(1-\mu)\left(x - \frac{1}{2}\right)^2}{2[2]_q(3\left(x - \frac{1}{2}\right)^2 - 4(x^2 - x + \frac{1}{6}))}.$$

Therefore, given (5) and (34), we conclude that the necessary inequality holds. □

6.Conclusions

We introduce and investigate new subclasses of bi-univalent functions in $\mathbb{U}$ associated with Bernoulli polynomials and satisfying subordination conditions. Moreover, we obtain upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and Fekete–Szegö problem $|a_3 - \mu a_2^2|$ for functions in these subclasses. The approach employed here has also been extended to generate new bi-univalent function subfamilies using the other special functions. The researchers may carry out the linked outcomes in practice.

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