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INVESTIGATING WEAK SOLUTIONS FOR A SINGULAR AND DEGENERATE SEMILINEAR PARABOLIC EQUATION WITH A NONLINEAR INTEGRAL CONDITION

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Abstract. This article aims to study the existence and uniqueness of a weak solution for a singular and degenerate nonlinear parabolic equation with a generalized nonlinear integral condition of the second type. The proof of the existence and uniqueness of the weak solution to such a problem will be proceeded with in three steps. In the same regard, the solvability of the linear case of the problem at hand will be handled with the use of the Faedo-Galerkin method, a priori estimate, and by imposing some nonlinear conditions of the second kind.

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1. Introduction

Degenerate parabolic equations are cropped up in mathematical modelling of a wide range of natural phenomena and dynamic processes in science and engineering, such as plate deviation theory, fluid flows in lungs, episodic vibration of elastic beams, physical flows including ice formation, noise removal and edge preservation in images, oscillation control in bridge slabs, floor systems, large window glasses, finance, and airplane wings [1, 2, 3, 4, 5].

The nonlinear process motivated a lot of researchers to develop new methods in most mathematics disciplines, especially partial differential equations and analysis [6, 7, 8, 9, 10, 11]. These pose challenges for the study of nonlinear evolution problems, which are supposed to describe a real phenomenon, and the most estimated in reality are singular and degenerate [12, 13, 14]. The preceding question gains in mathematical difficulty when we start from the well-posedness

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of the problem with the nonlinear integral condition, which makes the evolution problem complex localization effects. The subject of the integral conditions were studied and introduced by many researchers, see [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], and it may be applied in many problems like problems discussed in [30, 31] . It is, however, of the first type

$$\int_{\Omega} u(x,t)dx = E(t), \quad \int_{\Omega} k(x,t)u(x,t)dx = 0,$$

where $t \in (0,T)$ and $\Omega \subset \mathbb{R}^2$ for which $x \in \Omega = [0,1]$. In view of this perspective, we consider the following nonlinear problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - (x^{\alpha} u_x)_x = f(x,t,u), \quad \forall (x,t) \in Q, \\ &u(x,0) = \varphi(x), \quad \forall x \in (0,1), \\ &u_x(0,t) = \int_0^1 k_1(x)h(u)(x,t)dx, \quad \forall t \in (0,T), \\ &u_x(1,t) = \int_0^1 k_2(x)g(u)(x,t)dx, \quad \forall t \in (0,T), \end{aligned} \tag{1}$$

where $\varphi \in L^2(0.1)$, k_1, k_2 are two weight continuous functions on $[0,1]$, and $f \in L^2(Q)$ is Lipschitzian function, which means that there exists a positive constant k such that

$$\|f(x,t,u_1) - f(x,t,u_2)\|_{L^2(Q)} \leq k\left(\|u_1 - u_2\|_{L^2(Q)}\right),$$

for all $u_1, u_2 \in L^2(Q)$, while the functions g and h are respectively verify the following two inequalities:

$$\begin{aligned} \|g(x,t,u)\|_{L^2(Q)} &\leqslant C_0 \|u\|_{L^2(Q)}, \\ \|h(x,t,u)\|_{L^2(Q)} &\leqslant C_1 \|u\|_{L^2(Q)}, \end{aligned} \tag{2}$$

where C_0 and C_1 are two positive constants. It is necessary to point out that problem (1) is singular and degenerate because the coefficients of u_x and u_{xx} tend to ∞ and 0 as $x \to 0$.

2. Study of the linear problem

This section is divided into two subsections; the first one aims to set a corresponding linear problem to (1), whereas the second subsection aims to study the existence of a weak solution to such a new problem.

2.1. Position of the problem. Herein, we let the domain

$$Q = \left\{(x,t) \in \mathbb{R}^2,\ 0 < x < 1, 0 < t < T\right\}.$$

Also, we consider the following linear problem:

$$\begin{aligned} & u_t - a(x^{\alpha}u_x)_x = f(x,t), \quad \forall (x,t) \in Q, \\ & u(x,0) = \varphi(x), \quad \forall x \in (0,1), \\ & u_x(0,t) = \int_0^1 k_1(x)h(u)(x,t)dx, \quad \forall t \in (0,T), \\ & u_x(1,t) = \int_0^1 k_2(x)g(u)(x,t)dx, \quad \forall t \in (0,T), \end{aligned} \tag{3}$$

where $\alpha \in (0,1)$ and the functions f, φ, k_1, and k_2 are known functions. In particular, $f \in L^2(Q)$, $\varphi \in L^2(0.1)$, and k_1, k_2 are two weight continuous functions defined on $[0,1]$.

Defn 2.1. *For all $v \in V$ and $t \in [0,T]$, the weak solution of the problem (3) is a function that verifies*

(i) $u \in L^2\left(0,T;\ H^1_{x^{\frac{\alpha}{2}}}(\Omega)\right) \cap L^{\infty}\left(0,T;\ L^2(\Omega)\right)$,

(ii) *u admits a strong derivative* $\dfrac{\partial u}{\partial t} \in L^2\left(0,T;\ L^2(\Omega)\right)$,

(iii) $u(0) = \varphi$,

(iv) $(u_t, v) - a\left((x^{\alpha}u_x)_x, v\right) = (f, v)$.

2.2. Existence of weak solution of problem (3). In the part, we intend to study of the existence of the weak solution of problem (3). For this purpose, we introduce the next theoretical result.

Thm 2.2. *Suppose that $f \in L^2(0,T,L^2(\Omega))$ and $\varphi \in H^1(\Omega)$. For every $\varepsilon, \delta > 0$ satisfying*

$$\left(1 - \left(2\varepsilon + \frac{1}{2\delta}\right)\right) > 0$$

and

$$\left(\frac{a}{2} - 2\varepsilon\right) > 0,$$

then the main problem admits a solution u such that

$$u \in L^2(0,T,H^1_{x^{\frac{\alpha}{2}}}(\Omega)) \cap L^{\infty}(0,T,L^2(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,T,L^2(\Omega)).$$

The demonstration of the existence of the solution to the problem (3) is based on the Faedo-Galerkin method, which consists of carrying out the following three steps:

Step 1: Construction of the approximate solutions

Herein, the space V is separable, then there exists a sequence $w_1, w_2, \cdots, w_m$

possessing the following properties:

$$\begin{cases} w_i \in V, & \forall i, \\ w_1, w_2, \cdots, w_m & \text{are linearly independent,} \\ V_m = \langle \{w_1, w_2, \cdots, w_m\} \rangle & \text{is dense in } V. \end{cases} \tag{4}$$

In particular, we have

$$\forall \varphi \in V \implies \exists (\alpha_{km})_m \in IN^* \tag{5}$$

for which

$$\varphi_m = \sum_{k=1}^{m} \alpha_{km} w_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \tag{6}$$

Faedo Galerkin approximation consists in searching for any integer $m \geq 1$ for which the function

$$t \mapsto u_m(x,t) = \sum_{i=1}^{m} g_{im}(t) w_i(x)$$

verifies

$$\begin{cases} u_m(t) \in V_m, \quad \forall t \in [0,T], \\ ((u_m(t))_t, w_k) + A(u_m(t), w_k) = (f(t), w_k), \end{cases} \tag{7}$$

$\forall k = \overline{1, m}$, where

$$\begin{aligned} ((u_m(t))_t, w_k) &= \left(\left(\sum_{i=1}^{m} g_{im}(t) w_i \right)_t, w_k \right) \\ &= \left(\sum_{i=1}^{m} \frac{\partial g_{im}}{\partial t}(t) w_i(x), w_k \right) \\ &= \sum_{i=1}^{m} (w_i, w_k) \frac{\partial g_{im}}{\partial t}(t), \end{aligned} \tag{8}$$

$$\begin{aligned} A(u_m(t), w_k) &= \left(\sum_{i=1}^{m} g_{im}(t) (x^\alpha (w_i)_x)_x, w_k \right) \\ &= \sum_{i=1}^{m} g_{im}(t) \left[-\int_\Omega x^\alpha \frac{\partial w_i}{\partial x} \frac{\partial w_k}{\partial x} dx + \frac{\partial w_i}{\partial x}(1) w_k(1) \right] \\ &= -\sum_{i=1}^{m} g_{im}(t) \int_\Omega x^\alpha \frac{\partial w_i(x)}{\partial x} \frac{\partial w_k(x)}{\partial x} dx + a \sum_{i=1}^{m} g_{im}(t) \frac{\partial w_i}{\partial x}(1) w_k(1) \\ &= -a \sum_{i=1}^{m} \left(x^{\frac{\alpha}{2}} w_i, x^{\frac{\alpha}{2}} w_k \right) g_{im}(t) + a \sum_{i=1}^{m} g_{im}(t) \frac{\partial w_i}{\partial x}(1) w_k(1), \end{aligned} \tag{9}$$

and

$$u_m(0) = \sum_{i=1}^{m} g_{im}(0) w_i(x) = \varphi_m = \sum_{i=1}^{m} \alpha_{im} w_i(x).$$

As a sequence, we obtain the following system of first-order nonlinear differential equations:

$$\left\{\begin{array}{l} \sum_{i=1}^{m}(w_i,w_k)\frac{\partial g_{im}}{\partial t}(t)+a\sum_{i=1}^{m}\left(x^{\frac{\alpha}{2}}w_i,x^{\frac{\alpha}{2}}w_k\right)g_{im}(t) \\ =(f(t),w_k)+a\sum_{i=1}^{m}g_{im}(t)\dfrac{\partial w_i}{\partial x}(1)\,w_k(1), \\ g_{im}(0)=\alpha_{im},\quad \forall i=\overline{1,m}. \end{array}\right. \tag{10}$$

To deal with this system, we consider the vectors

$$g_m=(g_{1m}(t),\cdots,g_{mm}(t)),\ f_m=((f,w_1),\cdots,(f,w_m))$$

and the matrices

$$B_m=((w_i,w_j))_{\substack{1\le i\le m\\1\le j\le m}},\ A_m=\left(\left(x^{\frac{\alpha}{2}}w_i,x^{\frac{\alpha}{2}}w_j\right)\right)_{\substack{1\le i\le m\\1\le j\le m}}$$

and

$$C_m=\left(\frac{\partial w_i}{\partial x}(1)\cdot w_j(1)\right)_{\substack{1\le i\le m\\1\le j\le m}}.$$

Thus, we can rewrite problem (10) in the matrix form as follows:

$$\left\{\begin{array}{l} B_m\frac{\partial g_m}{\partial t}(t)+aA_mg_m=f_m+aC_mg_m, \\ g_m(0)=(\alpha_{im})_{1\le i\le m}. \end{array}\right.$$

Due to the matrix B_m is a diagonal, then its entries are linearly independent. This implies that $\det B_m\neq 0$, and so it is invertible. Hence, g_m is the solution of

$$\left\{\begin{array}{l} \dfrac{\partial g_m}{\partial t}(t)+\left(aB_m^{-1}A_m-aB_m^{-1}C_m\right)g_m=B_m^{-1}f_m, \\ g_m(0)=(\alpha_{im})_{1\le i\le m}. \end{array}\right. \tag{11}$$

With the help of applying Theorem 3.1 reported in [32] that addresses the usual existence and uniqueness issues for ordinary differential systems, we can have the matrix

$$aB_m^{-1}A_m+bB_m^{-1}B_m-aB_m^{-1}C_m$$

with constant coefficients and the vector $B_m^{-1}f_m$ for which the continuous functions are majorized by integrable functions on $(0,T)$. As a result, we can conclude that there exists a t_m depends only on $|\alpha_{im}|$ such that the nonhomogeneous problem (11) admits a unique local solution $g_m(t)\in C[0,t_m]$ in the interval $[0,t_m]$ for which $g_m^{'}(t)\in L^2[0,T]$. But due to the elements of the vector $B_m^{-1}f_m$ are majorized by integrable functions on $(0,T)$, the solution can then be extended to $[0,T]$.

Step 2: A priori estimate

To apply on the method of a priori estimate, we multiply the equation of (7) by

$g_{km}(t)$ and we sum the result over k to obtain

$$\sum_{k=1}^{m}\left((u_m(t))_t\,,w_k\right)g_{km}(t)+a\sum_{k=1}^{m}\left(\frac{\partial}{\partial x}\left(x^{\alpha}\frac{\partial u_m}{\partial x}\right)(t),\frac{\partial w_k}{\partial x}\right)g_{km}(t) \\ =\sum_{k=1}^{m}\left(f(t),w_k\right)g_{km}(t). \tag{12}$$

Consequently, we obtain

$$\left((u_m(t))_t\,,u_m(t)\right)+a\left(\frac{\partial}{\partial x}\left(x^{\alpha}\frac{\partial u_m}{\partial x}\right),u_m\right)=\left(f(t),u_m(t)\right). \tag{13}$$

Thus, we get

$$\frac{1}{2}\frac{d}{dt}\left\|u_m\right\|^2_{L^2(\Omega)}+a\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} \\ =\left(f(t),u_m(t)\right)+a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(1\right)\sum_{k=1}^{m}g_{km}(t)w_k(1). \tag{14}$$

By using Cauchy inequality with ε, i.e.,

$$|ab|\leq\frac{a^2}{2\varepsilon}+\frac{\varepsilon b^2}{2},$$

we get

$$\frac{1}{2}\frac{d}{dt}\left\|u_m\right\|^2_{L^2(\Omega)}+a\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} \\ \leq\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(\Omega)}+\frac{\varepsilon}{2}\left\|u_m\right\|^2_{L^2(\Omega)}+a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(1\right)\sum_{k=1}^{m}g_{km}(t)w_k(1). \tag{15}$$

Now, integrating the above inequality over 0 to t yields

$$\frac{1}{2}\left\|u_m\right\|^2_{L^2(\Omega)}-\frac{1}{2}\left\|u_m\left(0\right)\right\|^2_{L^2(\Omega)}+a\int_0^t\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}d\tau\ \leq\frac{1}{2\varepsilon}\int_0^t\left\|f\right\|^2_{L^2(\Omega)}d\tau \\ +\frac{\varepsilon}{2}\int_0^t\left\|u_m\right\|^2_{L^2(\Omega)}d\tau+a\int_0^t\left(\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(1\right)\sum_{k=1}^{m}g_{km}(t)w_k(1)\right)d\tau.$$

Herein, we should give an estimate of the third part of right-hand side of previous inequality, i.e.,

$$\int_0^t\left(\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(1\right)\sum_{k=1}^{m}g_{km}(t)w_k(1)\right)d\tau=\int_0^t\left(\frac{\partial u_m}{\partial x}\left(1,t\right)\cdot u_m\left(1,t\right)\right)d\tau.$$

By using Cauchy inequality once again with ε, we obtain

$$\int_0^t \left(\frac{\partial u_m}{\partial x}(1,\tau)\cdot u_m(1,\tau)\right)d\tau < \frac{\varepsilon}{2}\int_0^t u_m^2(1,\tau)\,d\tau + \frac{1}{2\varepsilon}\int_0^t (u_m)_x^2(1,\tau)\,d\tau$$
$$\leq \frac{\varepsilon}{2}\int_0^t \left[2\int_x^1 u_y^2 dy + 2u^2\right]d\tau + \frac{1}{2\varepsilon}\int_0^t \left[\int_0^1 k(x,t)\,g(u(x,t))dx\right]^2 d\tau.$$

Consequently. with the use of Bochner inquality as well as the Sobolev embding, we obtain

$$\int_0^t \left(\frac{\partial u_m}{\partial x}(1,\tau)\cdot u_m(1,\tau)\right)d\tau$$
$$\leq \varepsilon\left[\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \|u_m\|^2_{L^2(Q)}\right] + \frac{K}{2\varepsilon}\left\|\int_0^1 g(u_m)dx\right\|^2_{L^2(0,T)}$$
$$\leq \varepsilon\left[\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \|u_m\|^2_{L^2(Q)}\right] + \frac{K}{2\varepsilon}\left\|\int_0^1 g(u_m)dx\right\|^2_{L^2(0,T)}$$
$$\leq \varepsilon\left[\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \|u_m\|^2_{L^2(Q)}\right] + \frac{K}{2\varepsilon}\left(\int_0^1 \|g(u_m)\|_{L^2(0,T)}\right)^2$$
$$\leq \varepsilon\left[\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \|u_m\|^2_{L^2(Q)}\right] + \frac{K}{2\varepsilon}\left(\int_0^1 \|g(u_m)\|_{L^2(0,T)}\right)^2$$
$$\leq \varepsilon\left[\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \|u_m\|^2_{L^2(Q)}\right] + \frac{K}{2\varepsilon}\|g(u_m)\|^2_{L^2(Q)}$$
$$\leq \varepsilon\left[\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \|u_m\|^2_{L^2(Q)}\right] + \frac{C_0K}{2\varepsilon}\|u_m\|^2_{L^2(Q)},$$

where the constant

$$K = \max\int_Q k^2(x,t)\,dxdt$$

and C_0 is the constant of the condition (2). As a result, we obtain

$$\frac{1}{2}\|u_m\|^2_{L^2(\Omega)} + a\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} - \varepsilon\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)}$$
$$\leq \frac{1}{2\delta}\|f\|^2_{L^2(0,T;\,L^2(\Omega))} + \frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)} + \left(\frac{\delta}{2}+\varepsilon+\frac{C_0K}{2\varepsilon}\right)\|u_m\|^2_{L^2(0,T;\,L^2(\Omega))}.$$

By using Gronwell Lemma and take $\delta = 1$, we get

$$\|u_m\|^2_{L^\infty(0,T;\,L^2(\Omega))} + 2(a-\varepsilon)\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)}$$
$$\leq \exp\left(\left(\frac{1}{2}+2\varepsilon+\frac{C_0K}{\varepsilon}\right)T\right)\left[\|f\|^2_{L^2(0,T;\,L^2(\Omega))} + \|\varphi_m\|^2_{L^2(\Omega)}\right],$$

which implies

$$\|u_m\|^2_{L^\infty(0,T;\ L^2(\Omega))} + 2\left(a-\varepsilon\right)\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} \leq \frac{\exp\left(\left(\frac{1}{2}+2\varepsilon+\frac{C_0K}{\varepsilon}\right)T\right)}{\min\left\{1,2\left(a-\varepsilon\right)\right\}}\left(\|f\|^2_{L^2(0,T;\ L^2(\Omega))}+\|\varphi_m\|^2_{L^2(\Omega)}\right).$$

So, we get

$$\|u_m\|^2_{L^\infty(0,T;\ L^2(\Omega))} + \left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} \leq R_1\left(\|f\|^2_{L^2(0,T;\ L^2(\Omega))}+\|\varphi_m\|^2_{L^2(\Omega)}\right), \tag{16}$$

where

$$R_1 = \frac{\exp(\left(\frac{1}{2}+2\varepsilon+\frac{C_0K}{\varepsilon}\right)T)}{\min\left\{1,2\left(a-\varepsilon\right)\right\}}.$$

Now, by using the same formulation variational (3), multiplying the new equation by $g'_{km}(t)$, and then summing the result over k, we get

$$\int_Q\left(\frac{\partial u_m}{\partial t}\right)^2 dxdt + a\int_Q x^\alpha\frac{\partial u_m}{\partial x}\cdot\frac{\partial\left(u_m\right)_t}{\partial x}dx - a\int_0^\tau x^\alpha\frac{\partial u_m}{\partial x}\cdot\frac{\partial u_m}{\partial t}\bigg|_{x=0}^{x=1}dt = \int_Q f\cdot\frac{\partial u_m}{\partial t}dx.$$

Thus, we obtain

$$\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\left(\tau\right)\right\|^2_{L^2(\Omega)} = \int_0^t\left(f\left(t\right),\frac{\partial u_m}{\partial t}\right) + a\int_0^\tau\frac{\partial u_m}{\partial x}\left(l,t\right)\cdot\frac{\partial u_m}{\partial t}\left(l,t\right)dt + \frac{a}{2}\left\|x^{\frac{\alpha}{2}}\frac{\partial\varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}. \tag{17}$$

Hence, with using (17) and Cauchy inequality with ε, we get

$$\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\left(\tau\right)\right\|^2_{L^2(\Omega)} = \int_0^t\left(f\left(t\right),\frac{\partial u_m}{\partial t}\right) + a\int_0^\tau\left(\int_0^1 k(x,t)g\left(u_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}\left(1,t\right)dt + \frac{a}{2}\left\|\frac{\partial\varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}$$

Then, we get

$$
\begin{aligned}
&\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}(\tau)\right\|^2_{L^2(\Omega)} \leq \int_0^t \left(f(t), \frac{\partial u_m}{\partial t}\right) \\
&\quad + a\int_0^\tau \left(\int_0^1 k(x,t)g(u_m)(x,t)dx\right)\cdot \frac{\partial u_m}{\partial t}(1,t)\,dt + \frac{a}{2}\left\|x^{\frac{\alpha}{2}}\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} \\
&\leq \int_0^t \left(f(t), \frac{\partial u_m}{\partial t}\right) + \frac{a}{2}\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} \\
&\quad + a\left\|\int_0^1 k(x,t)g(u_m)(x,t)dx\right\|_{L^\infty(0,T)}\left\|\int_0^1 \frac{\partial^2 u_m}{\partial t\partial x} - \frac{\partial u_m}{\partial t}\right\|_{L^1(0,T)} \\
&\leq \int_0^t \left(f(t), \frac{\partial u_m}{\partial t}\right) + \frac{a}{2}\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} \\
&\quad + a\left[\frac{1}{2\varepsilon}\left\|\int_0^1 k(x,t)g(u_m)(x,t)dx\right\|^2_{L^\infty(0,T)} + \frac{\varepsilon}{2}\left\|\int_0^1 \frac{\partial^2 u_m}{\partial t\partial x} - \frac{\partial u_m}{\partial t}\right\|^2_{L^1(0,T)}\right] \\
&\leq \int_0^t \left(f(t), \frac{\partial u_m}{\partial t}\right) + a\left[\frac{1}{2\varepsilon}\left\|\int_0^1 k(x,t)g(u_m)(x,t)dx\right\|^2_{L^\infty(0,T)}\right. \\
&\quad \left. + \frac{a}{2}\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} + \varepsilon\left(\left\|\int_0^1 \frac{\partial^2 u_m}{\partial t\partial x}\right\|^2_{L^1(0,T)} + \left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^1(0,T)}\right)\right] \\
&\leq \int_0^t \left(f(t), \frac{\partial u_m}{\partial t}\right) + a\left[\frac{1}{2\varepsilon}\left\|\int_0^1 k(x,t)g(u_m)(x,t)dx\right\|^2_{L^2(0,T)}\right. \\
&\quad \left. + \frac{a}{2}\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} + \varepsilon\left(\left(\int_0^1 \frac{\partial u_m}{\partial x} - \int_0^1 \frac{\partial \varphi_m}{\partial x}\right)^2 + \left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(0,T)}\right)\right] \\
&\leq \int_0^t \left(f(t), \frac{\partial u_m}{\partial t}\right) + a\frac{K}{2\varepsilon}\|u_m\|^2_{L^2(Q)} + \frac{a}{2}\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} \\
&\quad + \varepsilon\left[2\left(\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(Q)}\right) + \left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}\right].
\end{aligned}
$$

In other words, we have

$$
\begin{aligned}
\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}(\tau)\right\|^2_{L^2(\Omega)} &\leq \int_0^t \left(f(t), \frac{\partial u_m}{\partial t}\right) + a\frac{K}{2\varepsilon}\|u_m\|^2_{L^2(Q)} \\
&+ 2\varepsilon\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + 2\varepsilon\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(0,T)} + \left(\frac{a}{2} + 2\varepsilon\right)\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Therefore, we get

$$\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}(\tau)\right\|^2_{L^2(\Omega)} \leq a\frac{K}{2\varepsilon}\|u_m\|^2_{L^2(Q)} + 2\varepsilon\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)}$$
$$+\left(2\varepsilon+\frac{1}{2\delta}\right)\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \left(\frac{a}{2}+2\varepsilon\right)\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} + \frac{\delta}{2}\|f\|^2_{L^2(Q)}$$

So, we finally obtain

$$\left(1-\left(2\varepsilon+\frac{1}{2\delta}\right)\right)\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \left(\frac{a}{2}-2\varepsilon\right)\left\|x^{\frac{\alpha}{2}}\frac{\partial u_m}{\partial x}(\tau)\right\|^2_{L^2(\Omega)}$$
$$\leq a\frac{K}{2\varepsilon}\|u_m\|^2_{L^2(Q)} + \left(\frac{a}{2}+2\varepsilon\right)\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} + \frac{\delta}{2}\|f\|^2_{L^2(Q)}$$
$$\leq a\frac{K}{2\varepsilon}R_1\left(\|f\|^2_{L^2(0,T;\ L^2(\Omega))} + \|\varphi_m\|^2_{L^2(\Omega)}\right) + \left(\frac{a}{2}+2\varepsilon\right)\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} + \frac{\delta}{2}\|f\|^2_{L^2}$$
$$\leq R_2,$$

where

$$R_2 = \frac{\max\left(a\frac{K}{2\varepsilon}R_1, \left(\frac{a}{2}+2\varepsilon\right), \frac{\delta}{2}\right)}{\min\left(\left(1-\left(2\varepsilon+\frac{1}{2\delta}\right)\right), \left(\frac{a}{2}-2\varepsilon\right)\right)}\left(\|f\|^2_{L^2(Q)} + \left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{L^2(\Omega)}\right)$$

Hence, we have

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(0,T;\ L^2(\Omega))} \leq R_2. \tag{18}$$

It follows from (16) that the solution to the initial value problem for the system of ODE (10) can be extended to $[0,T]$. This confirms what we have demonstrated in the first step. So, when $m \to +\infty$ in (18), we obtain

$$\begin{cases} u_m \text{ uniformly bounded in } L^\infty(0,T;\ L^2(\Omega)), \\ x^{\frac{\alpha}{2}}u_m \text{ uniformly bounded in } L^2(0,T;\ H^1(\Omega)), \\ (u_m)_t \text{ uniformly bounded in } L^2\left(0,T;\ L^2(\Omega)\right). \end{cases} \tag{19}$$

Step 3: Convergence and result of existence

To complete address our investigation, we state and prove the next theoretical result that concerns with the convergence of the existence theorem.

Thm 2.3. *There is a function*

$$u \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

with

$$\frac{\partial u}{\partial t} \in L^2\left(0,T;\ L^2(\Omega)\right)$$

and a subsequence denoted by $(u_{m_k})_k \subseteq (u_m)_m$ *such that*

$$\begin{cases} x^{\frac{\alpha}{2}}u_{m_k} \rightharpoonup x^{\frac{\alpha}{2}}u & \text{in } L^2(0,T\ ;H^1(\Omega)), \\ \dfrac{\partial u_{m_k}}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{in } L^2\left(0,T\ ;L^2(\Omega)\right), \end{cases}$$

when $m \longrightarrow +\infty$.

Proof. Herein, we consider (u_{m_k}), $\left(\dfrac{\partial u_{m_k}}{\partial t}\right)$ are subsequences of (u_m) and $(u_m)_t$, respectively, such that

$$x^{\frac{\alpha}{2}} u_{m_k} \rightharpoonup x^{\frac{\alpha}{2}} u \qquad \text{in } L^2(0,T;\ H^1(\Omega)), \tag{20}$$

and

$$\tfrac{\partial u_{m_k}}{\partial t} \rightharpoonup w \qquad \text{in } L^2\left(0,T;\ L^2(\Omega)\right). \tag{21}$$

We know, according to Relikh-Kondrachoff's Theorem, that the injection of $H^1(Q)$ into $L^2(Q)$ is compact. Thus, like the results of Rellich's Theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence which converges strongly in $L^2(Q)$, we then have

$$x^{\frac{\alpha}{2}} u_{m_k} \longrightarrow x^{\frac{\alpha}{2}} u \qquad \text{in } L^2(Q). \tag{22}$$

On the other hand, we can consider that there is a subsequence of $(u_{m_k})_k$ denoted by u_{m_k} that converges almost everywhere to u such that

$$x^{\frac{\alpha}{2}} u_{m_k} \longrightarrow x^{\frac{\alpha}{2}} u \qquad \text{almost everywhere } Q. \tag{23}$$

Thus, it remains to demonstrate that $w = \frac{\partial u}{\partial t}$, which suffices to prove

$$u(t) = \varphi + \int_0^t w(\tau) d\tau a \tag{24}$$

as

$$u_{m_k} \rightharpoonup u \qquad \text{in } L^2(0,T\ ;\ L^2(\Omega)).$$

In fact, the proof of (24) is equivalent to demonstrate that

$$u_{m_k} \rightharpoonup (\varphi + \chi) \qquad \text{in } L^2(0,T\ ;\ L^2(\Omega)),$$

which means

$$\lim\left((u_{m_k} - \varphi - \chi), v\right)_{L^2(0,T\ ;\ L^2(\Omega))} = 0,$$

$\forall v \in L^2(0,T\ ;\ L^2(\Omega))$, where

$$\chi(t) = \int_0^t w(\tau) d\tau.$$

Now, by using the equality

$$u_{m_k} - \varphi_{m_k} = \int_0^t \frac{\partial u_{m_k}}{\partial \tau} d\tau, \text{ for all } t \in [0,T],$$

and based on the assumptions

$$u_{m_k} \in L^2(0,T; V_{m_k}) \text{ and } (u_{m_k})_t \in L^2(0,T; V_{m_k}),$$

we can obtain

$$
\begin{aligned}
&\left(\left(u_{m_k}-\varphi-\int_0^t w(\tau)d\tau\right),v\right)_{L^2(0,T;L^2(\Omega))}\\
&=\left(\left(u_{m_k}-\varphi_{m_k}-\int_0^t w(\tau)d\tau\right),v\right)_{L^2(0,T;L^2(\Omega))}+\left(\left(\varphi_{m_k}-\varphi\right),v\right)_{L^2(0,T;L^2(\Omega))}\\
&=\left(\int_0^t\left(\frac{\partial u_{m_k}}{\partial\tau}-w(\tau)\right)d\tau,v\right)_{L^2(0,T;L^2(\Omega))}+\left(\left(\varphi_{m_k}-\varphi\right),v\right)_{L^2(0,T;L^2(\Omega))},
\end{aligned}
$$

$\forall t\in[0,T]$. By virtue of **(ii)**, it comes

$$
\begin{aligned}
&\left(\left(u_{m_k}-\varphi-\int_0^t w(\tau)d\tau\right),v\right)_{L^2(0,T\ ;\ L^2(\Omega))}\\
&=\int_0^t\left(\left(\frac{\partial u_{m_k}}{\partial\tau}-w(\tau)\right),v\right)_{L^2(0,T;L^2(\Omega))}d\tau+\left(x^{\frac{\alpha}{2}}\left(\varphi_{m_k}-\varphi\right),v\right)_{L^2(0,T;L^2(\Omega))}
\end{aligned}
$$

$\forall t\in[0,T]$. On the one hand, we have

$$
\lim_{k\longrightarrow\infty}\int_0^t\left(\left(\frac{\partial u_{m_k}}{\partial\tau}-w(\tau)\right),v\right)_{L^2(0,T\ ;\ L^2(\Omega))}d\tau=0, \tag{25}
$$

for $t\in[0,T]$. Also, we have

$$
\lim_{k\longrightarrow\infty}\left(\varphi_{m_k}-\varphi,v\right)_{L^2(0,T\ ;\ L^2(\Omega))}=0. \tag{26}
$$

So, we get

$$
\lim_{k\longrightarrow\infty}\left(\left(u_{m_k}-\varphi-\chi\right),v\right)_{L^2(0,T\ ;\ L^2(\Omega))}=0,
$$

for all $v\in L^2\left(0,T;L^2\left(\Omega\right)\right)$. □

Thm 2.4. *The function u of Theorem (2.3) is the a weak solution to problem (3) in the sense of Definition (2.1).*

Proof. From Theorem (2.3), we can show that the limit function u satisfies the first two conditions of Definition (2.1). In this regard, we intend to demonstrate **(iii)**. To this end, according to Theorem (2.3), we can have

$$
u_{m_k}(0)\rightharpoonup u(0)\qquad\text{in } L^2(\Omega).
$$

On the other hand, we have

$$
u_{m_k}(0)\longrightarrow\varphi\qquad\text{in } L^2(\Omega),
$$

which implies

$$
u_{m_k}(0)\rightharpoonup\varphi\qquad\text{in } L^2(\Omega).
$$

From the uniqueness of the limit, we get

$$
u(0)=\varphi.
$$

Still to demonstrate **(iv)**. For this purpose, we notice that

$$
\left(u_t,v\right)+a\left(u,v\right)=\left(f,v\right),\qquad\forall v\in V,\quad\forall t\in[0,T].
$$

Now, integrating (7) over $(0,T)$ yields

$$\int_0^t ((u_m(t))_t\,, w_k)\,d\tau - a\int_0^t ((x^\alpha\,(u_m)_x)_x, w_k)\,d\tau = \int_0^t (f(t), w_k)\,d\tau, \qquad (27)$$

for all $k=\overline{1,m}$ and $t\in[0,T]$. By using (18), recognizing that V_m is dense in V, and passing the result to the limit in (27), we obtain

$$\int_0^T (u_t, w_k)\,d\tau - a\int_0^T ((x^\alpha u_x)_x, w_k)\,d\tau = \int_0^T (f, w_k)\,d\tau, \quad \forall t\in[0,T]\,.$$

Hence, **(iv)** is verified. □

Cor 2.5. *The uniqueness of the solution of problem (3) comes straight through the estimate (16).*

3. Solvability of the weak solution of the nonlinear problem

This section is devoted to the proof of the existence and the uniqueness of the solution of the nonlinear problem (1):

$$\begin{cases} u_t - (x^\alpha u_x)_x = f\,(x,t,u)\,, & \forall\,(x,t)\in Q,\\ u\,(x,0) = \varphi\,(x)\,, & \forall x\in(0,1)\,,\\ u_x(0,t) = \int_0^1 k_1(x)h(u)(x,t)dx, & \forall t\in(0,T),\\ u_x(1,t) = \int_0^1 k_2(x)g(u)(x,t)dx, & \forall t\in(0,T)\,. \end{cases} \qquad (28)$$

In order to achieve this goal, we put

$$u = y + w,$$

such that w is a solution to the following nonlocal linear problem:

$$\begin{cases} w_t - (x^\alpha w_x)_x = 0, & \forall\,(x,t)\in Q,\\ w\,(x,0) = \varphi\,(x)\,, & \forall x\in(0,1)\,,\\ w_x(0,t) = \int_0^1 k_1(x)w(x,t)dx, & \forall t\in(0,T),\\ w_x(0,t) = \int_0^1 k_2(x)w(x,t)dx, & \forall t\in(0,T)\,. \end{cases}, \qquad (29)$$

and the solution

$$y = u - w$$

satisfies the following nonlocal nonlinear problem:

$$\mathcal{L}y = y_t - (x^\alpha y_x)_x = G\,(x,t,y)\,, \qquad (30)$$

with

$$y(x,0) = 0, \quad \forall x\in(0,1)\,, \qquad (31)$$

and

$$y_x(0,t) = y_x\,(1,t) = 0, \quad \forall t\in(0,t)\,, \qquad (32)$$

where

$$G\,(x,t,y) = f\,(x,t,y+w)\,.$$

Similarly to the function f, the function G is also Lipchitzian, which means that there is a positive constant k such that

$$\parallel G\left(x,t,u_1,\right)-G\left(x,t,u_2\right)\parallel_{L^2(Q)}\leq k\left(\|u_1-u_2\|_{L^2(0,T,H^1(0,1))}\right). \tag{33}$$

Herein, we build a recurring sequence starting with $y^{(0)}=0$. As a result, the sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ can be defined as follows: Given the element $y^{(n-1)}$, then for $n=1,2,3,\cdots$, we intend to solve the following problem:

$$\begin{cases} \dfrac{\partial y^{(n)}}{\partial t}-(x^\alpha y_x^{(n)})_x=G\left(x,t,y^{(n-1)}\right), \\ y^{(n)}\left(x,0\right)=0, \\ y^{(n)}(0,t)=y^{(n)}(1,t)=0. \end{cases} \tag{34}$$

According to the study of the previous linear problem, we fix the n each time. Then, problem (34) admits a unique solution $y^{(n)}\left(x,t\right)$. Now, by supposing

$$z^{(n)}\left(x,t\right)=y^{(n+1)}\left(x,t\right)-y^{(n)}\left(x,t\right),$$

we get the following new problem:

$$\begin{cases} \dfrac{\partial z^{(n)}}{\partial t}-(x^\alpha z_x^{(n)})_x=p^{(n-1)}(x,t), \\ z^{(n)}\left(x,0\right)=0, \\ z_x^{(n)}\left(0,t\right)=z_x^{(n)}(1,t)=0, \end{cases}, \tag{35}$$

where

$$p^{(n-1)}(x,t)=G\left(x,t,y^{(n)}\right)-G\left(x,t,y^{(n-1)}\right).$$

Now, multiply the equality

$$\frac{\partial z^{(n)}}{\partial t}-(x^\alpha z_x^{(n)})_x=p^{(n-1)}(x,t)$$

by $z^{(n)}$, and then integrate the result over Q_τ to get

$$\int_{Q_\tau}\frac{\partial z^{(n)}}{\partial t}\left(x,t\right)z^{(n)}(x,t)dxdt-\int_{Q_\tau}(x^\alpha z_x^{(n)})_x z^{(n)}\left(x,t\right)dxdt$$
$$=\int_{Q_\tau}p^{(n-1)}(x,t)z^{(n)}\left(x,t\right)dxdt.$$

By using the integration by parts and taking into account the initial and boundary conditions, we obtain

$$\frac{1}{2}\int_0^1(z^{(n)}\left(x,\tau\right))^2dx+\int_{Q_\tau}x^\alpha\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2dxdt$$
$$=\int_{Q_\tau}p^{(n-1)}(x,t)z^{(n)}\left(x,t\right)dxdt.$$

We apply now Cauchy Schwarz inequality on the second part of the above equation to obtain

$$\int_{Q_\tau} p^{(n-1)}(x,t)z^{(n)}(x,t)\,dxdt$$
$$\leqslant \frac{1}{2\varepsilon}\int_{Q^\tau} \mid p^{(n-1)}(x,t)\mid^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}(x,t)\right)^2 dxdt$$
$$\leqslant \frac{1}{2\varepsilon}\int_{Q^\tau} \mid G\left(x,t,y^{(n)}\right) - G\left(x,t,y^{(n-1)}\right)\mid^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}(x,t)\right)^2 dxdt.$$

Like G Lipschtizienne, we can have

$$\frac{k^2}{2\varepsilon}\int_{Q^\tau}(\mid y^{(n)} - y^{(n-1)}\mid)^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}(x,t)\right)^2 dxdt$$
$$\leqslant \frac{k^2}{2\varepsilon}\left\|z^{(n-1)}\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}(x,t)\right)^2 dxdt.$$

Consequently, we multiply the previous inequality by 2 and then apply Grenwell's Lemma to get

$$\int_0^1 (z^{(n)}(x,\tau))^2 dx + 2a\int_{Q_\tau} x^\alpha\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$\leqslant \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,L^2(0,1))}\exp\left(\int_0^\tau \varepsilon dt\right)$$
$$\leqslant \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,L^2(0,1))}\exp(\varepsilon T).$$

By integrating the above inequality over t, we obtain

$$\int_{Q_T}(z^{(n)}(x,\tau))^2 dxdt + 2aT\int_{Q_T} x^\alpha\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$\leqslant \frac{Tk^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(Q)}\exp(\varepsilon T),$$

which implies

$$\left\|z^{(n)}\right\|^2_{L^2(Q)} + \left\|\frac{\partial z^{(n)}}{\partial x}\right\|^2_{L^2_{\sqrt{x^\alpha}}(Q)} \leqslant \frac{Tk^2\exp(\varepsilon T)}{\varepsilon\min(1,2aT)}\left\|z^{(n-1)}\right\|^2_{L^2(Q)}$$

and so we obtain

$$\left\|z^{(n)}\right\|^2_{L^2(Q)} + \left\|\frac{\partial z^{(n)}}{\partial x}\right\|^2_{L^2_{\sqrt{x^\alpha}}(Q)}$$
$$\leqslant \frac{Tk^2\exp(\varepsilon T)}{\varepsilon\min(1,2T)}\left(\left\|z^{(n-1)}\right\|^2_{L^2(Q)} + \left\|\frac{\partial z^{(n-1)}}{\partial x}\right\|^2_{L^2_{\sqrt{x^\alpha}}(Q)}\right).$$

Putting

$$c = \frac{Tk^2 \exp(\varepsilon T)}{\varepsilon \min(1, 2T)}$$

yields

$$\left\| z^{(n)} \right\|_V^2 \leqslant c \left\| z^{(n-1)} \right\|_V^2, \tag{36}$$

where

$$V = \left\{ y,\ y \in L^2(Q), y_x \in L^2_{\sqrt{x^\alpha}}(Q) \right\}.$$

As a result, we have

$$\sum_{i=1}^{n-1} z^{(i)} = y^{(n)}.$$

According to the convergence criterion of the series, we notice that the series $\sum\limits_{n=1}^{\infty} z^{(n)}$ converges if $|c| < 1$. This, consequently, implies

$$\left| \frac{Tk^2 \exp(\varepsilon T)}{\varepsilon \min(1, 2T)} \right| < 1,$$

$$k \sqrt{\frac{T \exp(\varepsilon T)}{\varepsilon \min(1, 2T)}} < 1,$$

and

$$k < \sqrt{\frac{\varepsilon \min(1, 2T)}{T \exp(\varepsilon T)}}.$$

Therefore, $\left(y^{(n)}\right)_n$ converges on an element of V, say y. In this regard, it is necessary to show that

$$\lim_{n \longrightarrow \infty} y^{(n)}(x,t) = y(x,t)$$

is a solution to the problem (34) in V by showing that y satisfies

$$A(y, v) = \int_{Q_\tau} G(x,t,y) v(x,t) dxdt \quad \forall v \in O, \tag{37}$$

where

$$O = \left\{ v \in C^1(Q), v(0.t) = v(1.t) = 0, \forall t \in (0.T) \right\},$$

and

$$A\left(y^{(n)}, v\right) = \int_{Q_\tau} \frac{\partial y^{(n)}}{\partial t}(x,t) v(x,t) dxdt + \int_{Q_\tau} x^\alpha \frac{\partial y^{(n)}}{\partial x}(x,t) \frac{\partial v}{\partial x}(x,t) dxdt.$$

As a consequence, we have

$$\begin{aligned} A\left(y^{(n)} - y, v\right) = &\int_{Q_\tau} \frac{\partial (y^{(n)} - y)}{\partial t}(x,t) v(x,t) dxdt \\ &+ \int_{Q_\tau} x^\alpha \frac{\partial (y^{(n)} - y)}{\partial x}(x,t) \frac{\partial v}{\partial x}(x,t) dxdt. \end{aligned}$$

So, by applying Cauchy Schwartz inequality, we can get

$$A\left(y^{(n)}-y,v\right) \leqslant \| v \|_V \left\|(y^{(n)}-y)_t\right\|_V + \| v \|_V \left\|(y^{(n)}-y)_x\right\|_V .$$

On the other hand, due to

$$y^{(n)} \longrightarrow y \qquad \text{in } V,$$

we obtain

$$\begin{array}{lll} y^{(n)} & \longrightarrow & y \quad \text{in } L^2(Q), \\ y_t^{(n)} & \longrightarrow & y_t \quad \text{in } L^2(Q), \\ y_x^{(n)} & \longrightarrow & y_x \quad \text{in } L^2_{\sqrt{x^\alpha}}(Q). \end{array}$$

Hence, by taking the limit as $n \longrightarrow +\infty$, we get

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)}-y,v\right)=0.$$

3.1. Uniqueness of the weak solution. In order to address the uniqueness of the weak solution of the problem at hand, we let y_1 and y_2 be two solutions in V. Then, we can have

$$Z = y_1 - y_2,$$

which satisfies $Z \in V$ and

$$\mathcal{L}Z = \frac{\partial Z}{\partial t} - (x^\alpha Z_x(x,t))_x = \eta(x,t), \tag{38}$$

with

$$Z(x,0)=0, \quad \forall x \in (0,1), \tag{39}$$

and

$$Z(1,t)=Z(1,t)=0, \quad \forall t \in (0,t), \tag{40}$$

where

$$\eta(x,t) = G(x,t,u_1) - G(x,t,u_2).$$

Now, multiplying (38) by Z gives

$$\frac{1}{2}\int_0^1 (Z(x,\tau))^2 dx + \int_{Q_\tau} (Z_x(x,t))^2\, dxdt = \int_{Q_\tau} \eta(x,t) Z(x,t)\, dxdt.$$

With the use of Cauchy Schawartz inequality and Gronwall Lemma, we obtain

$$\int_{Q_T} (Z(x,\tau))^2 dxdt + \int_{Q_T} \left(x^{\frac{\alpha}{2}} Z_x(x,t)\right)^2 dxdt \leqslant c\,\|Z\|_V,$$

which gives

$$\|Z\|_V \leqslant c\,\|Z\|_V .$$

Due to $c<1$, we obtain

$$(1-c)\,\|Z\|_V \leqslant 0.$$

This, consequently, implies

$$Z = 0 \Longrightarrow y_1 = y_2,$$

which confirms the uniqueness of the weak solution.

4. Conclusion

This paper has investigated the existence and uniqueness of a weak solution with a generalized nonlinear integral condition of the second type for a singular and degenerate nonlinear parabolic equation. The Faedo-Galerkin approach, an a priori estimate, and the imposition of certain nonlinear constraints of the second sort have successfully been used to address the solvability of the studied problem.

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