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L-FUZZY 2−ABSORBING IDEALS IN AN ADL

NATNAEL TESHALE AMARE* AND S. NAGESWARA RAO

Abstract. In this study, we present the concepts of $\mathcal{L}-$fuzzy 2−absorbing ideals and 2−absorbing $\mathcal{L}-$fuzzy ideals of an ADL. We define and analyse the concept of $\mathcal{L}-$fuzzy 2−absorbing ideals using the t-cut method. Furthermore, we demonstrate that both the image and the pre-image of $\mathcal{L}-$fuzzy 2−absorbing ideals remain $\mathcal{L}-$fuzzy 2−absorbing ideals. The main focus of the study is to establish the correlation between the $\mathcal{L}-$fuzzy prime ideal and the $\mathcal{L}-$fuzzy 2−absorbing ideal, the prime $\mathcal{L}-$fuzzy ideal and the 2−absorbing $\mathcal{L}-$fuzzy ideal, and the 2−absorbing $\mathcal{L}-$fuzzy ideal and the $\mathcal{L}-$fuzzy 2−absorbing ideals. In conclusion, we depict 2−absorbing $\mathcal{L}-$fuzzy ideals in terms of the 2−absorbing ideal of an ADL and the 2−absorbing element in a frame.

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1. Introduction

In the field of fuzzy set theory, L.A. Zadeh [14] introduced the concept of fuzzy subsets, which are functions from a set X to $[0, 1]$. J.A. Goguen [1] expanded on this idea by using a complete lattice $\mathcal{L}$ instead of the valuation set $[0, 1]$ leading to the study of $\mathcal{L}-$fuzzy sets. This generalized approach attracted the interest of algebraists who explored fuzzy subalgebras in various algebraic structures. In 1982, W.J. Liu [3] conducted research on fuzzy subrings and fuzzy ideals in rings. Following this, several researchers, such as Malik and Mordeson [4, 5], Kukharjee and Sen [6], Lehmke [2], Swamy and Raju [13], Swamy [10], Nimbhorker and Patil [8], Swamy, Raj and Natnael [11], delved into the study of fuzzy subrings, fuzzy ideals, fuzzy prime ideals in lattices and ADLs. These researchers contributed to the advancement of knowledge in these areas.

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The idea of an ADL was first conceptualized by Swamy and Rao [12]. More recently, Natnael [7] has expanded on this idea by introducing the notion of weakly 2-absorbing ideals of an ADL. In this research paper, we further explore this concept by presenting the idea of $\mathcal{L}-$fuzzy 2-absorbing ideal of an ADL which is the generalization of the notion of $\mathcal{L}-$fuzzy prime ideals of an ADL introduced by Swamy, Raj and Natnael [9]. Our discussion revolves around the characterization of $\mathcal{L}-$fuzzy 2-absorbing ideal using the t-cut framework. Assuming R and G are ADLs, and a lattice homomorphism, h, exists from R to G. If we consider an $\mathcal{L}-$fuzzy 2-absorbing ideal ψ of G, it can be inferred that $h^{-1}(\psi)$ will result in an L-fuzzy 2-absorbing ideal of R. Furthermore, if h is an isomorphism and η is an $\mathcal{L}-$fuzzy 2-absorbing ideal of R, then the result of $h(\eta)$ will be an $\mathcal{L}-$fuzzy 2-absorbing ideal of G. We also characterize all 2-absorbing $\mathcal{L}-$fuzzy ideals of an ADL R in terms of both the 2-absorbing ideal of an ADL and the $2-$absorbing element in a frame $\mathcal{L}$. Mainly, we extensively delve into the relationship between the $\mathcal{L}-$fuzzy prime ideal and $\mathcal{L}-$fuzzy 2-absorbing ideal, prime $\mathcal{L}-$fuzzy ideal and 2-absorbing $\mathcal{L}-$fuzzy ideal, and $\mathcal{L}-$fuzzy ideal and $\mathcal{L}-$fuzzy 2-absorbing ideal. We also demonstrate that the intersection of any two $\mathcal{L}-$fuzzy prime ideals results in an $\mathcal{L}-$fuzzy 2-absorbing ideal. However, the intersection of any two $\mathcal{L}-$fuzzy 2-absorbing ideals may not necessarily yield an $\mathcal{L}-$fuzzy 2-absorbing ideal. Similarly, the intersection of two prime $\mathcal{L}-$fuzzy ideals can produce a $2A-\mathcal{L}-$fuzzy ideal, but the intersection of any two $2A-\mathcal{L}-$fuzzy ideals may not result in a $2A-\mathcal{L}-$fuzzy ideal. We validate these claims by providing counter examples.

Throughout this paper, R stands for an ADL $(A, \wedge, \vee, 0)$ with a maximal element and $\mathcal{L}-$ stands for a complete lattice $(L, \wedge, \vee, 0, 1)$ satisfying the infinite meet distributive law and this type of a lattice is called a frame. More over, in this paper we denote $2-$absorbing ideal of R by $2A-$ideal of R.

2. Preliminaries

In this section, we recall some definitions and basic results mostly taken from [12] and [9].

Definition 2.1. An algebra $R = (R, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice(abbreviated as ADL) if it satisfies the following conditions for all p, q and $r \in R$.

(1) $0 \wedge p = 0$
(2) $p \vee 0 = p$
(3) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
(4) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
(5) $(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$
(6) $(p \vee q) \wedge q = q$

Any bounded below distributive lattice is an ADL.

Example 2.2. Any nonempty set Y can be made into an ADL by fixing an arbitrarily chosen element 0 in Y and fix an arbitrary element $x_0 \in Y$. For any $x, y \in Y$, define $\wedge$ and $\vee$ on X by,

$$x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0 \end{cases} \text{ and } x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases}$$

Then $(Y, \wedge, \vee, x_0)$ is an ADL with x_0 as its zero element. This ADL is called the **discrete ADL**.

Definition 2.3. Let $R = (R, \wedge, \vee, 0)$ be an ADL. For any a and $b \in R$, define $a \leq b$ if $a = a \wedge b$ ($\Leftrightarrow a \vee b = b$). Then $\leq$ is a partial order on R with respect to which 0 is the smallest element in R.

Theorem 2.4. *The following hold for any p, q and r in R.*

(1) $p \wedge 0 = 0$ *and* $0 \vee p = p$
(2) $p \wedge p = p = p \vee p$
(3) $p \wedge q \leq q \leq q \vee p$
(4) $p \wedge q = p \Leftrightarrow p \vee q = q$
(5) $p \wedge q = q \Leftrightarrow p \vee q = p$
(6) $(p \wedge q) \wedge r = p \wedge (q \wedge r)$ *(i.e., $\wedge$ is associative)*
(7) $p \vee (q \vee p) = p \vee q$
(8) $p \leq q \Rightarrow p \wedge q = p = q \wedge p \ \big(\Leftrightarrow p \vee q = q = q \vee p \big)$
(9) $(p \wedge q) \wedge r = (q \wedge p) \wedge r$
(10) $(p \vee q) \wedge r = (q \vee p) \wedge r$
(11) $p \wedge q = q \wedge p \Leftrightarrow p \vee q = q \vee p$
(12) $p \wedge q = \inf\{p, q\} \Leftrightarrow p \wedge q = q \wedge p \Leftrightarrow p \vee q = \sup\{p, q\}$.

Definition 2.5. Let R and G be ADLs. A mapping $h : R \to G$ is called a homomorphism if the following are satisfied, for any $p, q \in R$.
(1). $h(p \wedge q) = h(p) \wedge h(q)$
(2). $h(p \vee q) = h(p) \vee h(q)$
(3). $h(0) = 0$.

Definition 2.6. Let R and G be ADLs and form the set $R \times G$ by $R \times G = \{(p, q) : p \in R \text{ and } q \in G\}$. Define $\wedge$ and $\vee$ in $R \times G$ by,

$$(p, q) \wedge (u, r) = (p \wedge u, q \wedge r)$$
and $(p, q) \vee (u, r) = (p \vee u, q \vee r)$, for any $(p, q), (u, r) \in R \times G$. Then
$$(R \times G, \wedge, \vee, 0)$$

is an ADL under the pointwise operations and $0 = (0, 0)$ is the zero element in $R \times G$.

Definition 2.7. Let I be a non empty subset of R. Then I is called an ideal of R if $a, b \in I$ implies $a \vee b \in I$ and $a \wedge p \in I$ for all $p \in R$.

As a consequence, for any ideal I of R, $p \wedge a \in I$ for all $a \in I$ and $p \in R$.

Definition 2.8. Let $R = (R, \wedge, \vee, 0)$ be an ADL. A proper ideal P of R is said to be a $2A-$ideal of R if $p \wedge q \wedge r \in P \Rightarrow p \wedge q \in P$ or $q \wedge r \in P$ or $p \wedge r \in P$, for any $p, q, r \in R$.

Definition 2.9. An L-fuzzy subset η of X is a mapping from X into L, where L is a complete lattice satisfying the infinite meet distributive law. If L is the unit interval $[0, 1]$ of real numbers, then these are the usual fuzzy subsets of X.

Definition 2.10. An $\mathcal{L}-$fuzzy subset η of R is said to be an $\mathcal{L}-$fuzzy ideal of R, if $\eta(0) = 1$ and $\eta(p \vee q) = \eta(p) \wedge \eta(q)$, for all $p, q \in R$.

Definition 2.11. A proper $\mathcal{L}-$fuzzy ideal η of R is called a prime $\mathcal{L}-$fuzzy ideal if for any $\mathcal{L}-$fuzzy ideals ν and μ of R, $\nu \wedge \mu \leq \eta$ implies either $\nu \leq \eta$ or $\mu \leq \eta$.

Definition 2.12. A proper $\mathcal{L}-$fuzzy ideal η of R is called an $\mathcal{L}-$fuzzy prime ideal of R if for any $p, q \in R, \eta(p \wedge q) = \eta(p)$ or $\eta(q)$.

3. Fuzzy $2A-$ideals

In this section, we introduce the notion of $\mathcal{L}-$fuzzy $2A-$ideals of an Almost Distributive Lattice (ADL) which fuzzified the concept of $2A-$ideals of an ADL. Recall that a proper ideal P of an ADL R is called a $2A-$ideal of R if whenever $p, q, r \in R$ and $p \wedge q \wedge r \in P$, then either $p \wedge q \in P$ or $q \wedge r \in P$ or $p \wedge r \in P$. Now, we have the following.

Definition 3.1. A proper $\mathcal{L}-$fuzzy ideal η of an ADL R is an $\mathcal{L}-$fuzzy $2A-$ideal of R if for all p, q and $r \in R$, $\eta(p \wedge q \wedge r) = \eta(p \wedge q)$ or $\eta(p \wedge r)$ or $\eta(q \wedge r)$.

Example 3.2. Let $R = \{0, p, q, r\}$ and $\vee$ and $\wedge$ be binary operations on R defined by:

$\vee$	0	p	q	r
0	0	p	q	r
p	p	p	p	p
q	q	q	q	q
r	r	p	q	r

$\wedge$	0	p	q	r
0	0	0	0	0
p	0	p	q	r
q	0	p	q	r
r	0	r	r	r

Then, $(R, \wedge, \vee, 0)$ is an ADL which is not a lattice $(p \wedge q \neq q \wedge p)$. Define an $\mathcal{L}-$fuzzy subset $\eta : R \to [0, 1]$ by $\eta(0) = 1, \eta(p) = 0, \eta(q) = 0$ and $\eta(r) = 0.5$. Then, Clearly η is an $\mathcal{L}-$fuzzy ideal of R. Now, $\eta(p \wedge q \wedge r) = \eta(q \wedge r)$ or $\eta(p \wedge q \wedge r) = \eta(p \wedge r)$ or $\eta(p \wedge q \wedge r) = \eta(p \wedge q)$. Therefore, η is an $\mathcal{L}-$fuzzy $2A-$ideal of R.

Next, we characterize the notion of $\mathcal{L}-$fuzzy $2A-$ideal in terms of t-cut. Recall that for any $t \in L$, the set $\eta_t = \{x \in R : t \leq \eta(p)\}$ is called the t-cut of η.

Theorem 3.3. *Let η be an $\mathcal{L}-$fuzzy ideal of R. Then η_t is a $2A-$ideal of R, for all $t \in L$ if and only if η is an $\mathcal{L}-$fuzzy $2A-$ideal of R.*

Proof. Suppose η_t is a $2A-$ideal, for all $t \in L$. Let $p, q, r \in R$ and put $t = \eta(p \wedge q \wedge r)$. Then $p \wedge q \wedge r \in \eta_t$. Assume $t \nleq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$.

Then $t \nleq \eta(p \wedge q)$, $t \nleq \eta(p \wedge r)$ and $t \nleq \eta(q \wedge r)$, implies that $p \wedge q \notin \eta_t$, $p \wedge r \notin \eta_t$ and $q \wedge r \notin \eta_t$. Which gives a contradiction, since η_t is a $2A-$ideal. Thus $\eta(p \wedge q \wedge r) = t \leq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$. Therefore, η is an $\mathcal{L}-$fuzzy $2A-$ideal of R. Conversely suppose that η is an $\mathcal{L}-$fuzzy $2A-$ideal of R. Then for all $p, q, r \in R$, $t = \eta(p \wedge q \wedge r) \Rightarrow p \wedge q \wedge r \in \eta_t$
$\Rightarrow t \leq \eta(p \wedge q \wedge r) = \eta(p \wedge q)$
$\Rightarrow t \leq \eta(p \wedge q)$
$\Rightarrow p \wedge q \in \eta_t$. Similarly, $p \wedge r \in \eta_t$ or $q \wedge r \in \eta_t$.
Thus η_t is a $2A-$ideal of R, for all $t \in L$. □

Corollary 3.4. *An ideal P of R is a $2A-$ideal of R if and only if its characteristics map χ_P is an $\mathcal{L}-$fuzzy $2A-$ideal of R.*

Now, we characterize the concept of $\mathcal{L}-$fuzzy $2A-$ideal of an ADL.

Theorem 3.5. *Let η be an $\mathcal{L}-$fuzzy ideal of R. Then η is an $\mathcal{L}-$fuzzy $2A-$ideal of R iff $\eta(p \wedge q \wedge r) \leq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$, for all $p, q, r \in R$.*

Proof. Let η be an $\mathcal{L}-$fuzzy $2A-$ideal of R. Let $p, q, r \in R$. Then $\eta(p \wedge q \wedge r) = \eta(p \wedge q) \leq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$. Therefore, $\eta(p \wedge q \wedge r) \leq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$. Conversely suppose $\eta(p \wedge q \wedge r) \leq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$, for all $p, q, r \in R$. Since η is an $\mathcal{L}-$fuzzy ideal, $\eta(0) = 1$ and hence $0 \in \eta_t$. Thus $\eta_t \neq \emptyset$. Let $t = \eta(p \wedge q \wedge r)$. Then $p \wedge q \wedge r \in \eta_t$. Now, $t \leq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$, which implies that $t \leq \eta(p \wedge q)$ or $t \leq \eta(p \wedge r)$ or $t \leq \eta(q \wedge r)$ and hence $p \wedge q \in \eta_t$ or $p \wedge r \in \eta_t$ or $q \wedge r \in \eta_t$. Thus η_t is a $2A-$ideal in R. By Theorem 3.3, η is an $\mathcal{L}-$fuzzy $2A-$ideal. □

Theorem 3.6. *Let η be an $\mathcal{L}-$fuzzy ideal of R such that $\eta(p \wedge q \wedge r) \leq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$, for all $p, q, r \in R$. Define $\langle \eta \rangle : R \to L$ by*

$$\langle \eta \rangle(p) = \begin{cases} 1 & \text{if } p = 0 \\ \eta(p) & \text{if } p \neq 0. \end{cases}$$

Then $\langle \eta \rangle$ is the smallest $\mathcal{L}-$fuzzy $2A-$ideal of R containing η.

Proof. Let $p, q, r \in R$.
Now, $p \wedge q \wedge r = 0 \Rightarrow$ either $p \wedge q = 0$ or $p \wedge r = 0$ or $q \wedge r = 0$
$\Rightarrow \langle \eta \rangle(p \wedge q \wedge r) = 1$ and $\langle \eta \rangle(p \wedge q) = 1$ or $\langle \eta \rangle(p \wedge r) = 1$ or $\langle \eta \rangle(q \wedge r) = 1$
$\Rightarrow \langle \eta \rangle(p \wedge q \wedge r) = \langle \eta \rangle(p \wedge q)$ or $\langle \eta \rangle(p \wedge r)$ or $\langle \eta \rangle(q \wedge r)$,
$p \wedge q \wedge r \neq 0 \Rightarrow p \wedge q \neq 0, p \wedge r \neq 0$ and $q \wedge r \neq 0$
$\Rightarrow \langle \eta \rangle(p \wedge q \wedge r) = \eta(p \wedge q \wedge r)$ and $\langle \eta \rangle(p \wedge q) = \eta(p \wedge q), \langle \eta \rangle(p \wedge r) = \eta(p \wedge r)$ and $\langle \eta \rangle(q \wedge r) = \eta(q \wedge r) \Rightarrow \langle \eta \rangle(p \wedge q \wedge r) = \langle \eta \rangle(p \wedge q)$ or $\langle \eta \rangle(p \wedge r)$ or $\langle \eta \rangle(q \wedge r)$. Thus $\langle \eta \rangle$ is an $\mathcal{L}-$fuzzy $2A-$ideal of R. Let ψ be an $\mathcal{L}-$fuzzy $2A-$ideal of R containing η. That is, $\eta(p) \leq \psi(p)$, for all $p \in R$. If $p = 0$, then $\eta(p) = \psi(p)$. Assume $p \neq 0$. Then $\langle \eta \rangle(p) = \eta(p) \leq \psi(p)$ and hence $\langle \eta \rangle(p) \leq \psi(p)$. So, $\langle \eta \rangle \leq \psi$. Therefore, $\langle \eta \rangle$ is the smallest $\mathcal{L}-$fuzzy $2A-$ideal of R containing η. □

We have shown that the t-cuts of any $\mathcal{L}-$fuzzy $2A-$ideal of an ADL R are $2A-$ideals of R. In the following, We demonstrate below that the $\mathcal{L}-$fuzzy $2A-$ideal is completely determined by these t-cuts.

Theorem 3.7. *Let $\{P_t\}_{t\in L}$ be a class of $2A-$ideals of R such that $\bigcap\limits_{t\in M} P_t = P_{SupM}$, for any $M \subseteq L$. Define $\eta : R \to L$ by $\eta(p) = Sup\ \{t \in L : p \in P_t\}$. Then η is an $\mathcal{L}-$fuzzy $2A-$ideal of R such that P_t is equal to the t-cut of η, for each $t \in L$. Conversely, every $\mathcal{L}-$fuzzy $2A-$ideal of R can be obtained as above.*

Proof. Suppose that $\{P_t\}_{t\in L}$ is a $2A-$ideals of R such that $\bigcap\limits_{t\in M} P_t = P_{SupM}$, for any $M \subseteq L$. Then for any $p \in R$ and $t \in L$, we have $p \in P_t \Rightarrow t \le \eta(p) \Rightarrow p \in \eta_t$. Thus $P_s \subseteq \eta_s$, for all $s \in L$. Also for any $t, s \in L$ such that $s \le t$.
$s \le t \Rightarrow t = t \vee s \Rightarrow P_t = P_{t\vee s} = P_t \cap P_s \Rightarrow P_t \subseteq P_s$.
Now, $p \in \eta_s \Rightarrow s \le \eta(p) = \vee\{t \in L : p \in P_t\}$
$\Rightarrow s = s \wedge \big(\vee \{t \in L : p \in P_t\}\big)$
$\Rightarrow s = \vee\{s \wedge t : p \in P_t\}$ (by the inifinite meet distributivity in $\mathcal{L}$)
$\Rightarrow P_s = P_{\bigvee\limits_{p\in P_t} s\wedge t} = \bigcap\limits_{p\in P_t} P_{s\wedge t}$
$\Rightarrow p \in \bigcap\limits_{p\in P_t} P_{s\wedge t} = P_s$ (Since $s \wedge t \le t \Rightarrow P_t \subseteq P_{s\wedge t}$)
$\Rightarrow p \in P_s$.
Therefore, $\eta_s \subseteq P_s$, for all $s \in L$. Therefore, $\eta_s = P_s$, for all s in $\mathcal{L}$. Since each P_s is a $2A-$ideal of R and we have that η is an $\mathcal{L}-$fuzzy $2A-$ideal of R. Conversely suppose that η is an $\mathcal{L}-$fuzzy $2A-$ideal of R. Then each t-cut, $\eta_t = \{p \in R \mid t \le \eta(p)\}$ is a $2A-$ideal of R and $\bigcap\limits_{t\in M} \eta_t = \eta_{\bigvee\limits_{t\in M} t}$ for any $M \subseteq L$. Also, for any $p \in R$, $\eta(p) = \bigvee \{t \in L : p \in \eta_t\}$. □

Theorem 3.8. *For any $\mathcal{L}-$fuzzy subset η of R, define $\bar{\eta}$ by $\bar{\eta}(p) = Sup\ \{t \in L : p \in \langle\eta_t\rangle\}$, for all $p \in R$. Then $\bar{\eta}$ is an $\mathcal{L}-$fuzzy $2A-$ideal of R and $\bar{\eta} = \langle\eta\rangle$.*

Theorem 3.9. *The set of all $\mathcal{L}-$fuzzy $2A-$ideals of R is a complete lattice in which the supremum and infimum of the family of $\{\eta_i\}_{i\in\Delta}$ of R are given by $\bigvee\limits_{i\in\Delta} \eta_i = \langle \bigcup\limits_{i\in\Delta} \eta_i\rangle$ and $\bigwedge\limits_{i\in\Delta} \eta_i = \bigcap\limits_{i\in\Delta} \eta_i$.*

Next, we introduce t-level $\mathcal{L}-$fuzzy $2A-$ideals of R.

Theorem 3.10. *Let P be a $2A-$ideal of R. Then for any $t \in L$, the mapping $t_P : R \to L$ defined by*

$$t_{P}(p) = \begin{cases} 1 & \text{if } x \in P \\ t & \text{if } x \notin P \end{cases}$$

is an $\mathcal{L}-$fuzzy $2A-$ideal of R and called the t-level $\mathcal{L}-$fuzzy $2A-$ideal of R corresponding to a $2A-$ideal P of R.

Theorem 3.11. *For a given $t \in L$, the mapping $P \longmapsto t_P$ is an isomorphism of the lattice of all $2A-$ideals of R onto the lattice of all t-level $\mathcal{L}-$fuzzy $2A-$ideals of R.*

In the following, we facilitate the inter-relationship between $\mathcal{L}-$fuzzy prime ideals and $\mathcal{L}-$fuzzy $2A-$ideals of ADL.

Theorem 3.12. *Every $\mathcal{L}-$fuzzy prime ideal of R is an $\mathcal{L}-$fuzzy $2A-$ideal of R.*

Proof. Let η be an $\mathcal{L}-$fuzzy prime ideal of R and $p, q, r \in R$. Then
$\eta(p \wedge q \wedge r) \leq \eta(p \wedge q) \vee \eta(r)$ or
$\eta(p \wedge q \wedge r) \leq \eta(p) \vee \eta(q \wedge r)$ or
$\eta(p \wedge q \wedge r) \leq \eta(q) \vee \eta(p \wedge r)$ (since η is an $\mathcal{L}-$fuzzy ideal)
which implies that $\eta(p \wedge q \wedge r) \leq \eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)$ (since $q \wedge r \leq r \Rightarrow \eta(r) \leq \eta(q \wedge r)$ and so on). Hence the result. □

The converse of the above result is not true; consider the following example.

Example 3.13. Let $R = \{0, p, q, r, 1\}$ be the lattice represented by the Hasse diagram given below:

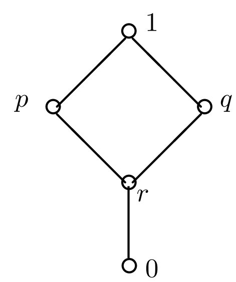

Figure 1: The complete lattice diagram.

Now define $\eta : R \rightarrow [0, 1]$ by $\eta(0) = 1, \eta(r) = 2/3, \eta(q) = 1/3, \eta(p) = 0$ and $\eta(1) = 0$. For any $a, b \in R$, we have $a \leq b$ imply that $\eta(b) \leq \eta(a)$ and hence η is an antitone map. Clearly η is an $\mathcal{L}-$fuzzy ideal of R, since $\eta(p \vee q) = \eta(1) = 0 = \eta(p) \wedge \eta(q)$. Now, $\eta(p \wedge q \wedge r) = \eta(r) = \eta(p \wedge q)$. Therefore, η is an $\mathcal{L}-$fuzzy $2A-$ideal of R, but η is not an $\mathcal{L}-$fuzzy prime ideal of R, since $\eta(p \wedge q) = \eta(r) = 2/3 \neq \eta(p)$ and $\eta(q)$.

Theorem 3.14. *Let η and ψ be $\mathcal{L}-$fuzzy prime ideals of R. Then $\eta \cap \psi$ is an $\mathcal{L}-$fuzzy $2A-$ideal.*

Proof. Let η and ψ be $\mathcal{L}-$fuzzy prime ideals of R and $p, q, r \in R$. Then
$(\eta \cap \psi)(p \wedge q \wedge r) = \eta(p \wedge q \wedge r) \wedge \psi(p \wedge q \wedge r)$
$\leq \Big(\eta(p \wedge q) \vee \eta(p \wedge r) \vee \eta(q \wedge r)\Big) \wedge \Big(\psi(p \wedge q) \vee \psi(p \wedge r) \vee \psi(q \wedge r)\Big)$
$\leq \Big(\eta(p) \vee \eta(q) \vee \eta(r)\Big) \wedge \Big(\psi(p) \vee \psi(q) \vee \psi(r)\Big)$ (by assumption)
$= \big(\eta(p) \wedge \psi(p)\big) \vee \big(\eta(p) \wedge \psi(q)\big) \vee \big(\eta(p) \wedge \psi(r)\big) \vee \big(\eta(q) \wedge \psi(p)\big) \vee \big(\eta(q) \wedge \psi(q)\big) \vee \big(\eta(q) \wedge \psi(r)\big) \vee \big(\eta(r) \wedge \psi(p)\big) \vee \big(\eta(r) \wedge \psi(q)\big) \vee \big(\eta(r) \wedge \psi(r)\big)$
$\leq \big(\eta \cap \psi\big)(p) \vee \big(\eta \cap \psi\big)(p \wedge q) \vee \big(\eta \cap \psi\big)(p \wedge r) \vee \big(\eta \cap \psi\big)(q \wedge p) \vee \big(\eta \cap \psi\big)(q) \vee \big(\eta \cap \psi\big)(q \wedge r) \vee \big(\eta \cap \psi\big)(r \wedge p) \vee \big(\eta \cap \psi\big)(r \wedge q) \vee \big(\eta \cap \psi\big)(r)$

$\leq \big(\eta\cap\psi\big)(p\wedge q)\vee\big(\eta\cap\psi\big)(p\wedge r)\vee\big(\eta\cap\psi\big)(q\wedge r)$ (since $\eta(q)\leq\eta(p\wedge q)=\eta(q\wedge p)$ and $\psi(q)\leq\psi(p\wedge q)=\eta(q\wedge p)$). Therefore, $\eta\cap\psi$ is an $\mathcal{L}-$fuzzy $2A-$ideal of R. □

The intersection of any two $\mathcal{L}-$fuzzy $2A-$ideals of an ADL R need not be an $\mathcal{L}-$fuzzy $2A-$ideal of R; consider the following example.

Example 3.15. Let $R=\{0,a,b,c,d,e,f,,g,h,i,1\}$ be a lattice whose Hasse diagram is given below:

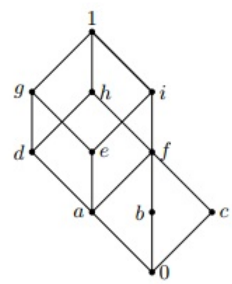

Figure 2: The Boolean lattice diagram.

Define $\mathcal{L}-$fuzzy subsets $\eta:R\to[0,1]$ and $\psi:R\to[0,1]$ by $\eta(0)=1,\ \eta(a)=0.8,$ $\eta(b)=0.8,\ \eta(c)=1,\ \eta(d)=0.8$ and $\eta(e)=\eta(f)=\eta(g)=\eta(h)=\eta(i)=\eta(1)=0$, and $\psi(0)=1,\ \psi(a)=0.9,\ \psi(b)=1,\ \psi(c)=0.7,\ \psi(d)=0,\ \psi(e)=0.9$ and $\psi(f)=\psi(g)=\psi(h)=\psi(i)=\psi(1)=0$. Clearly η and ψ are $\mathcal{L}-$fuzzy $2A-$ideals, but $\eta\cap\psi$ is not an $\mathcal{L}-$fuzzy $2A-$ideal, since
$(\eta\cap\psi)(0)=\eta(0)\wedge\psi(0)=1,$
$(\eta\cap\psi)(a)=\eta(a)\wedge\psi(a)=0.8,$
$(\eta\cap\psi)(b)=\eta(b)\wedge\psi(b)=0.8,$
$(\eta\cap\psi)(c)=\eta(c)\wedge\psi(c)=0.7,$
$(\eta\cap\psi)(d)=\eta(d)\wedge\psi(d)=0,$
$(\eta\cap\psi)(e)=\eta(e)\wedge\psi(e)=0,$
$(\eta\cap\psi)(f)=\eta(f)\wedge\psi(f)=0,$
$(\eta\cap\psi)(g)=\eta(g)\wedge\psi(g)=0,$
$(\eta\cap\psi)(h)=\eta(h)\wedge\psi(h)=0,$
$(\eta\cap\psi)(i)=\eta(i)\wedge\psi(i)=0,$
$(\eta\cap\psi)(1)=\eta(1)\wedge\psi(1)=0$ and hence
$(\eta\cap\psi)(g\wedge h\wedge i)=(\eta\cap\psi)(a)=0.8\nleq 0=(\eta\cap\psi)(g\wedge h)=(\eta\cap\psi)(g\wedge i)=(\eta\cap\psi)(h\wedge i).$

Definition 3.16. Let η_1 and η_2 be $\mathcal{L}-$fuzzy subsets of R and G respectively. Then the product of η_1 and η_2 is denoted by $\eta_1\times\eta_2$ and defined by $(\eta_1\times\eta_2)(a,b)=\eta_1(a)\wedge\eta_2(b)$, for all $(a,b)\in R\times G$.

Theorem 3.17. *Let η_1 and η_2 be $\mathcal{L}-$fuzzy ideals of R and G respectively. Then $\eta_1\times\eta_2$ is an $\mathcal{L}-$fuzzy ideal of $R\times G$.*

Proof. Suppose η_1 and η_2 are $\mathcal{L}-$fuzzy ideals of R and G respectively. Then $(0,0) \in R \times G$, $(\eta_1 \times \eta_2)(0,0) = \eta_1(0) \wedge \eta_2(0) = 1$. Also, for all $a, b \in r$ and $c, d \in G$,
$(\eta_1 \times \eta_2)((a,c) \vee (b,d)) = (\eta_1 \times \eta_2)(a \vee b, c \vee d)$
$= \eta_1(a \vee b) \wedge \eta_2(c \vee d)$
$= \eta_1(a) \wedge \eta_1(b) \wedge \eta_2(c) \wedge \eta_2(d)$
$= \eta_1(a) \wedge \eta_2(c) \wedge \eta_1(b) \wedge \eta_2(d)$
$= (\eta_1 \times \eta_2)(a,c) \wedge (\eta_1 \times \eta_2)(b,d)$.
Therefore, $\eta_1 \times \eta_2$ is an $\mathcal{L}-$fuzzy ideal of $R \times G$. □

Theorem 3.18. *Let η_1 and η_2 be $\mathcal{L}-$fuzzy prime ideals of R and G respectively. Then $\eta_1 \times \eta_2$ is an $\mathcal{L}-$fuzzy $2A-$ideal of $R \times G$.*

Proof. For any $p, q, r \in R$ and $a, b, c \in G$,
$(\eta_1 \times \eta_2)((p,a) \wedge (q,b) \wedge (r,c)) = (\eta_1 \times \eta_2)(p \wedge q \wedge r, a \wedge b \wedge c)$
$= \eta_1(p \wedge q \wedge r) \wedge \eta_2(a \wedge b \wedge c)$
$\leq \Big[\eta_1(p \wedge q) \vee \eta_1(r)\Big] \wedge \Big[\eta_2(a \wedge b) \vee \eta_1(c)\Big]$ or
$\leq \Big[\eta_1(p \wedge r) \vee \eta_1(q)\Big] \wedge \Big[\eta_2(a \wedge c) \vee \eta_1(b)\Big]$ or
$\leq \Big[\eta_1(q \wedge r) \vee \eta_1(p)\Big] \wedge \Big[\eta_2(b \wedge c) \vee \eta_1(a)\Big]$
$\leq \big[\eta_1(p \wedge q) \wedge \eta_2(a \wedge b)\big] \vee \big[\eta_1(p \wedge r) \wedge \eta_2(a \wedge c)\big] \vee \big[\eta_1(q \wedge r) \wedge \eta_2(b \wedge c)\big]$
$= \big[\eta_1 \times \eta_2(p \wedge q, a \wedge b)\big] \vee \big[\eta_1 \times \eta_2(p \wedge r, a \wedge c)\big] \vee \big[\eta_1 \times (q \wedge r, b \wedge c)\big]$.
Hence the result. □

Theorem 3.19. *Let R and G be ADLs, and $h : R \to G$ be a lattice homomorphism. If ψ is an $\mathcal{L}-$fuzzy $2A-$ideal of G, then $h^{-1}(\psi)$ is an $\mathcal{L}-$fuzzy $2A-$ideal of R. If h is an isomorphism and η is an $\mathcal{L}-$fuzzy $2A-$ideal of R, then $h(\eta)$ is an $\mathcal{L}-$fuzzy $2A-$ideal of G.*

Proof. Let h be a lattice homomorphism of ADLs R and G and let ψ be an $\mathcal{L}-$fuzzy $2A-$ideal of G. For any $p, q, r \in R$. Then $h^{-1}(\psi)(p \wedge q \wedge r) = \psi(h(p \wedge q \wedge r)) = \psi(h(p) \wedge h(q) \wedge h(r)$
$\leq \psi(h(p) \wedge h(q) \vee \psi(h(p) \wedge h(r) \vee \psi(h(q) \wedge h(r)$
$= \psi(h(p \wedge q) \vee \psi(h(p \wedge r)) \vee \psi(h(q \wedge r))$.
Therefore, $h^{-1}(\psi)$ is an $\mathcal{L}-$fuzzy $2A-$ideal of R. Also, let h be an epimorphism and η an $\mathcal{L}-$fuzzy $2A-$ideal of R. Consider, $h(\eta)(p\wedge q) \vee h(\eta)(p\wedge r) \vee h(\eta)(q\wedge r) = \Big(\bigvee_{a\wedge b \in h^{-1}(p\wedge q)} \eta(a \wedge b)\Big) \vee \Big(\bigvee_{a\wedge c \in h^{-1}(p\wedge r)} \eta(a \wedge c)\Big) \vee \Big(\bigvee_{b\wedge c \in h^{-1}(q\wedge r)} \eta(b \wedge c)\Big) \geq \bigvee_{a\wedge b\wedge c \in h^{-1}(p\wedge q\wedge r)} \eta(a\wedge b\wedge c) = h(\eta)(p\wedge q\wedge r)$. Thus, $h(\eta)$ is an $\mathcal{L}-$fuzzy $2A-$ideal of G. □

4. $2A-\mathcal{L}-$fuzzy ideals

In this section, we discuss the concept of $2A-\mathcal{L}-$fuzzy ideal of an ADL R which is weaker than that of a prime $\mathcal{L}-$fuzzy ideal of R and we discuss the relationship between $2A-\mathcal{L}-$fuzzy ideal and $\mathcal{L}-$fuzzy $2A-$ideal of R.

Definition 4.1. A proper $\mathcal{L}-$fuzzy ideal η of R is a $2A-\mathcal{L}-$fuzzy ideal of R if for all $\mathcal{L}-$fuzzy ideals η_1, η_2, η_3 of R, $\eta_1 \wedge \eta_2 \wedge \eta_3 \leq \eta \Rightarrow$ either $\eta_1 \wedge \eta_2 \leq \eta$ or $\eta_1 \wedge \eta_3 \leq \eta$ or $\eta_2 \wedge \eta_3 \leq \eta$.

An element t in a frame $\mathcal{L}$ is a $2-$absorbing element in $\mathcal{L}$ if there exists $t_1 \wedge t_2 \wedge t_3 \leq t$ implies either $t_1 \wedge t_2 \leq t$ or $t_1 \wedge t_3 \leq t$ or $t_2 \wedge t_3 \leq t$.

In the following, we characterize all $2A-\mathcal{L}-$fuzzy ideals of R interms of the $2A-$ideal of R and the $2-$absorbing element t in $\mathcal{L}$.

Theorem 4.2. *Let P be a proper ideal of R. Then t_P is a $2A-\mathcal{L}-$fuzzy ideal of R if and only if P is a $2A-$ideal of R and t is a $2-$absorbing element in $\mathcal{L}$.*

Proof. Suppose t_P is an $2A-\mathcal{L}-$fuzzy ideal of R. Let P_1, P_2 and P_3 be ideals of R such that $P_1 \cap P_2 \cap P_3 \subseteq P$. Then $t_{P_1} \wedge t_{P_2} \wedge t_{P_3} = t_{P_1 \cap P_2} \wedge t_{P_3} \leq t_P$ and hence $t_{P_1 \cap P_2} \leq t_P$ or $t_{P_3} \leq t_P$. So that, $P_1 \cap P_2 \subseteq P$ or $P_3 \subseteq P$. Similarly, either $P_1 \cap P_3 \subseteq P$ or $P_2 \cap P_3 \subseteq P$. Thus, P is a $2A-$ideal of R. Also, for any $t_1, t_2, t_3 \in L$ such that $t_1 \wedge t_2 \wedge t_3 \leq t$. Then $(t_1 \wedge t_2 \wedge t_3)_P \leq t_P$ imply that $(t_1)_P \wedge (t_2)_P \wedge (t_3)_P \leq t_P$ and since t_P is a $2A-\mathcal{L}-$fuzzy ideal, either $(t_1)_P \wedge (t_2)_P \leq t_P$ or $(t_1)_P \wedge (t_3)_P \leq t_P$ or $(t_2)_P \wedge (t_3)_P \leq t_P$ imply that $\big(t_1 \wedge t_2\big)_P \leq t_P$ or $\big(t_1 \wedge t_3\big)_P \leq t_P$ or $\big(t_2 \wedge t_3\big)_P \leq t_P$ and hence $t_1 \wedge t_2 \leq t$ or $t_1 \wedge t_3 \leq t$ or $t_2 \wedge t_3 \leq t$. Therefore, t is a $2-$absorbing element in $\mathcal{L}$. Conversely suppose that P is a $2A-$ideal of R and t is a $2-$absorbing element in $\mathcal{L}$. Let η_1, η_2 and η_3 be $\mathcal{L}-$fuzzy ideals of R such that $\eta_1 \nleq t_P, \eta_2 \nleq t_P$ and $\eta_3 \nleq t_P$. Now there exists $p, q, r \in R$ such that $\eta_1(p \wedge q) \nleq t_P(p \wedge q), \eta_2(p \wedge r) \nleq t_P(p \wedge r)$ and $\eta_3(q \wedge r) \nleq t_P(q \wedge r)$. So that, $t_P(p \wedge q) = t_P(p \wedge r) = t_P(q \wedge r) = t$ and hence $p \wedge q, p \wedge r$ and $q \wedge r \notin P$. Since P is a $2A-$ideal, $p \wedge q \wedge r \notin P$. Also, since t is a $2-$absorbing element in $\mathcal{L}$ and $\eta_1(p \wedge q) \nleq t, \eta_2(p \wedge r) \nleq t$ and $\eta_3(q \wedge r) \nleq t$, we have $\eta_1(p \wedge q) \wedge \eta_2(p \wedge r) \wedge \eta_3(q \wedge r) \nleq t$. Now $(\eta_1 \wedge \eta_2 \wedge \eta_3)(p \wedge q \wedge r) =$
$\eta_1(p \wedge q \wedge r) \wedge \eta_2(p \wedge q \wedge r) \wedge \eta_3(p \wedge q \wedge r)$
$\geq \eta_1(p \wedge q) \wedge \eta_2(p \wedge r) \wedge \eta_3(q \wedge r)$
and hence $(\eta_1 \wedge \eta_2 \wedge \eta_3)(p \wedge q \wedge r) \nleq t = t_P(p \wedge q \wedge r)$. So, $\eta_1 \wedge \eta_2 \wedge \eta_3 \nleq t_P$. Therefore, t_P is a $2A-\mathcal{L}-$fuzzy ideal of R. □

Lemma 4.3. *An $\mathcal{L}-$fuzzy ideal η of R is a $2A-\mathcal{L}-$fuzzy ideal of R if and only if η is two valued and there exists $0 \in R$ such that $\eta(0) = 1$ and η_1 is a $2A-$ideal of R.*

Proof. Assume that η is a $2A-\mathcal{L}-$fuzzy ideal. Suppose that η assumes more than two values. Then there exists $p, q, r \in R$ and $\alpha \neq \beta \neq \gamma \in L - \{1\}$ such that $\eta(p) = \alpha$, $\eta(q) = \beta$ and $\eta(r) = \gamma$. Now, define L-fuzzy subsets η_1, η_2 and

η_3 of R as follows:

$$\eta_1(b) = \begin{cases} 1 & \text{if } b \in \langle p] \\ 0 & \text{otherwise,} \end{cases} \quad \eta_2(b) = \begin{cases} 1 & \text{if } b = 0 \\ \alpha & \text{otherwise} \end{cases} \text{ and } \eta_3(b) = \begin{cases} 1 & \text{if } b = 0 \\ \beta & \text{otherwise.} \end{cases}$$

Then, clearly $\eta_1 = 0_{\langle p]}$, $\eta_2 = \alpha_{\langle 0]}$, $\eta_3 = \beta_{\langle 0]}$ and hence, by 4.2, η_1, η_2 and η_3 are L-fuzzy ideals of R. Also, $(\eta_1 \wedge \eta_2 \wedge \eta_3)(b) \leq \eta(b)$, for all $b \in R$; for,
$b = 0 \Rightarrow \big(\eta_1 \wedge \eta_2 \wedge \eta_3\big)(b) = \eta_1(b) \wedge \eta_2(b) \wedge \eta_3(b) = 1 \wedge 1 \wedge 1 = 1 = \eta(0) = \eta(b)$
$0 \neq b \in \langle p] \Rightarrow \eta_1(b) \wedge \eta_2(b) \wedge \eta_3(b) = 1 \wedge \alpha \wedge \beta = \alpha \wedge \beta = \eta(p) \wedge \eta(q) = \eta(p \vee q) \leq \eta(b)$
and $b \notin \langle p] \Rightarrow \eta_1(b) \wedge \eta_2(b) \wedge \eta_3(b) = 0 \wedge \alpha \wedge \beta = 0 \leq \eta(b)$.
Therefore, $\eta_1 \wedge \eta_2 \wedge \eta_3 \leq \eta$. By assumption, we have that $\eta_1 \wedge \eta_2 \leq \eta$ or $\eta_1 \wedge \eta_3 \leq \eta$ or $\eta_2 \wedge \eta_3 \leq \eta$. But $\eta_2 \wedge \eta_3 \leq \eta$ (since $\eta_1(p) \wedge \eta_2(p) = 1$, $\eta(p) = \alpha$ and $1 \nleq \alpha$, and $\eta_1(q) \wedge \eta_3(q) = 1$, $\eta(q) = \beta$ and $1 \nleq \beta$). Therefore, $\eta_2 \wedge \eta_3 \leq \eta$, in particular, $\eta_2(p) \wedge \eta_3(p) \leq \eta(p) = \alpha$ (or, $\eta_2(q) \wedge \eta_3(p) \leq \eta(q) = \beta$). Since $\eta(p) \neq \eta(0)$ (or, $\eta(q) \neq \eta(0)$), it follows that $p \neq 0$ (or, $q \neq 0$) and hence $\eta_2(p) \wedge \eta_3(p) = \alpha \wedge \beta$ (or, $\eta_2(q) \wedge \eta_3(q) = \alpha \wedge \beta$). Since $\alpha \neq \beta$, then either $\alpha \leq \beta$ or $\beta \leq \alpha$. Thus $\beta = \alpha \wedge \beta \leq \alpha$. Therefore $\alpha = \beta$. Similarly, if we define an L-fuzzy subset η_3 of R by

$$\eta_3(b) = \begin{cases} 1 & \text{if } b = 0 \\ \gamma & \text{if } b \neq 0, \end{cases}$$

then it can be verified that $\beta \leq \gamma$. Thus, either $\alpha = \beta$ or $\beta = \gamma$ and hence $\eta(p) = \eta(q)$ or $\eta(q) = \eta(r)$. Which gives a contradiction to our assumption. Thus η is two valued. Consider the set $P = \{r \in R : \eta(r) = 1\}$. Then P is proper ideal of R, since η is proper. Let t be the other value of η. Then

$$\eta(r) = \begin{cases} 1 & \text{if } r \in P \\ t & \text{otherwise} \end{cases}$$

and hence $\eta = t_P$. By 4.2, we get that P is a $2A-$ideal of R. The converse is clear. □

Next, we discuss the relationship between $2A-\mathcal{L}-$fuzzy ideal and $\mathcal{L}-$fuzzy $2A-$ideal.

Theorem 4.4. *Every $2A-\mathcal{L}-$fuzzy ideal of R is an $\mathcal{L}-$fuzzy $2A-$ideal of R.*

Proof. Suppose that η is a $2A-\mathcal{L}-$fuzzy ideal of R. Then there exists a $2A-$ideal P of R and $2-$absorbing element t in $\mathcal{L}$ such that $\eta = t_P$. Then η is an $\mathcal{L}-$fuzzy $2A-$ideal of R. □

The converse of the above theorem is not true; that is there are $\mathcal{L}-$fuzzy $2A-$ideals of ADLs which are not $2A-\mathcal{L}-$fuzzy ideals; even when ADL is a lattice. Consider the following.

Example 4.5. Let $R = \{0, p, q, r\}$ be an ADL defined in 3.2 and $L = \{0, s, 1\}$ with $0 < s < 1$. Let η be an $\mathcal{L}-$fuzzy ideal of R defined by $\eta(0) = 1, \eta(p) = 0, \eta(q) = t$ and $\eta(r) = s$. Clearly, η is an $\mathcal{L}-$fuzzy $2A-$ideal of R while η is not a $2A-\mathcal{L}-$fuzzy ideal, since η is not two valued.

Lemma 4.6. *Let P be an ideal of R. If the characteristics map χ_P is a $2A-\mathcal{L}-$fuzzy ideal of R, then P is a $2A-$ideal of R.*

Theorem 4.7. *Every prime $\mathcal{L}-$fuzzy ideal of R is a $2A-\mathcal{L}-$fuzzy ideal of R and the converse of this is not true.*

Proof. Suppose η is a prime $\mathcal{L}-$fuzzy ideal of R. Let η_1, η_2 and η_3 be $\mathcal{L}-$fuzzy ideals of R such that $\eta_1 \wedge \eta_2 \wedge \eta_3 \leq \eta$. Then, either $\eta_1 \wedge \eta_2 \leq \eta$ or $\eta_3 \leq \eta$, or
$\eta_1 \wedge \eta_3 \leq \eta$ or $\eta_2 \leq \eta$, or
$\eta_2 \wedge \eta_3 \leq \eta$ or $\eta_1 \leq \eta$, since η is prime.
Which implies that either $\eta_1 \wedge \eta_2 \leq \eta$ or $\eta_1 \wedge \eta_3 \leq \eta$ or $\eta_2 \wedge \eta_3 \leq \eta$. Therefore, η is a $2A-\mathcal{L}-$fuzzy ideal of R. □

Example 4.8. Let $D = \{0, x, y\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, a, b, c, 1\}$ be the lattice represented by the Hasse diagram given below:

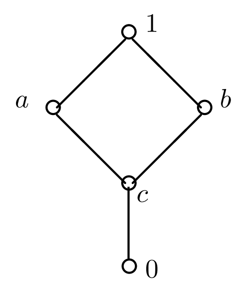

Figure 3: The complete lattice diagram.

Consider $D \times L = \{(t, s) : t \in D \text{ and } s \in L\}$. Then $(D \times L, \wedge, \vee, 0)$ is an ADL under the point-wise operations $\wedge$ and $\vee$ on $D \times L$ and $0 = (0, 0)$, the zero element in $D \times L$. Now define $\eta : D \times L \to [0, 1]$ by

$$\eta(t, s) = \begin{cases} 1 & \text{if } t = 0 \textit{ and } s = 0 \\ 0.5 & \text{otherwise.} \end{cases}$$

for all $(t, s) \in D \times L$. Clearly η is an $\mathcal{L}-$fuzzy ideal of $D \times L$ (note that $D \times L$ is not a lattice). Let $P = (0, 0)$ and put $\eta = 0.5_P$. Thus η is a $2A-\mathcal{L}-$fuzzy ideal of $D \times L$ but ϕ is not a prime $\mathcal{L}-$fuzzy ideal, since P is a $2A-$ideal of $D \times L$ which is not prime ideal; for, $(0, a) \wedge (x, b) = (0, 0)$.

Theorem 4.9. *Let η and ϕ be prime $\mathcal{L}-$fuzzy ideals of R. Then $\eta \cap \phi$ is a $2A-\mathcal{L}-$fuzzy ideal of R.*

Proof. For all $r \in R$, $(\eta \cap \phi)(r) = \eta(r) \wedge \phi(r)$. Suppose η and ϕ are prime $\mathcal{L}-$fuzzy ideals of R. Let η_1, η_2 and η_3 be $\mathcal{L}-$fuzzy ideals of R such that $\eta_1 \wedge \eta_2 \wedge \eta_3 \leq \eta \cap \psi$. Then $\eta_1(r) \wedge \eta_2(r) \wedge \eta_3(r) \leq \eta(r) \wedge \psi(r) \Rightarrow \eta_1(r) \wedge \eta_2(r) \wedge \eta_3(r) \leq \eta(r)$ and $\eta_1(r) \wedge \eta_2(r) \wedge \eta_3(r) \leq \psi(r)$
$\Rightarrow \eta_1(r) \wedge \eta_2(r) \leq \eta(r)$ or $\eta_3(r) \leq \eta(r)$ or $\eta_2(r) \wedge \eta_3(r) \leq \eta(r)$ and

$\eta_1(r) \wedge \eta_2(r) \leq \psi(r)$ or $\eta_3(r) \leq \psi(r)$ or $\eta_2(r) \wedge \eta_3(r) \leq \psi(r)$,
(since η and ϕ are prime $\mathcal{L}-$fuzzy ideals)
$\Rightarrow \eta_1(r) \wedge \eta_2(r) \leq \eta(r)$ and $\eta_1(r) \wedge \eta_2(r) \leq \psi(r)$, or $\eta_3(r) \leq \eta(r)$ and $\eta_3(r) \leq \psi(r)$, or $\eta_2(r) \wedge \eta_3(r) \leq \eta(r)$ and $\eta_2(r) \wedge \eta_3(r) \leq \psi(r)$
$\Rightarrow \eta_1(r) \wedge \eta_2(r) \leq \eta(r) \wedge \psi(r)$, or $\eta_3(r) \leq \eta(r) \wedge \psi(r)$, or $\eta_2(r) \wedge \eta_3(r) \leq \eta(r) \wedge \psi(r)$.
Which implies that, $\eta_1 \wedge \eta_2 \leq \eta \cap \psi$ or $\eta_1 \wedge \eta_3 \leq \eta \cap \psi$ or $\eta_2 \wedge \eta_3 \leq \eta \cap \psi$. Therefore, $\eta \cap \psi$ is a $2A-\mathcal{L}-$fuzzy ideal of R. □

Example 4.10. Let $R = \{0, a, b, c, d, e, f, 1\}$ be a lattice whose Hasse diagram is given below:

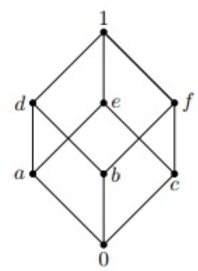

Figure 4: The Boolean lattice diagram.

Define $\mathcal{L}-$fuzzy subsets $\eta : R \to [0,1]$ and $\psi : R \to [0,1]$ by $\eta(0) = \eta(a) = 1, \eta(p) = 0$ if $p \in A - \{0, a\}$ and $\psi(0) = \psi(c) = 1, \psi(p) = 0$ if $x \in A - \{0, c\}$. From this we have that, η is two valued, $\eta(0) = 1$ and $\eta_1 = \{0, a\}$ is a $2A-$ideal of R. Similarly, ψ is two valued, $\psi(0) = 1$ and $\psi_1 = \{0, c\}$ is a $2A-$ideal of R. By 4.3, η and ψ are $2A-\mathcal{L}-$fuzzy ideals of R. Let $P = \{0, a\}$ and $Q = \{0, c\}$. Then $P \cap Q = \{0\}$. Thus, $\eta_1 \cap \psi_1 = P \cap Q$ is not a $2A-$ideal of R, since $d \wedge e \wedge f = 0 \in P \cap Q$ but $d \wedge e = a \notin P \cap Q$, $d \wedge f = b \notin P \cap Q$ and $e \wedge f = c \notin P \cap Q$. Thus, $\chi_{P \cap Q}$ is not a $2A-\mathcal{L}-$fuzzy ideal of R. Therefore, $\eta \cap \psi$ is not a $2A-\mathcal{L}-$fuzzy ideal of R.

5. Conclusion

This work introduces and derives several findings from the ideas of $\mathcal{L}-$fuzzy $2A-$ideal, $2A-\mathcal{L}-$fuzzy ideal of an almost distributive lattice, and their direct product. Our next work will concentrate on the Stone space of fuzzy $2A-$ideals.

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Natnael Teshale Amare received PhD from Andhra University, since 2019, he is working as an Associate Professor in the department of Mathematics, University of Gondar, Ethiopia. His research interests include Algebra and Fuzzy algebra.

Department of Mathematics, University of Gondar, Ethiopia.
e-mail: yenatnaelteshale@gmail.com

S. Nageswara Rao received Ph.D. from Andhra University, since 2022. He is currently an an Assistant professor at Mallareddy College of Engineering, Hyderabad. His research interests are algebra, pseudo complemented ADL and fuzzy algebra.

Department of Mathematics, Mallareddy College of Engineering, Hyderabad, India-500014.
e-mail: snraomaths@gmail.com