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SUM OF PICTURE FUZZY IDEALS OF Γ-NEAR RINGS

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ABSTRACT. In the present paper we introduce the concept on sum of picture fuzzy ideals of a Γ-near ring and the direct sum of picture fuzzy ideals of a Γ-near ring and investigated several properties. Also, we have discussed their relations of sum and direct sum of picture fuzzy ideals of a Γ-near ring.

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1. Introduction

In 1965, Zadeh [11] has initiated the notion of fuzzy set. Then many researchers were applying it in various branches of Mathematics. The algebraic system Γ-near ring was introduced by Satyanarayana [9]. Later several mathematicians like [1, 2, 5, 6] worked on this algebraic system. The notion of an anti fuzzy ideals of Γ-near-ring was studied by Srinivas et. al., [10]. Kim and Jun [4] has studied the concept of an anti fuzzy ideals in near-rings. The sum of the fuzzy ideals of a near-ring was studied by Narasimha Swamy [7]. Now we are introducing the sum of picture fuzzy ideals of a Γ-near-ring and also the sum of anti picture fuzzy ideals of a Γ-near-ring. Also studied the concept of direct sum in both cases.

2. Preliminaries

A non-empty set N with two binary operations " $+$ " and " $\cdot$ " is said to be a left near-ring, if it satisfies the following three conditions; (i) $(N,+)$ is a group (not necessarily abelian), (ii) $(N,\cdot)$ is a semigroup, (iii) $l\cdot(m+z) = x\cdot m + l\cdot z$ for all $l, m, z \in N$. We will use the word "near-ring" to mean "left near-ring". We

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denote lm instead of $l \cdot m$. Moreover, a near-ring N is said to be a zero-symmetric if $0 \cdot n = 0$ for all $n \in N$, where 0 is the additive identity in N.

Definition 2.1. [9] Let M be a Γ-near ring (briefly, ΓNR). A normal subgroup $(I, +)$ of $(M, +, \Gamma)$ is called

(i) a right ideal, if $(l+i)\alpha m - l\alpha m \in I$ for all $l, m \in M,\ \alpha \in \Gamma,\ i \in I$,
(ii) a left ideal, if $l\alpha i \in I$ for all $l \in M,\ \alpha \in \Gamma, i \in I$,
(iii) an ideal, if it is both a left ideal and a right ideal.

Definition 2.2. [9] A Γ-near ring M is said to be a zero symmetric Γ-near ring if $0\alpha n = 0$ for every $n \in M,\ \alpha \in \Gamma$, where 0 is the additive identity in M.

A fuzzy set B_ζ on a non-empty A is a mapping $B_\zeta : A \to [0, 1]$.

Definition 2.3. [3] A fuzzy set B_ζ of a Γ-near ring M is called a fuzzy ideal (briefly, FI) of M if

(i) $B_\zeta(l - m) \geq \min\{B_\zeta(l), B_\zeta(m)\}$,
(ii) $B_\zeta(m + l - m) \geq B_\zeta(l)$,
(iii) $B_\zeta((l+i)\alpha m - l\alpha m) \geq B_\zeta(i)$ (or equivalently, $B_\zeta(z\alpha m - l\alpha m) \geq B_\zeta(z-l)$),
(iv) $B_\zeta(l\alpha m) \geq B_\zeta(m)$ for all $l, m, z, i \in M$ and $\alpha \in \Gamma$. If B_ζ satisfies (i), (ii) and (iii) then B_ζ is called a fuzzy right ideal of M. If B_ζ satisfies (i), (ii) and (iv) then B_ζ is called a fuzzy left ideal of M.

Definition 2.4. [8] Let B_ζ and B_η be two fuzzy ideals of a zero symmetric Γ-near ring M. Then the sum $B_\zeta + B_\eta$ is a fuzzy subset of M defined by

$$(B_\zeta + B_\eta)(l) = \begin{cases} sup(\min(B_\zeta(m)\ B_\eta(z))) : l = m + z \\ 0 : \qquad \text{otherwise}, \end{cases}$$

3. Sum and direct sum of picture fuzzy ideals

Definition 3.1. A $PFS\ \zeta$ in a $\Gamma NR\ M$ is called a picture fuzzy Γ near ring (briefly, $PF\Gamma NR$) of M if

(i) $B_\zeta(l - m) \geq \min(B_\zeta(l), B_\zeta(m)),\ A_\zeta(l - m) \leq \max(A_\zeta(l), A_\zeta(m)),\ F_\zeta(l - m) \leq \max(F_\zeta(l), F_\zeta(m))$.
(ii) $B_\zeta(l\alpha m) \geq B_\zeta(m),\ A_\zeta(l\alpha m) \leq A_\zeta(m),\ F_\zeta(l\alpha m) \leq F_\zeta(m)$.

Definition 3.2. Let $(M, +, \Gamma)$ or simply M be a ΓNR. A $PFS\ \zeta$ in a $\Gamma NR\ M$ is called a picture fuzzy Γ ideal (briefly, $PF\Gamma I$) of M if

(i) $B_\zeta(l - m) \geq min(B_\zeta(l), B_\zeta(m)),\ A_\zeta(l - m) \leq max(A_\zeta(l), A_\zeta(m))$ and $F_\zeta(l - m) \leq max(F_\zeta(l), F_\zeta(m))$.
(ii) $B_\zeta(m+l-m) \geq B_\zeta(l)$, $A_\zeta(m+l-m) \leq A_\zeta(l)$ and $F_\zeta(m+l-m) \leq F_\zeta(l)$.
(iii) $B_\zeta((l+i)\alpha m - l\alpha m) \geq B_\zeta(i)$, $A_\zeta((l+i)\alpha m - l\alpha m) \leq A_\zeta(i)$ and $F_\zeta((l+i)\alpha m - l\alpha m) \leq F_\zeta(i)$.
(iv) $B_\zeta(l\alpha m) \geq B_\zeta(m)$, $A_\zeta(l\alpha m) \leq A_\zeta(m)$ and $F_\zeta(l\alpha m) \leq F_\zeta(m)$.

A picture fuzzy subset with the above conditions (i)-(iii) is called a picture fuzzy Γ right ideal of M, whereas a picture fuzzy subset with (i), (ii) and (iv) is called a picture fuzzy Γ left ideal of M.

Definition 3.3. Let ζ and η are two picture fuzzy ideals of a zero symmetric Γ-near ring M. Then the sum $\zeta+\eta$ is a picture fuzzy set of M defined by,

$$(\zeta+\eta)(l)=\begin{cases} sup(\min(\zeta(m),\eta(z))) : & l=m+z \\ 0 : & \text{otherwise.}\end{cases}$$

Theorem 3.4. If ζ and η are two picture fuzzy ideals of a zero symmetric Γ-near ring M, then $\zeta+\eta$ is also an picture fuzzy ideal of M.

Proof. Let $l, m, z \in M$ and $\alpha \in \Gamma$.

(i) Put $l = l_1 + l_2$ and $m = m_1 + m_2$ where $l_1, l_2, m_1, m_2 \in M$. Then

$$\begin{aligned} l-m &= l_1+l_2-(m_1+m_2) \\ &= l_1-m_1+m_1+l_2-(m_1+m_2). \end{aligned}$$

Now,

$$\begin{aligned} (B_\zeta+B_\eta)(l-m) &= (B_\zeta+B_\eta)(l_1-m_1+m_1+l_2-m_1-m_2) \\ &= sup\{\min\{(B_\zeta(l_1-m_1),B_\eta(m_1+l_2-m_1-m_2))\}\} \\ &\geq sup\{\min\{\min(B_\zeta(l_1),B_\zeta(m_1)),\min(B_\eta(m_1+l_2-m_1),B_\eta(m_2))\}\} \\ &\geq sup\{\min\{\min(B_\zeta(l_1),B_\zeta(m_1)),\min(B_\eta(l_2),B_\eta(m_2))\}\} \\ &\geq \min\{sup\{\min(B_\zeta(l_1),B_\zeta(l_2)),sup(\min(B_\eta(m_1),B_\eta(m_2)))\}\} \\ &= \min\{(B_\zeta+B_\eta)(l),(B_\zeta+B_\eta)(m)\}. \end{aligned}$$

$$\begin{aligned} (A_\zeta+A_\eta)(l-m) &= (A_\zeta+A_\eta)(l_1-m_1+m_1+l_2-m_1-m_2) \\ &= sup\{\min\{(A_\zeta(l_1-m_1),A_\eta(m_1+l_2-m_1-m_2))\}\} \\ &\leq sup\{\min\{\max(A_\zeta(l_1),A_\zeta(m_1)),\max(A_\eta(m_1+l_2-m_1),A_\eta(m_2))\}\} \\ &\leq sup\{\min\{\max(A_\zeta(l_1),A_\zeta(m_1)),\max(A_\eta(l_2),A_\eta(m_2))\}\} \\ &\leq \max\{sup\{\min(A_\zeta(l_1),A_\zeta(l_2)),sup(\min(A_\eta(m_1),A_\eta(m_2)))\}\} \\ &= \max\{(A_\zeta+A_\eta)(l),(A_\zeta+A_\eta)(m)\}. \end{aligned}$$

Similarly, $(F_\zeta+F_\eta)(l-m) \leq \max\{(F_\zeta+F_\eta)(l),(F_\zeta+F_\eta)(m)\}$.

(ii) Put $l = l_1 + l_2$ where $l_1, l_2 \in M$. Then
$m+l-m = m+l_1+l_2-m = m+l_1-m+m+l_2-m$.

$$\begin{aligned} (B_\zeta+B_\eta)(m+l-m) &= sup\{\min(B_\zeta(m+l_1-m),B_\eta(m+l_2-m))\} \\ &\geq sup\{\min\{B_\zeta(l_1),B_\eta(l_2)\}\} \\ &= (B_\zeta+B_\eta)(l). \end{aligned}$$

$$\begin{aligned} (A_\zeta+A_\eta)(m+l-m) &= sup\{\min(A_\zeta(m+l_1-m),A_\eta(m+l_2-m))\} \\ &\leq sup\{\min(A_\zeta(l_1),A_\eta(l_2))\} \\ &= (A_\zeta+A_\eta)(l). \end{aligned}$$

Similarly, $(F_\zeta+F_\eta)(l) \leq (F_\zeta+F_\eta)(l)$.

(iii) Let $i = i_1 + i_2$ where $i_1, i_2 \in M$. Then
$(l+i)\alpha m - l\alpha m = (l+i_1+i_2)\alpha m - (l+i_2)\alpha m + (l+i_2)\alpha m - l\alpha m$.

$$\begin{aligned}
&(B_\zeta{+}B_\eta)((l+i)\alpha m - l\alpha m)\\
&\quad = sup\{\min[B_\zeta((l+i_1+i_2)\alpha m - (l+i_2)\alpha m), B_\eta((l+i_2)\alpha m - l\alpha m)]\}\\
&\quad \geq sup\{\min[B_\zeta(i_1), B_\eta(i_2)]\}\\
&\quad = (B_\zeta + B_\eta)(i).
\end{aligned}$$

$$\begin{aligned}
&(A_\zeta{+}A_\eta)((l+i)\alpha m - l\alpha m)\\
&\quad = (A_\zeta + A_\eta)\{(l+i_1+i_2)\alpha m - (l+i_2)\alpha m + (l+i_2)\alpha m - l\alpha m\}\\
&\quad = sup\{\min[A_\zeta((l+i_1+i_2)\alpha m - (l+i_2)\alpha m), A_\eta((l+i_2)\alpha m - l\alpha m)]\}\\
&\quad \leq sup\{\min[A_\zeta(i_1), A_\eta(i_2)]\}\\
&\quad = (A_\zeta + A_\eta)(i).
\end{aligned}$$

Similarly, $(F_\zeta + F_\eta)((l+i)\alpha m - l\alpha m) \leq (F_\zeta + F_\eta)(i)$.

(iv) Put $m = m_1 + m_2; m_1, m_2 \in M$. Then

$$\begin{aligned}
(B_\zeta + B_\eta)(l\alpha m) &= (B_\zeta + B_\eta)(l\alpha(m_1 + m_2))\\
&= (B_\zeta + B_\eta)(l\alpha m_1 + l\alpha m_2)\\
&= sup\{\min[B_\zeta(l\alpha m_1), B_\eta(l\alpha m_2)]\}\\
&\geq sup\{\min[B_\zeta(m_1), B_\eta(m_2)]\}\\
&= (B_\zeta + B_\eta)(m).
\end{aligned}$$

$$\begin{aligned}
(A_\zeta + A_\eta)(l\alpha m) &= (A_\zeta + A_\eta)(l\alpha(m_1 + m_2))\\
&= (A_\zeta + A_\eta)(l\alpha m_1 + l\alpha m_2)\\
&= sup\{\min[A_\zeta(l\alpha m_1), A_\eta(l\alpha m_2)]\}\\
&\leq sup\{\min[A_\zeta(m_1), A_\eta(m_2)]\}\\
&= (A_\zeta + A_\eta)(m).
\end{aligned}$$

Similarly, $(F_\zeta + F_\eta)(l\alpha m) = (F_\zeta + F_\eta)(m)$.

Hence $(\zeta + \eta)$ is a picture fuzzy ideal of ΓNR of M. □

Example 3.5. Let $M = \{0, a, b, c\}$, clearly M is a zero symmetric Γ-near ring. Define two picture fuzzy sets $\zeta = \{(l, B_\zeta(l), A_\zeta(l), F_\zeta(l))\}$ and $\eta = \{(l, B_\eta(l), A_\eta(l), F_\eta(l))\}$ as

$B_\zeta : M \to [0,1]$, $A_\zeta : M \to [0,1]$, $F_\zeta : M \to [0,1]$
$B_\eta : M \to [0,1]$, $A_\eta : M \to [0,1]$ and $F_\eta : M \to [0,1]$

$$B_\zeta(0) = 0.7\ A_\zeta(0) = 0.02\ F_\zeta(0) = 0.02$$
$$B_\zeta(a) = 0.05\ A_\zeta(a) = 0.6\ F_\zeta(a) = 0.03$$
$$B_\zeta(b) = 0.05\ A_\zeta(b) = 0.6\ F_\zeta(b) = 0.3$$

$$B_\zeta(c) = 0.05 \ A_\zeta(c) = 0.6 \ F_\zeta(c) = 0.3$$

and,

$$\begin{gathered} B_\zeta(0) = 0.5 \ A_\zeta(0) = 0.2 \ F_\zeta(0) = 0.2 \\ B_\zeta(a) = 0.4 \ A_\zeta(a) = 0.3 \ F_\zeta(a) = 0.3 \\ B_\zeta(b) = 0.4 \ A_\zeta(b) = 0.3 \ F_\zeta(b) = 0.3 \\ B_\zeta(c) = 0.4 \ A_\zeta(c) = 0.3 \ F_\zeta(c) = 0.3 \end{gathered}$$

The routine calculation shows that, ζ and η are picture fuzzy ideals of M. Now, for any $m, z \in M$,

$$\begin{aligned} B_{\zeta+\eta}(0) &= \bigvee_{0=m+z} \{\min(B_\zeta(m), B_\eta(z))\} \\ &= \bigvee \{\min(B_\zeta(0), B_\eta(0)), \min(B_\zeta(a), B_\eta(a)), \min(B_\zeta(b), B_\eta(b)), \\ &\qquad \min(B_\zeta(c), B_\eta(c))\} \\ &= \bigvee \{0.5, 0.05, 0.05, 0.05\} \\ &= 0.5, \end{aligned}$$

$$\begin{aligned} B_{\zeta+\eta}(a) &= \bigvee_{a=m+z} \{\min(B_\zeta(m), B_\eta(z))\} \\ &= \bigvee \{\min(B_\zeta(0), B_\eta(a)), \min(B_\zeta(a), B_\eta(0)), \min(B_\zeta(b), B_\eta(c)), \\ &\qquad \min(B_\zeta(c), B_\eta(b))\} \\ &= \bigvee \{0.4, 0.05, 0.05, 0.05\} \\ &= 0.4, \end{aligned}$$

$$\begin{aligned} B_{\zeta+\eta}(b) &= \bigvee_{b=m+z} \{\min(B_\zeta(m), B_\eta(z))\} \\ &= \bigvee \{\min(B_\zeta(0), B_\eta(b)), \min(B_\zeta(b), B_\eta(0)), \min(B_\zeta(a), B_\eta(c)), \\ &\qquad \min(B_\zeta(c), B_\eta(a))\} \\ &= \bigvee \{0.4, 0.05, 0.05, 0.05\} \\ &= 0.4, \end{aligned}$$

$$\begin{aligned} B_{\zeta+\eta}(c) &= \bigvee_{c=m+z} \{\min(B_\zeta(m), B_\eta(z))\} \\ &= \bigvee \{\min(B_\zeta(0), B_\eta(c)), \min(B_\zeta(c), B_\eta(0)), \min(B_\zeta(a), B_\eta(b)), \end{aligned}$$

$$\min(B_\zeta(b), B_\eta(a))\}$$
$$= \bigvee\{0.4, 0.05, 0.05, 0.05\}$$
$$= 0.4,$$

$$\begin{aligned}
A_{\zeta+\eta}(0) &= \bigwedge_{0=m+z}\{\max(A_\zeta(m), A_\eta(z))\}\\
&= \bigwedge\{\max(A_\zeta(0), A_\eta(0)), \max(A_\zeta(a), A_\eta(a)), \max(A_\zeta(b), A_\eta(b)),\\
&\qquad \max(A_\zeta(c), A_\eta(c))\}\\
&= \bigwedge\{0.2, 0.6, 0.6, 0.6\}\\
&= 0.2,
\end{aligned}$$

$$\begin{aligned}
A_{\zeta+\eta}(a) &= \bigwedge_{a=m+z}\{\max(A_\zeta(m), A_\eta(z))\}\\
&= \bigwedge\{\max(A_\zeta(0), A_\eta(a)), \max(A_\zeta(a), A_\eta(0)), \max(A_\zeta(b), A_\eta(c)),\\
&\qquad \max(A_\zeta(c), A_\eta(b))\}\\
&= \bigwedge\{0.3, 0.6, 0.6, 0.6\}\\
&= 0.3,
\end{aligned}$$

$$\begin{aligned}
A_{\zeta+\eta}(b) &= \bigwedge_{b=m+z}\{\max(A_\zeta(m), A_\eta(z))\}\\
&= \bigwedge\{\max(A_\zeta(0), A_\eta(b)), \max(A_\zeta(b), A_\eta(0)), \max(A_\zeta(a), A_\eta(c)),\\
&\qquad \max(A_\zeta(c), A_\eta(a))\}\\
&= \bigwedge\{0.3, 0.6, 0.6, 0.6\}\\
&= 0.3,
\end{aligned}$$

$$\begin{aligned}
A_{\zeta+\eta}(c) &= \bigwedge_{c=m+z}\{\max(A_\zeta(m), A_\eta(z))\}\\
&= \bigwedge\{\max(A_\zeta(c), A_\eta(0)), \max(A_\zeta(0), A_\eta(c)), \max(A_\zeta(a), A_\eta(b)),\\
&\qquad \max(A_\zeta(b), A_\eta(a))\}\\
&= \bigwedge\{0.6, 0.3, 0.6, 0.6\}\\
&= 0.3,
\end{aligned}$$

$$\begin{aligned}F_{\zeta+\eta}(0) &= \bigwedge_{0=m+z}\{\max(F_\zeta(m),F_\eta(z))\}\\ &= \bigwedge\{\max(F_\zeta(0),F_\eta(0)),\max(F_\zeta(a),F_\eta(a)),\max(F_\zeta(b),F_\eta(b)),\\ &\qquad \max(F_\zeta(c),F_\eta(c))\}\\ &= \bigwedge\{0.2,0.3,0.3,0.3\}\\ &= 0.2,\end{aligned}$$

$$\begin{aligned}F_{\zeta+\eta}(a) &= \bigwedge_{a=m+z}\{\max(F_\zeta(m),F_\eta(z))\}\\ &= \bigwedge\{\max(F_\zeta(0),F_\eta(a)),\max(F_\zeta(a),F_\eta(0)),\max(F_\zeta(b),F_\eta(c)),\\ &\qquad \max(F_\zeta(c),F_\eta(b))\}\\ &= \bigwedge\{0.3,0.3,0.3,0.3\}\\ &= 0.3,\end{aligned}$$

$$\begin{aligned}F_{\zeta+\eta}(b) &= \bigwedge_{b=m+z}\{\max(F_\zeta(m),F_\eta(z))\}\\ &= \bigwedge\{\max(F_\zeta(0),F_\eta(b)),\max(F_\zeta(b),F_\eta(0)),\max(F_\zeta(a),F_\eta(c)),\\ &\qquad \max(F_\zeta(c),F_\eta(a))\}\\ &= \bigwedge\{0.3,0.3,0.3,0.3\}\\ &= 0.3,\end{aligned}$$

$$\begin{aligned}F_{\zeta+\eta}(c) &= \bigwedge_{c=m+z}\{\max(F_\zeta(m),F_\eta(z))\}\\ &= \bigwedge\{\max(F_\zeta(0),F_\eta(c)),\max(F_\zeta(c),F_\eta(0)),\max(F_\zeta(a),F_\eta(b)),\\ &\qquad \max(F_\zeta(b),F_\eta(a))\}\\ &= \bigwedge\{0.3,0.3,0.3,0.3\}\\ &= 0.3,\end{aligned}$$

Therefore, $B_{\zeta+\eta}(l) = \begin{cases}0.5: & l=0\\ 0.4: & \text{otherwise.}\end{cases}$

$A_{\zeta+\eta}(l) = \begin{cases}0.2: & l=0\\ 0.3: & \text{otherwise.}\end{cases}$

$$F_{\zeta+\eta}(l) = \begin{cases} 0.3: & l = 0 \\ 0.3: & \text{otherwise.} \end{cases}$$

The routine calculation shows that, $\zeta + \eta$ is a picture fuzzy ideal of M.

Now we extend the above Theorem 3.2 to the sum of finite number of fuzzy ideals of a zero symmetric Γ-near-ring M.

Definition 3.6. Let M be a zero symmetric Γ-near ring and let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be the picture fuzzy ideals of a Γ-near ring M. For any $l \in M$, put $S(l) = \min\{\zeta_1(l_1), \zeta_2(l_2) \ldots \zeta_n(l_n) : l = l_1 + l_2 + \cdots + l_n,\ l_i \in M, i = 1 \text{ to } n\}$. Define $(\zeta_1 + \zeta_2 + \cdots + \zeta_n)(l) = sup\ S(l) = sup\{\min(\zeta_1(l_1), \zeta_2(l_2), \ldots, \zeta_n(l_n)) : l = l_1 + l_2 + \cdots + l_n\}$.

Remark 3.1. Let $l = l_1 + l_2 + \cdots + l_n$. Consider a transposition of the indices $(1, k),\ k > 1$. Then

$$\begin{aligned} l =& l_1 + l_2 + \cdots + l_{k-1} + l_k + l_{k+1} + \cdots + l_n \\ =& y + l_k - m + l_1 + l_2 + \cdots + l_{k-1} + l_{k+1} + \cdots + l_n \\ & (\text{ where } m = l_1 + l_2 + \cdots + l_{k-1}) \\ =& (m + l_k - m) + z - z + l_1 + z + l_{k+1} + \cdots + l_n \\ & (\text{ where } z = l_2 + \cdots + l_{k-1}) \\ =& l_k^{'} + z + l_1^{'} + l_{k+1} + ... + l_n \ (\text{ where } l_k^{'} = m + l_k - m,\ l_1^{'} = -z + l_1 + z) \\ =& l_k^{'} + l_2 + \cdots + l_{k-1} + l_1^{'} + l_{k+1} + \cdots + l_n. \end{aligned}$$

Thus
$\min\{\zeta_1(l_k^{'}), \zeta_2(l_2), \ldots \zeta_{k-1}(l_{k-1}), \zeta_k(l_1^{'}), \zeta_{k+1}(l_{k+1}), \ldots, \zeta_n(l_n)\} = \min\{\zeta_1(l_k), \zeta_2(l_2), \ldots, \zeta_{k-1}(l_{k-1}), \zeta_k(l_1), \zeta_{k+1}\ (l_{k+1}), \zeta_n(l_n)\} \in S(l)$.

This is true for every transposition (i, j) of the indices. Since every permutation is a product of transpositions, then for any permutation $\begin{pmatrix} 1\ 2 \ldots n \\ i_1\ i_2 \ldots i_n \end{pmatrix}$ we have $\min\{\zeta_1(l_{i1}), \zeta_2(l_{i2}), ..., \zeta_n(l_{in})\}$ belongs to $S(l)$ for $l = l_1 + l_2 + \cdots + l_n$. Hence $\zeta_1 + \zeta_2 + \cdots + \zeta_n = \zeta_{i1} + \zeta_{i2} + \cdots + \zeta_{in}$.

Theorem 3.7. Let M be a zero symmetric Γ-near ring. If $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the picture fuzzy ideals of M, then $\zeta_1 + \zeta_2 + \cdots + \zeta_n$ is also a picture fuzzy ideal of M.

Proof. Put $\zeta = \zeta_1 + \zeta_2 + \cdots + \zeta_n$.

(i) Let $l = l_1 + l_2 + \cdots + l_n, m = m_1 + m_2 + \cdots + m_n; l_i, m_i \in M,\ i = 1, 2, \ldots n$. Then $l - m = l_1 + l_2 + \cdots + l_n - m_1 - m_2 - m_3 - \cdots - m_n$. This can be expressed as $l - m = l_1^{'} - m_1^{'} + l_2^{'} - m_2^{'} + \cdots + l_n^{'} - m_n^{'}$, where $l_i^{'}$ is a conjugate of l_i and $m_i^{'}$ is a conjugate of m_i. Therefore $l - m = (l_1^{'} - m_1^{'}) + (l_2^{'} - m_2^{'}) + \cdots + (l_n^{'} - m_n^{'})$. Which implies,

$B_\zeta(l - m)$

$$
\begin{aligned}
&= B_\zeta((l_1^{'} - m_1^{'}) + (l_2^{'} - m_2^{'}) + \cdots + (l_n^{'} - m_n^{'})) \\
&= sup\{\min(B_{\zeta_1}(l_1^{'} - m_1^{'}), B_{\zeta_2}(l_2^{'} - m_2^{'}), \ldots B_{\zeta_n}(l_n^{'} - m_n^{'}))\} \\
&\geq sup\{\min(\min(B_{\zeta_1}(l_1^{'}), B_{\zeta_1}(m_1^{'})), \min(B_{\zeta_2}(l_2^{'}), B_{\zeta_2}(m_2^{'})), \ldots \min(B_{\zeta_n}(l_n^{'}), \\
&\qquad B_{\zeta_n}(m_n^{'})))\} \\
&= \min\{sup\{\min(B_{\zeta_1}(l_1^{'}), B_{\zeta_2}(l_2^{'}), \ldots B_{\zeta_n}(l_n^{'}))\}\}, sup\{\min(B_{\zeta_1}(m_1^{'}), B_{\zeta_2}(m_2^{'}), \\
&\qquad \ldots B_{\zeta_n}(m_n^{'}))\} \\
&= \min\{sup\ S(l), sup\ S(m)\} \\
&= \min\{B_\zeta(l), B_\zeta(m)\}.
\end{aligned}
$$

$$
\begin{aligned}
&A_\zeta(l - m) \\
&= A_\zeta\{(l_1^{'} - m_1^{'}) + (l_2^{'} - m_2^{'}) + \cdots + (l_n^{'} - m_n^{'})\} \\
&= sup\{\min(A_{\zeta_1}(l_1^{'} - m_1^{'}), A_{\zeta_2}(l_2^{'} - m_2^{'}), \ldots, A_{\zeta_n}(l_n^{'} - m_n^{'}))\} \\
&\leq sup\{\min(\max(A_{\zeta_1}(l_1^{'}), A_{\zeta_1}(m_1^{'}))), \max(A_{\zeta_2}(l_2^{'}), A_{\zeta_2}(m_2^{'})), \ldots, \max(A_{\zeta_n}(l_n^{'}), \\
&\qquad A_{\zeta_n}(m_n^{'}))\} \\
&= \max\{sup\{\min(A_{\zeta_1}(l_1^{'}), A_{\zeta_2}(l_2^{'}), \ldots, A_{\zeta_n}(l_n^{'}))\}\}, sup\{\min(A_{\zeta_1}(m_1^{'}), A_{\zeta_2}(m_2^{'}), \\
&\qquad \ldots A_{\zeta_n}(m_n^{'}))\} \\
&= \max\{A_\zeta(l), A_\zeta(m)\}.
\end{aligned}
$$

Similarly, $F_\zeta(l - m) \leq \max\{F_\zeta(l), F_\zeta(m)\}$.

(ii) Let $l, m \in M$ and $l = l_1 + l_2 + \cdots + l_n; l_i \in M,\ i = 1, 2, \ldots, n$.

Then $m + l - m = m + l_1 + l_2 + \cdots + l_n - m = y + l_1 - m + m + l_2 - m + m + l_3 - m + \cdots + m + l_n - m$. This implies that,

$$
\begin{aligned}
B_\zeta(m + l - m) &= B_\zeta(m + l_1 - m + m + l_2 - m + \cdots + m + l_n - m) \\
&= sup\{\min\{B_\zeta(m + l_1 - m), B_\zeta(m + l_2 - m), \ldots, B_\zeta(m + l_n - m)\}\} \\
&\leq sup\{\min\{B_\zeta(l_1), B_\zeta(l_2), \ldots, B_\zeta(l_n)\}\} \\
&= sup\ S(l) \\
&= B_\zeta(l).
\end{aligned}
$$

$$
\begin{aligned}
A_\zeta(m + l - m) &= A_\zeta(m + l_1 - m + m + l_2 - m + \cdots + m + l_n - m) \\
&= sup\{\min\{A_\zeta(m + l_1 - m), A_\zeta(m + l_2 - m), \ldots, A_\zeta(m + l_n - m)\}\} \\
&\leq sup\{\min\{A_\zeta(l_1), A_\zeta(l_2), \ldots, A_\zeta(l_n)\}\} \\
&= sup\ S(l) \\
&= A_\zeta(l).
\end{aligned}
$$

Similarly $F_\zeta(m + l - m) \leq F_\zeta(l)$.

(iii) Let $l, m, i \in M$ and $\alpha \in \Gamma$. And let $i = i_1 + i_2 + \cdots + i_n; i_i \in M, i = 1, 2, \ldots, n$. Which implies
$l+i = (l+i_1+i_2+\cdots+i_n)\alpha m-(i_2+i_3+\cdots+i_n+l)\alpha m+(i_2+i_3+\cdots+i_n+l)\alpha m-(i_3+i_4+\cdots+i_n+l)\ \alpha m+(i_3+i_4+\cdots+i_n+l)\alpha m-\cdots-(i_n+l)\alpha m+\cdots+(i_n+l)\alpha m$.
Now,

$$\begin{aligned}
&B_\zeta((l+i)\alpha m - l\alpha m)\\
&= B_\zeta\{(l+i_1+i_2+\cdots+i_n)\alpha m - (i_2+i_3+\cdots+i_n+l)\alpha m\\
&\qquad\qquad + (i_2+i_3+\cdots+i_n+l)\alpha m\\
&\ - (i_3+i_4+\cdots+i_n+l)\alpha m + (i_3+i_4+\cdots+i_n+l)\alpha m - \ldots\\
&\qquad\qquad - (i_n+l)\alpha m - l\alpha m\}\\
&= sup\{\min\{B_\zeta((l+i_1+i_2+\cdots+i_n)\alpha m - (i_2+i_3+\cdots+i_n+l)\alpha m),\\
&\ B_\zeta((i_2+i_3+\cdots+i_n+l)\alpha m - (i_3+i_4+\cdots+i_n+l)\alpha m), \ldots\\
&\qquad\qquad B_\zeta((i_n+l)\alpha m - l\alpha m)\}\}\\
&\geq sup\{\min(B_\zeta(i_1), B_\zeta(i_2), B_\zeta(i_3), \ldots B_\zeta(i_n))\}\\
&= sup\ S(l)\\
&= B_\zeta(i).
\end{aligned}$$

$$\begin{aligned}
&A_\zeta((l+i)\alpha m - l\alpha m)\\
&= A_\zeta\{(l+i_1+i_2+\cdots+i_n)\alpha m - (i_2+i_3+\cdots+i_n+l)\alpha m\\
&\qquad\qquad + (i_2+i_3+\cdots+i_n+l)\alpha m\\
&\ - (i_3+i_4+\cdots+i_n+l)\alpha m + (i_3+i_4+\cdots+i_n+l)\alpha m + \ldots\\
&\qquad\qquad + (i_n+l)\alpha m - l\alpha m\}\\
&= sup\{\min\{A_\zeta((l+i_1+i_2+\cdots+i_n)\alpha m - (l+i_2+i_3+\cdots+i_n)\alpha m),\\
&\ A_\zeta((l+i_2+i_3+\cdots+i_n)\alpha m - (l+i_3+i_4+\cdots+i_n)\alpha m), \ldots,\\
&\qquad\qquad A_\zeta((i_n+l)\alpha m - l\alpha m)\}\}\\
&\leq sup\{\min(A_\zeta(i_1), A_\zeta(i_2), A_\zeta(i_3), \ldots, A_\zeta(i_n))\}\\
&= sup\ S(l)\\
&= A_\zeta(i).
\end{aligned}$$

Similarly $F_\zeta((l+i)\alpha m - l\alpha m) = F_\zeta(i)$.

(iv) Let $l, m \in M$ and $\alpha \in \Gamma$. Put $m = m_1 + m_2 + \cdots + m_n; m_i \in M, i = 1, 2, \ldots, n$. Then

$$\begin{aligned}
B_\zeta(l\alpha m) &= B_\zeta(l\alpha(m_1+m_2+\cdots+m_n))\\
&= B_\zeta(l\alpha m_1 + = \alpha m_2 + \cdots + l\alpha m_n)\\
&= sup\{\min(B_\zeta(l\alpha m_1), B_\zeta(l\alpha m_2), \ldots, B_\zeta(l\alpha m_n))\}\\
&\leq sup\{\min(B_\zeta(m_1), B_\zeta(m_2), \ldots, B_\zeta(m_n))\}
\end{aligned}$$

$$= sup\ S(m)$$
$$= B_\zeta(m).$$

$$\begin{aligned} A_\zeta(l\alpha m) &= A_\zeta(l\alpha(m_1+m_2+\cdots+m_n)) \\ &= A_\zeta(l\alpha m_1 + l\alpha m_2 + \cdots + l\alpha m_n) \\ &= sup\{\min(A_\zeta(l\alpha m_1), A_\zeta(l\alpha m_2), \ldots, A_\zeta(l\alpha m_n))\} \\ &\leq sup\{\min(A_\zeta(m_1), A_\zeta(m_2), \ldots, A_\zeta(m_n))\} \\ &= sup\ S(m) \\ &= A_\zeta(m). \end{aligned}$$

Similarly $F_\zeta(l\alpha m) \leq F_\zeta(m)$. Hence B_ζ is a picture fuzzy ideal of M. □

Definition 3.8. Let M be a zero symmetric Γ -near ring and $\zeta_1, \zeta_2, \ldots, \zeta_n$ be the picture fuzzy ideals of M. Then the sum $\zeta = \zeta_1 + \zeta_2 + \cdots + \zeta_n$ is said to be direct, if

$$\min\{(B_{\zeta_1} + B_{\zeta_2} + \cdots + B_{\zeta_{i-1}} + B_{\zeta_{i+1}} + \cdots + B_{\zeta_n}), B_{\zeta_i}\} = 0$$
$$\max\{(A_{\zeta_1} + A_{\zeta_2} + \cdots + A_{\zeta_{i-1}} + A_{\zeta_{i+1}} + \cdots + A_{\zeta_n}), A_{\zeta_i}\} = 0$$
$$\max\{(F_{\zeta_1} + F_{\zeta_2} + \cdots + F_{\zeta_{i-1}} + F_{\zeta_{i+1}} + \cdots + F_{\zeta_n}), F_{\zeta_i}\} = 0$$

Theorem 3.9. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be the direct sum of Γ-near rings $M_1, M_2, \ldots, M_n$ with left or right identity $e = (e_1, e_2, \ldots, e_n)$ and ζ be a picture fuzzy ideal of M. Then there exists picture fuzzy ideals $\zeta_1, \zeta_2, \ldots, \zeta_n$ of M such that $\zeta = \zeta_1 \oplus \zeta_2 \oplus \cdots \oplus \zeta_n$.

Proof. Let $l_i = (0, 0, \ldots, 0, l_i, 0, \ldots, 0)$ and $e_i = (0, 0, \ldots, 0, e_i, 0, \ldots, 0), \alpha \in \Gamma$. Then for $l = (l_1, l_2, \ldots, l_n) = l_1 + l_2 + \cdots + l_n$, we have

$$B_\zeta(l) = B_\zeta(l_1 + l_2 + \cdots + l_n) \geq \min\{B_\zeta(l_1), B_\zeta(l_2), \ldots, B_\zeta(l_n)\}.$$
$$A_\zeta(l) = A_\zeta(l_1 + l_2 + \cdots + l_n) \leq \max\{A_\zeta(l_1), A_\zeta(l_2), \ldots, A_\zeta(l_n)\}.$$
$$F_\zeta(l) = F_\zeta(l_1 + l_2 + \cdots + l_n) \leq \max\{F_\zeta(l_1), F_\zeta(l_2), \ldots, F_\zeta(l_n)\}.$$

But

$$B_\zeta(l_i) = B_\zeta(e_i \alpha l) \geq B_\zeta(l),$$
$$A_\zeta(l_i) = A_\zeta(e_i \alpha l) \leq A_\zeta(l),$$
$$F_\zeta(l_i) = F_\zeta(e_i \alpha l) \leq F_\zeta(l),$$

for $i = 1, 2, \ldots, n$. That is,

$$\min\{B_\zeta(l_1), B_\zeta(l_2), \ldots, B_\zeta(l_n)\} \geq B_\zeta(l),$$
$$\max\{A_\zeta(l_1), A_\zeta(l_2), \ldots, A_\zeta(l_n)\} \leq A_\zeta(l),$$
$$\max\{F_\zeta(l_1), F_\zeta(l_2), \ldots, F_\zeta(l_n)\} \leq F_\zeta(l).$$

Thus

$$B_\zeta(l) \le \min\{B_\zeta(l_1), B_\zeta(l_2), \ldots, B_\zeta(l_n)\},$$
$$A_\zeta(l) \ge \max\{A_\zeta(l_1), A_\zeta(l_2), \ldots, A_\zeta(l_n)\},$$
$$F_\zeta(l) \ge \max\{F_\zeta(l_1), F_\zeta(l_2), \ldots, F_\zeta(l_n)\}.$$

Define, $(B_{\zeta_i}, A_{\zeta_i}, F_{\zeta_i})$ on M by

$$B_{\zeta_i}(l) = \begin{cases} B_\zeta(l) : l \in M_i \\ 0 \ : \quad \text{otherwise} \end{cases}$$

$$A_{\zeta_i}(l) = \begin{cases} A_\zeta(l) : \ l \in M_i \\ 0 \quad : \quad \text{otherwise} \end{cases}$$

$$F_{\zeta_i}(l) = \begin{cases} F_\zeta(l) : l \in M_i \\ 0 \ : \quad \text{otherwise.} \end{cases}$$

Hence $\zeta_1 \oplus \zeta_2 \oplus \cdots \oplus \zeta_n = \zeta$. □

4. Conclusion

In this paper, we have introduced the concept on sum of picture fuzzy ideals of a Γ-near ring and the direct sum of picture fuzzy ideals of a Γ-near ring and investigated several properties. Also, the relations between sum and direct sum of picture fuzzy ideals of a Γ-near rings are also discussed. Further, this can extend to sum of anti picture fuzzy ideals of a Γ-near ring and the direct sum of anti picture fuzzy ideals of a Γ-near ring in future research work.

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